An overview of processes with branching

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Galton–Watson processes

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Branching Random Walks

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Crump–Mode–Jagers processes

$\downarrow$

Fragmentation Chains

$\downarrow$

Fragmentation processes

$\downarrow$

Continuous-time Galton–Watson processes

$\downarrow$

Continuous-state branching processes

$\downarrow$

Branching Brownian Motion

$\downarrow$

Super-Brownian motion

1. Galton–Watson processes

- A model for asexual reproduction represented by the Markov chain $\{Z_n : n \geq 0\}$ and $k \in \mathbb{N}$,
- Z_n is the number of individuals in the n-th generation.
- Take, $Z_0 = k \in \mathbb{N}$. [Usually assume $k = 1$].
- Thereafter, iterate from generation n to $n+1$ via

$$Z_{n+1} = \sum_{j=1}^{Z_n} A_j^{(n+1)},$$

  where $\{A_j^{(n+1)}\}$ are independent of $\{Z_1, \cdots, Z_n\}$ and have a common distribution $\{p_i : i \geq 0\}$, known as the *offspring distribution*. [Assume $p_1 = 0$ and distribution is not defective].
- What makes this a *branching* process?
  Momentarily incorporate the the initial value k into the notation:

$$Z_n^{(k)} =^d Z_n^{(1)}(1) + Z_n^{(1)}(2) + \cdots + Z_n^{(1)}(k),$$

  where $Z_n^{(1)}(j)$ is an i.i.d. copy of $Z_n^{(1)}$.
- Note Markov property: for $k, n, n' \in \mathbb{N}$

$$Z_{n+n'}^{(k)} = \widetilde{Z}_{n'}^{(Z_n^{(k)})}$$

  where $\widetilde{Z}_\cdot^{(\cdot)}$ is an independent copy of $Z_\cdot^{(\cdot)}$.

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2. Criticality

- Suppose that $q = \mathbb{P}(Z_n = 0 \text{ for some } n \in \mathbb{N})$
- Let $\mathcal{F}_n = \sigma(Z_0, \cdots, Z_n)$, then we have a martingale:

$$\mathbb{P}(Z_n = 0 \text{ for some } n \in \mathbb{N} | \mathcal{F}_n) = q^{Z_n}$$

- The constant q is a fixed point of the p.g.f. of the offspring distribution

$$q = \sum_{j=0}^{\infty} q^i p_i$$

- Conversely, any fixed point, q, of the p.g.f. of the offspring distribution makes a martingale: $\{q^{Z_n} : n \geq 0\}$
- Note that $q_1 = 1$ is always a root.
- Note, moreover, that

$$\left. \sum_{j=0}^{\infty} s^i p_i \right|_{s=0+} = p_0$$

and a little argument shows that the p.g.f. is strictly convex and hence there is a second root $q_2 \in (0, 1)$ if and only if

$$m := \sum_{i=1}^{\infty} i p_i > 1.$$

- Always assume that $m < \infty$

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- Either $m \leq 1$, in which case $q = 1$ is the only fixed point, i.e. extinction is certain.
- Or $m > 1$, in which case $q_2^{Z_n}$ is a uniformly integrable martingale which has a non-trivial limit with mean $q_2 < 1$.
- This means $\mathbb{P}(Z_n = 0 \text{ for some } n \geq 1) \in [0, 1)$ and hence, as this probability is also a fixed point, it must be equal to q_2.
- Theorem:
  (i) If $m \leq 1$, (sub)critical, then $\mathbb{P}(Z_n = 0 \text{ for some } n \geq 1) = 1$
  (ii) If $m > 1$, supercritical, then $\mathbb{P}(Z_n = 0 \text{ for some } n \geq 1) = q$ where $q = \sum_{i=0}^{\infty} q^i p_i$.

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4. Supercritical growth

- Another martingale (with unit mean):

$$M_n := \frac{Z_n}{m^n}, \qquad n \geq 0.$$

- Note

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = \frac{1}{m^{n+1}}\mathbb{E}[\widetilde{Z}_1^{(Z_n)}|\mathcal{F}_n]$$

so it is enough to prove that

$$\mathbb{E}[Z_1^{(\ell)}] = \ell m,$$

but this is obvious.

- As a positive martingale, M_n has an almost sure limit, say M_∞. If the latter is non-trivial, then

$$Z_n^{(k)} \sim m^n M_\infty \text{ as } n \to \infty.$$

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5. Supercritical growth

- Theorem (Kesten–Stigum): Suppose that $m > 1$. The martingale M_n is L^1 convergent (in particular $\mathbb{E}(M_\infty) = 1$ and hence M_∞ is not trivial if and only

$$\sum_{i=1}^{\infty} i \log i p_i < \infty.$$

Otherwise $M_\infty \equiv 0$.

- In fact, when there is L^1 convergence $\{M_\infty = 0\} = \{Z_n = 0 \text{ for some } n\}$

6. Branching Random Walk

- We want to build a spatial 'branching' out of the Galton–Watson process. We think of the population in generation n as random measure on $\mathbb{R}^d$ with atomic support, each atom having unit mass: i.e. a process $X = \{X_n(\cdot) : n \geq 0\}$, where

$$X_n(\cdot) = \sum_{i=1}^{Z_n} \delta_{x_i^n}(\cdot),$$

  and $\{x_i^n : i = 1, \cdots, Z_n\}$ are the positions and number of particles making up the support of X_n.
- Consider a point process $\xi(\cdot)$ on $\mathbb{R}^d$.
- Let $X_0(\cdot) = \delta_0(\cdot)$ and, given $X_n(\cdot)$,

$$X_{n+1}(\cdot) = \sum_{i=1}^{Z_n} \xi_i^{(n+1)}(\cdot - x_i^n),$$

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- X is still a branching process, in the sense of independent additivity, and Markovian, as a measure-valued process.

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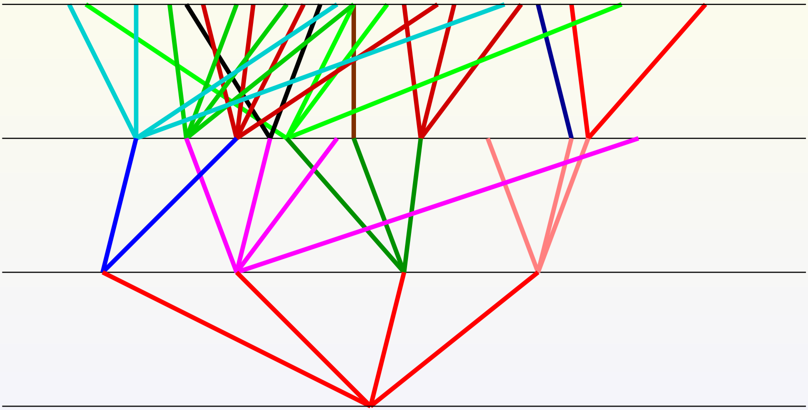

8. Branching Random Walk (BRW)

- Note that the total mass of X_n, is integer valued and satisfies

$$\langle 1, X_n \rangle = \int_{\mathbb{R}^d} 1 \; X_n(\mathrm{d}x) = Z_n$$

- Pre-emptive choice of notation: Z_n is again a Galton–Watson process.
- Drop to one dimension, fix $\theta \in \mathbb{R}$ and let

$$m(\theta) = \mathbb{E}\left[\sum_{i=1}^{Z_1} \mathrm{e}^{-\theta x_i^n}\right]$$

  Note that when $\theta = 0$ then $m(0) = m = \mathbb{E}(Z_1)$, as before.
- Another martingale:

$$W_n(\theta) := \frac{1}{m(\theta)^n} \sum_{i=1}^{Z_n} \mathrm{e}^{-\theta x_i^n}, \qquad n \geq 0.$$

  Note that when $\theta = 0$, then $W_n(0) = Z_n/m^n$.
- Again, as a non-negative martingale, it has an almost sure limit, say $W_\infty(\theta)$.

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9. Biggins' Martingale Convergence Theorem

- Let $\theta_1 = \inf\{\theta : m(\theta) < \infty\}$ and $\theta_2 = \sup\{\theta : m(\theta) < \infty\}$.
- Theorem (Biggins 1977): Suppose that $\theta_1 < \theta_2$. Then there exists an interval (θ_*, θ^*) such that, for all $\theta \in (\theta_1, \theta_2)$, $W_\infty(\theta)$ is an L^1 limit if and only if
$$\mathbb{E}[W_1(\theta)|\log W_1(\theta)|] < \infty \text{ and } \theta \in (\theta_*, \theta^*),$$
otherwise $W_\infty(\theta) \equiv 0$.
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9. Biggins' Martingale Convergence Theorem

- Let $\theta_1 = \inf\{\theta : m(\theta) < \infty\}$ and $\theta_2 = \sup\{\theta : m(\theta) < \infty\}$.
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10. Right most particle

- By differentiating $W_n(\theta)$ across its conditional expectation, one quickly establishes that

$$\partial W_n(\theta) := -\frac{\partial}{\partial \theta} W_n(\theta), \qquad n \geq 0,$$

is also a martingale (albeit signed).

- Theorem (Biggins and K. 2004) (A continuation of Biggins' Martingale Convergence Theorem). For $\theta \in [\theta_*, \theta^*]$ the derivative martingale limit exists almost surely (denoted by $\partial W_\infty(\theta)$) and

$$\partial W_\infty(\theta) \equiv 0 \text{ when } \theta \in (\theta_*, \theta^*),$$

moreover, under some additional mild moment conditions,

$$\partial W_\infty(\theta_*) > 0 \text{ and } \partial W_\infty(\theta^*) > 0$$

on $\{Z_n = 0 \text{ for some } n\}^c$ and both have infinite mean.

- Theorem (Aidekon 2012): Under mild conditions,

$$\mathbb{P}(R_n - \gamma^* n + \frac{3}{2} c^* \log n \leq x) \to \mathbb{E}[\exp\{-C^* \partial W_\infty(\theta^*)\}]$$

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11. Crump–Mode–Jagers processes

- In the special case that $\mathrm{supp}\,\xi = [0, \infty)$, the BRW describes a Crump–Mode-Jagers process.
- Think of 'spatial displacement' as 'birth time'.
- Rather than studying the CMJ indexed by generation, it is now more natural to study the evolution the process as it evolves in 'time'.
- For example, the 'coming generation':

  $\mathcal{C}(t) = \{$individuals born after time t whose parents were born before time $t\}$

  Denote their birth times by $\{\sigma_u : u \in \mathcal{C}(t)\}$.
- Malthusian Parameter: The constant $\alpha > 0$ such that

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\alpha x}\xi(\mathrm{d}x)\right] = 1.$$

- In fact

$$\Lambda_t(\alpha) := \sum_{u\in\mathcal{C}(t)} \mathrm{e}^{-\alpha\sigma_u}, \qquad t \geq 0$$

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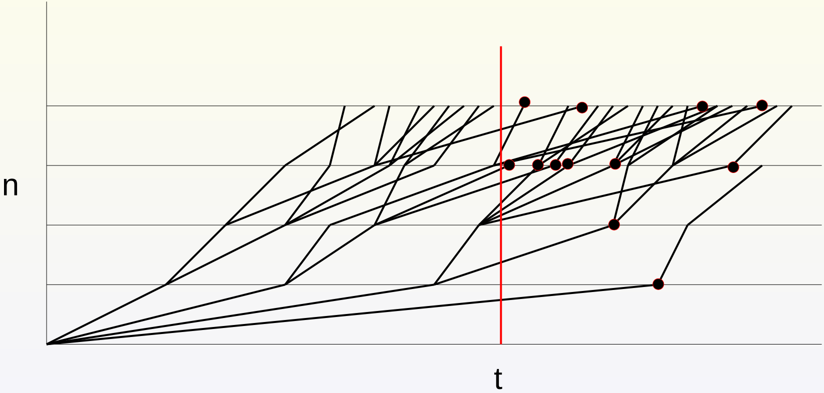

13. Fragmentation Chains

- Consider the unit interval $[0,1]$ fragmented randomly into smaller pieces (intervals), and the pieces arranged in descending order of their lengths: $B_1, B_2, \cdots,$

$$\sum_{i\geq 1} |B_i| = 1$$

- Use independent samples from the distribution of $(B_1, B_2, \cdots)$ to fragment each of these pieces further into further pieces. e.g. given a fragment interval I, it can be dislocated further into fragments $(IB'_1, IB'_2, \cdots)$, where $(B'_1, B'_2, \cdots)$ is an independent sample from the distribution of $(B_1, B_2, \cdots)$.
- Suppose that $(I_1^n, I_2^n, \cdots)$ are the pieces (intervals) in the n-th generation of fragmentations, arranged in decreasing order of size. Then

$$X_n(\cdot) := \sum_{j=1}^{\infty} \delta_{-\log I_j^n}(\cdot), \qquad n \geq 0,$$

is a C-M-J process.
- Note that the process can equally be represented by a sequence of ordered length (or mass) partitions of $[0,1]$, indexed by generations of fragmentation:

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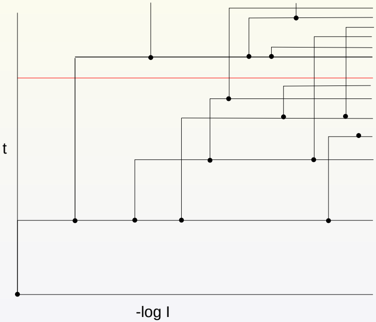
- Instead of considering dislocations, at each generation, we can set the process in real time by applying an independent and identically distributed exponential holding time to each fragment before it dislocates.



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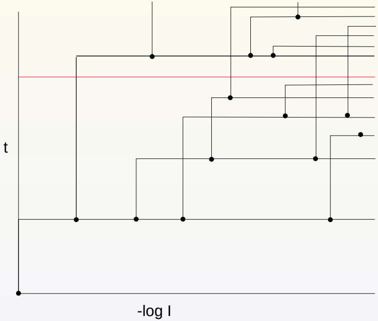

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15. Fragmentation Chains

- A self-similar fragmentation chain has the property that a fragment of size s has an independent exponentially distributed holding time with rate which is proportional to s^α. Here, $\alpha \in \mathbb{R}$ is the index of self-similarity.
- $\alpha = 0$ is the homogenous case considered on the previous slide.
- In general, the resulting fragmentation chain has the property that, for any $c \in (0, 1)$,

$$\{cI(c^\alpha t) : t \geq 0\} \text{ with } I(0) = (1, 0, 0, \cdots),$$

  is equal in law to

$$\{I(t) : t \geq 0\} \text{ with } I(0) = (c, 0, 0, \cdots).$$

- Branching and Markov properties still to be found: The law of $I(t + s)$ given $\{I(u) : u \leq s\}$ is equal in law to the ordering of the collective mass partitions produced by an independent sequence of mass partitions

$$c_1 I(c_1^\alpha s), c_2 I(c_2^\alpha s), c \ldots,$$

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16. Fragmentation processes

- A (self-similar) mass fragmentation process is a stochastic process which is valued on the space of ordered mass partitions of $[0,1]$ and which satisfies the branching and Markov properties in the previous bullet point.
- A fragmentation chain fits the description of a fragmentation process, but there are more processes to be found in the latter class.
- In general, one can find fragmentation processes for which dislocation times form a dense set of of $[0,\infty)$.
- Fragmentation chains are to fragmentation processes what compound Poiss on processes are to Lévy processes.

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17. Continuous-time Galton–Watson processes

- Following the example of fragmentation chains, we can convert a Galton–Watson process to a continuous-time branching process by giving holding times (life lengths) to individuals before they branch, which are independent and identically exponentially distributed with parameter, say, β.
- Write $\{Z(t) : t \geq 0\}$ for the number of individuals at time t.
- Temporarily introducing extra notation for the number of initial individuals: $Z^{(k)}(t)$ satisfies $Z^{(k)}(0) = 0$.
- We still have the branching property

$$Z^{(k)}(t) =^d Z_1^{(1)}(t) + \cdots + Z_k^{(1)}(t), \qquad t \geq 0,$$

  where $Z_i^{(1)}(\cdot)$ are i.i.d. copies of $Z^{(1)}(\cdot)$.
- The lack of memory property for each life length gives us the Markov property

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- Following the example of fragmentation chains, we can convert a Galton–Watson process to a continuous-time branching process by giving holding times (life lengths) to individuals before they branch, which are independent and identically exponentially distributed with parameter, say, β.
- Write $\{Z(t) : t \geq 0\}$ for the number of individuals at time t.
- Temporarily introducing extra notation for the number of initial individuals: $Z^{(k)}(t)$ satisfies $Z^{(k)}(0) = 0$.
- We still have the branching property

$$Z^{(k)}(t) =^d Z_1^{(1)}(t) + \cdots + Z_k^{(1)}(t), \qquad t \geq 0,$$

where $Z_i^{(1)}(\cdot)$ are i.i.d. copies of $Z^{(1)}(\cdot)$.

- The lack of memory property for each life length gives us the Markov property

$$Z(t+s) =^d \widetilde{Z}^{(Z_t)}(s), \qquad t \geq 0,$$

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18. Continuous-time Galton–Watson processes

- Lots of familiar properties when we compare to the discrete-time Galton–Watson process
- If $m \leq 1$ then $\mathbb{P}(Z(t) = 0 \text{ for some } t > 0) = 1$
- If $m > 1$ then $q := \mathbb{P}(Z(t) = 0 \text{ for some } t > 0) < 1$ and $Z(t) \to \infty$ on $\{Z(t) = 0 \text{ for some } t > 0\}^c$.
- $q^{Z(t)}$, $t \geq 0$, is a martingale.
- When $Z(0) = 1$,
$$\mathbb{E}[Z(t)] = \mathrm{e}^{\beta(m-1)t}$$
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- Exercise: guess the analogue of the Kesten–Stigum Theorem.

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19. Lamperti time change

- Introduce a new distribution on $\{\pi_i : i = -1, 0, 1, 2, \cdots\}$, where $\pi_i = p_{i+1}$. (The number of GW offspring minus 1).
- Write, for $t \geq 0$,
$$J_t = \int_0^t Z(s)\mathrm{d}s,$$
set
$$\varphi(t) = \inf\{u > 0 : J_u > t\}$$
(with the usual $\inf \emptyset = \infty$) and define
$$L(t) = Z(\varphi(t)), \qquad t \geq 0.$$
- Consider what happens up to the first branching time T_1:
- If $Z(0) = k$, then T_1 is the minimum of k independent exponentially distributed random variables, each with rate q. i.e. $T_1 \sim \exp(k\beta)$.
- And hence, $J_{T_1} = kT_1 \sim \exp(\beta)$.
- Apply Markov property at time T_1, when the number of individuals moves from k to $k + i$ with probability π_i, and use this same reasoning again until the second branching time.
- The time change $Z(\varphi(t))$ has the effect of spacing out branching events with independent and identical exponentially distributed random times.
- Said another way: $\{L(t) : t \geq 0\}$ is a compound Poisson process with arrival rate q and jump distribution $F(\mathrm{d}x) = \sum_{i=-1}^{\infty} \pi_i \delta_i(\mathrm{d}x)$

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- The converse is also true: Suppose that L_t is a compound Poisson process with arrival rate q and jump distribution $F(\mathrm{d}x) = \sum_{i=-1}^{\infty} \pi\delta_i(\mathrm{d}x)$. Let

$$K_t = \int_0^t \frac{1}{L(s)} \mathrm{d}s, \qquad t \geq 0,$$

set

$$\theta(t) = \inf\{u > 0 : K_u > t\}$$

and define

$$Z(t) = L(\theta(t) \wedge \tau_0), \qquad t \geq 0,$$

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21. Continuous-state branching process (CSBP)

- A $[0,\infty]$-valued strong Markov process $Z = \{Z(t) : t \geq 0\}$ with probabilities $\{P_x : x \geq 0\}$ is called a *continuous-state branching process* if it has paths that are right-continuous with left limits and its law observes the branching property: for all $\theta \geq 0$ and $x, y \geq 0$,

$$\mathbb{E}_{x+y}(\mathrm{e}^{-\theta Z(t)}) = \mathbb{E}_x(\mathrm{e}^{-\theta Z(t)})\mathbb{E}_y(\mathrm{e}^{-\theta Z(t)}).$$

- The same time change using the additive functional

$$\int_0^t Y(s)\mathrm{d}s, \qquad t \geq 0$$

makes $Z(\varphi(t))$, $t \geq 0$ a Lévy process with no negative jumps.

- Similarly, given a Lévy process $\{L(t) : t \geq 0\}$ with no negative jumps, the same transform as before using the additive functional

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22. CSBP semi-group

- Recall that a (finite mean) Lévy process with no negative jumps is characterised through its Laplace exponent:

$$\mathbb{E}[\mathrm{e}^{-\lambda L(t)}] = \exp\{\psi(\lambda)t\}, \qquad t \geq 0,$$

where

$$\psi(\lambda) = -a\lambda + \sigma\lambda^2 + \int_{(0,\infty)} (\mathrm{e}^{-\lambda x} - 1 + \lambda x)\Pi(\mathrm{d}x), \qquad \lambda \geq 0,$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$.

- Not easy to see the CSBP Z through a path wise construction, but some information in its semi-group: For $\theta \geq 0$, $x > 0$,

$$\mathbb{E}_x(\mathrm{e}^{-\theta Z(t)}) = \mathrm{e}^{-u_t(\theta)x},$$

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- For comparison, consider the semi-group of the continuous-time G–W process:

$$\mathbb{E}_x(\mathrm{e}^{-\theta Z(t)}) = v_t(\theta)x,$$

  where, for $t, \theta \geq 0$,

$$\frac{\partial}{\partial t} v_t(\theta) = G(v_t(\theta)), \qquad \text{and } u_0(\theta) = \mathrm{e}^{-\theta}.$$

  where $G(s) = \beta\left(\sum_{j=0}^{\infty} s^j p_j - s\right)$.

23. Growth, extinguishing and extinction

- As before

$$\mathbb{E}_x(Z(t)) = x\mathrm{e}^{at} \text{ and } Z(t)\mathrm{e}^{-at} \text{ is a martingale.}$$

- (Sub)critical if $a \le 0$. Supercritical if $a > 0$.
- Extinguishing: $\{Z(t) \to 0\}$
- Extinction: $\{Z(t) = 0$ for some $t \ge 0\}$ (implies extinguishing).
- $\mathbb{P}_x(\text{Extinguishing}) = \exp\{-\psi^{-1}(0)x\}$ (< 1 if and only if $a > 0$).
- $\{\text{Extinguishing}\}^c = \{Z(t) \to \infty\}$.
- If $\{\text{Extinction}\} \neq \emptyset$ then $\{\text{Extinguishing}\backslash\text{Extinction}\} = \emptyset$ (a.s.)
- Extinction if and only if

$$\int^{\infty} \frac{1}{\psi(\theta)} \mathrm{d}\theta < \infty$$

- Note that continuous-time G–W processes cannot undergo extinguishing as they are integer valued! The reason why there are two types of 'dying out' for CSBPs can be see through particle approximation via continuous-state G–W processes (see later).

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- Note that continuous-time G–W processes cannot undergo extinguishing as they are integer valued! The reason why there are two types of 'dying out' for CSBPs can be see through particle approximation via continuous-state G–W processes (see later).

23. Growth, extinguishing and extinction

- As before

$$\mathbb{E}_x(Z(t)) = x\mathrm{e}^{at} \text{ and } Z(t)\mathrm{e}^{-at} \text{ is a martingale.}$$

- (Sub)critical if $a \leq 0$. Supercritical if $a > 0$.
- Extinguishing: $\{Z(t) \to 0\}$
- Extinction: $\{Z(t) = 0 \text{ for some } t \geq 0\}$ (implies extinguishing).
- $\mathbb{P}_x(\text{Extinguishing}) = \exp\{-\psi^{-1}(0)x\}$ (< 1 if and only if $a > 0$).
- $\{\text{Extinguishing}\}^c = \{Z(t) \to \infty\}$.
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24. Branching Brownian Motion

- Take a continuous-time Galton–Watson processes and make each individual execute an independent (d-dimensional) Brownian motion from its space-time moment of birth until branching.
- Similarly to a BRW, we can describe the process as a continuous-time atomic-valued Markov process

$$X_t(\cdot) = \sum_{i=1}^{Z(t)} \delta_{x_i(t)}(\cdot)$$

- Total mass: $\langle 1, X(t) \rangle = \int_{\mathbb{R}^d} 1 X_t(\mathrm{d}x) = Z(t)$
- Martingales:

$$\mathrm{e}^{-\beta(m-1)t} \sum_{i=1}^{N_t} \mathrm{e}^{-\lambda x_i(t) - \lambda^2 t/2}, \qquad t \geq 0$$

- Right most particle for $d = 1$: $R_t := \sup\{x \in \mathbb{R} : X_t(x, \infty) > 0\}$

$$\frac{R_t}{t} \to \sqrt{2\beta} \text{ as } t \to \infty \text{ on } \{\text{Extinction}\}^c$$

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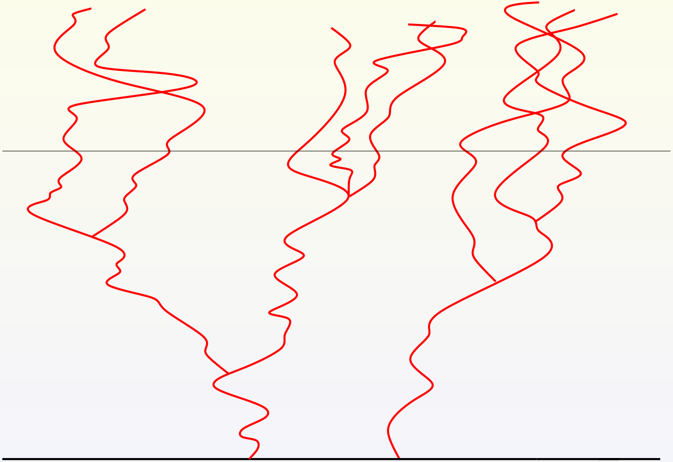
25. Branching Brownian Motion (BBM)



26. ψ-superBrownian motion

- Want to construct a Markov process with values in the $\mathcal{M}_F$, the space of finite measures (on $\mathbb{R}^d$).
- Fix $\mu \in \mathcal{M}_F$, fix $n \in \mathbb{N}$. Consider a BBM with an initial number of particles which are scattered in space according to a Poisson random field with intensity $n\mu(\cdot)$.
- Normally BBM assigns unit mass to each individual. Now assign mass $1/n$ to each individual.
- Fix the branching rate in BBM at n.
- Impose a special offspring distribution such that the generator

$$G(s) = \text{``}\ \beta\left(\sum_{j=0}^{\infty} s^j p_j - s\right)\ \text{''} = n\left(\frac{1}{n}\psi(n(1-s))\right)$$

where

$$\psi(\lambda) = \sigma\lambda^2 + \int_{(0,\infty)} (e^{-\lambda y} - 1 + \lambda x)\nu(\mathrm{d}x)$$

- Now take a 'weak' limit of the resulting BBM as $n \to \infty$ and we get a measure-valued Markov process $\mathcal{X} := \{\mathcal{X}_t(\cdot) : t \geq 0\}$ valued in $\mathcal{M}_F$. [Note $\mathcal{X}_0(\cdot) = \mu(\cdot)$].

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- We can characterise the evolution of $\mathcal{X}$ through its semi-group: For all bounded measurable f, $t \geq 0$ and $x \in \mathbb{R}^d$ and $\mu \in \mathcal{M}_F$,

$$\mathbb{E}_{\delta_x}(\mathrm{e}^{-\langle f, \mathcal{X}_t \rangle}) =: \mathrm{e}^{-w(x,t)}$$

(branching)

$$\mathbb{E}_{\mu}(\mathrm{e}^{-\langle f, \mathcal{X}_t \rangle}) = \mathrm{e}^{-\int_{\mathbb{R}^d} w(x,t)\mu(\mathrm{d}x)}$$

and

$$\frac{\partial}{\partial t} w(x,t) = \frac{1}{2}\frac{\partial}{\partial x^2} w(x,t) - \psi(w(x,t)).$$

- It is straightforward to check that $\{\langle 1, \mathcal{X}_t \rangle : t \geq 0\}$ is a CSBP. Inspection of ψ shows that it is a critical CSBP.
- An adaptation of this reasoning can produce a supercritical (subcritical) ψ-superBrownian motion.
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