Spines, backbones and orthopedic surgery.

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- It is also a natural question to ask how such a process behaves as the strip becomes thinner.

- Specifically, is there a critical width below which there is no possibility of surviving and how does the process behave at criticality?
Branching Brownian motion in a strip \((0, K)\)

Particles execute Brownian motion with killing on exiting \((0, K)\).

Particles undergo dyadic branching at constant rate \(\beta > 0\).
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- \(Z = \{Z_t(\cdot) : t \geq 0\}\), where \(Z_t(\cdot) = \sum_{i=1}^{N_t} \delta_{x_i(t)}(\cdot)\), is the sequence of random measures which describes the evolution of particles.
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- \(Z = \{Z_t(\cdot) : t \geq 0\}\), where \(Z_t(\cdot) = \sum_{i=1}^{N_t} \delta_{x_i(t)}(\cdot)\), is the sequence of random measures which describes the evolution of particles.
- The process becomes extinct at time \(\zeta^K := \inf\{t > 0 : Z_t(0, K) = 0\}\).
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- A straightforward exercise to show that $\lambda_c(K) = \beta - \pi^2/2K^2$ [coming from the ‘ground state’ positive eigen-function $\sin(\pi x/K)$] and hence $K^* = \pi/\sqrt{2\beta}$. 
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- **Theorem:** (i) When $K > K^*$ then $\phi_K \in (0, 1)$ on $(0, K)$ and is the unique solution to the ODE

\[
\frac{1}{2}f'' + \beta(f^2 - f) = 0 \text{ on } (0, K) \text{ and } f(0) = f(K) = 1. \tag{1}
\]

(ii) When $K \leq K^*$ then $\phi_K \equiv 1$ and the ODE (1) has no solutions valued in $[0, 1]$ other than the trivial ones.
Intuition: finding a spine is equivalent to survival.
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- Martingale density to condition a Brownian motion \( \{B_t : t \geq 0\} \) to stay in the interval \((0, K)\) is

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e^{\frac{\pi^2 t}{2K^2}} \sin\left(\frac{\pi B_t}{K}\right) 1_{\{t < \tau(0, K)\}}, \quad t \geq 0.
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- Martingale density to condition \( Z \) to survive
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  M_t := \int_{(0, K)} e^{(\frac{\pi^2}{2K^2} - \beta)t} \sin(\frac{\pi x}{K}) Z_t(dx), \quad t \geq 0,
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  induces a spine decomposition:
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  induces a spine decomposition:

- (i) Run a Brownian motion conditioned to stay in \((0, K)\) - the spine.
- (ii) At rate \(2\beta\) dress the path of the spine with independent copies of \(\mathbb{P}^K\)-BBMs.
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- $M$ is $L^1(\mathbb{P}^K)$-convergent if and only if $K > K^*$ and if this condition fails then $M_\infty \equiv 0$ a.s.
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  have the property that for $x \in (0, K)$

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  \prod_{i=1}^{N_t} \phi_K(x_i(t)) \text{ and } \prod_{i=1}^{N_t} \psi_K(x_i(t))
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  are bounded martingales and hence both $\phi_K$ and $\psi_K$ solve (1).
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  are bounded martingales and hence both $\phi_K$ and $\psi_K$ solve (1).
- Conversely, for any solution $f$ to (1),

  \[ \prod_{i=1}^{N_t} f(x_i(t)) \]

  is a bounded martingale with limit $1_{\{\zeta^K < \infty\}}$. 
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- When $K = K^*$ we have $\{M_\infty = 0\} = \{\zeta^K < \infty\}$ almost surely $\Rightarrow$ cannot condition on survival and get a spine decomposition.

- Look instead for a quasi-stationary type result and try to understand if there is any meaning to the limit

$$\lim_{K \downarrow K^*} \mathbb{P}_x^K (\cdot | \zeta^K = \infty)$$
Blue and Red for $K > K^*$
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- Does the blue tree describe a branching diffusion?
- Do the red subtrees describe branching diffusions?
- What happens if there is no blue tree?
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\frac{1}{2} \triangle - \frac{\phi'_K}{1 - \phi_K} \frac{d}{dx} \left( = L^w_0 := L^w - \frac{Lw}{w} \text{ where } L = \frac{1}{2} \triangle \text{ and } w = 1 - \phi_K \right)
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- Can be shown that **Red** describes $\mathbb{P}^K(\cdot|\zeta^K < \infty)$. 
Backbone decomposition for $K > K^*$

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- If ‘tails’ then grow a Blue tree and with rate $2\beta \phi_K(\cdot)$ ‘dress’ the spatial paths of the Blue tree with independent Red trees.
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- $\mathbb{P}^K_x (\cdot | \zeta^K = \infty)$ has the same law as observing a dressed **Blue** tree.
**Backbone decomposition for** \( K > K^* \)

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A significance convenience from this construction:
- \( \mathbb{P}_x^K (\cdot | \zeta^K = \infty) \) has the same law as observing a dressed Blue tree.
- Equivalently \( \mathbb{P}_x^K (\cdot | \zeta^K = \infty) \) has the same law as the backbone construction conditioned on throwing a ’tail’.
Orthopedic surgery \((K > K' > K^*)\)
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- (Blue motion) $\frac{1}{2} \Delta - \frac{\phi_K'}{1-\phi_K} \frac{d}{dx} \rightarrow \frac{1}{2} \Delta + \frac{(\sin \frac{\pi x}{K^*})'}{\sin \frac{\pi x}{K^*}} \frac{d}{dx}$
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Theorem.
The backbone becomes a spine through orthopedic surgery and gives the quasi-stationary result:

$$\lim_{K \downarrow K^*} P_K x(\cdot|\zeta_K = \infty) = P^* x(\cdot),$$

for $x \in (0, K^*)$ where $P^* x$ is the law of a particle system consisting of a spine behaving as a Brownian motion conditioned to stay in the interval $(0, K^*)$, dressing of the spine at rate $2\beta$ with $P_K \cdot$ branching diffusions.
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- **(Blue branching rate)** $\beta(1 - \phi_K) \rightarrow 0$
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- **(rate of dressing Red on to Blue)** $2\beta \phi_K \rightarrow 2\beta$. 

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The backbone becomes a spine through orthopedic surgery and gives the quasi-stationary result:

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- **(Blue branching rate)** \( \beta(1-\phi_K) \rightarrow 0 \)
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\lim_{K \downarrow K^*} \mathbb{P}_x^K (\cdot | \zeta^K = \infty) = P^*_x(\cdot),
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for \( x \in (0, K^*) \) where \( P^*_x \) is the law of a particle system consisting of
- a spine behaving as a Brownian motion conditioned to stay in the interval \((0, K^*)\),
- dressing of the spine at rate \( 2\beta \) with \( \mathbb{P}^{K^*} \) branching diffusions.
Quasi-stationary limit as $K \downarrow K^*$

- $\phi_K(\cdot) \uparrow 1$: $1 - \phi_K(x) \sim c_K \sin(\pi x / K)$ as $K \downarrow K^*$.
- (Blue motion) $\frac{1}{2} \Delta - \frac{\phi'_K}{1 - \phi_K} \frac{d}{dx} \xrightarrow{\sim} \frac{1}{2} \Delta + \frac{(\sin \pi x / K^*)'}{\sin \pi x / K^*} \frac{d}{dx}$
- (Blue branching rate) $\beta(1 - \phi_K) \to 0$
- (Red motion) $\frac{1}{2} \Delta + \frac{\phi'_K}{\phi_K} \frac{d}{dx} \xrightarrow{\sim} \frac{1}{2} \Delta$
- (Red branching rate) $\beta \phi_K \to \beta$
- (rate of dressing Red on to Blue) $2\beta \phi_K \to 2\beta$.

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