# De Finetti's control problem and spectrally negative Lévy processes

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- The ruin problem looks at the behaviour of the surplus process up to and on the event

$$\{\tau_0^+ < \infty\}$$

where

$$\tau_0^+ = \inf\{t > 0 : X_t < 0\}.$$

under the assumption that $c - \lambda\mathbb{E}(\xi_1) > 0$, i.e. $\lim_{t\uparrow\infty} X_t = \infty$.

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- In this talk, you have the option to think of $X = \{X_t : t \geq 0\}$ as a spectrally negative Lévy process.
- In either case, for $\theta \geq 0$ we may work with the Laplace exponent

$$\psi(\theta) := \log \mathbb{E}_0(e^{\theta X_1}),$$

which is strictly convex, respects the condition $\psi'(0+) > 0$, passes though the origin and so tends to $+\infty$ at $\infty$.

# de Finetti's view of the ruin problem

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- de Finetti's control problem: find the value function and matching dividend strategy $L^*$ such that

$$v(x) = \sup_L \mathbb{E}_x \left( \int_0^{\sigma^L} e^{-qt} dL_t \right) = \mathbb{E}_x \left( \int_0^{\sigma^{L^*}} e^{-qt} dL^*_t \right)$$

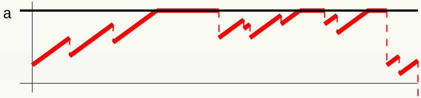
  where $q > 0$ and the supremum is taken over all admissible dividend strategies.

# Reflection strategies

- It has been shown that the optimal strategy is of a 'barrier type with reflection':

$$L_t^a = (a \vee \sup_{s \le t} X_s) - a$$

for some optimal level $a$. Below a realisation of $X_t - L_t^a$

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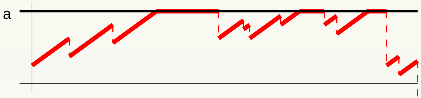


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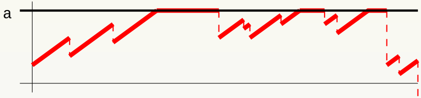


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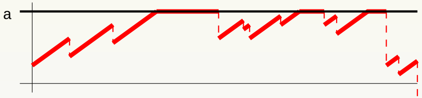


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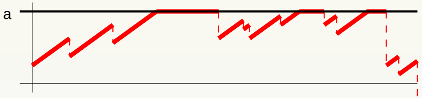


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3. (Ascue & Muler 2005) Cramér-Lundberg process with gamma distributed jumps having density proportional to $xe^{-x}$.

# Scale functions are a natural tool

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- For each $q \geq 0$ there exists a function $W^{(q)} : [0, \infty) \to [0, \infty)$ defined by its Laplace transform

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q}$$

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- For all $a > 0$,

$$v^a(x) := \mathbb{E}_x \left( \int_0^{L^a} e^{-qx} dL_t^a \right) = \begin{cases} \frac{W^{(q)}(x)}{W^{(q)\prime}(a)} & \text{when } x \leq a \\ (x-a) + \frac{W^{(q)}(a)}{W^{(q)\prime}(a)} & \text{when } x > a \end{cases}$$

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  2. The above condition is satisfied if the distribution of the i.i.d. claims $\{\xi_i : i \geq 1\}$ has a density $f$ which is completely monotone.[1] i.e. $(-1)^n d^n f/dx^n \geq 0$ for all $n \geq 1$.

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- The latter condition expands vastly the claim distributions in the Cramér-Lundberg model for which the reflection barrier strategy is optimal.
- Moreover, it gives some hint as to why the Azcue & Muler example fails: In that case the claim distribution has a density which is not completely monotone!

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# Restricted class of control strategies

- Many variations on this theme have been examined for the case of diffusions (Jeanblanc & Shiryaev 1995, Elena Boguslavskaya's Ph.D. thesis) as well as the Cramér-Lundberg case with exponential jumps (Gerber & Shiu 2006) including the following:

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- The class of admissible strategies is further restricted to the case that

$$L_t = \int_0^t \phi(s)\,ds$$

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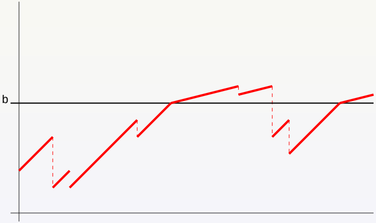
- What was the optimal strategy appeared in the aforementioned articles?

# Refraction strategies

- A refraction strategy refers to the control $\phi(x) = \delta \mathbf{1}_{(x>b)}$ for some threshold level $b \geq 0$. Thus the controlled process would need to solve the stochastic differential equation

$$U_t = X_t - \delta \int_0^t \mathbf{1}_{(U_s>b)}\, ds.$$



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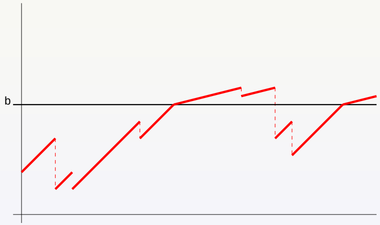


- Note in the case that $X$ is a general spectrally negative Lévy process the above SDE is highly non-trivial if there is no Gaussian component.

# K., and Loeffen (2010)

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- Write $\mathbb{W}^{(q)}$ for the scale function associated with $X_t - \delta t$.
- Suppose that

$$\kappa_0^- := \inf\{t > 0 : U_t < 0\}.$$

For $q \geq 0$ and $x \geq 0$

$$v^b(x) := \mathbb{E}_x\left(\int_0^{\kappa_0^-} e^{-qt}\delta \mathbf{1}_{\{U_t > b\}}\, ds\right)$$

$$= -\delta \int_0^{(x-b)\vee 0} \mathbb{W}^{(q)}(z)\, dz + \frac{W^{(q)}(x) + \delta \mathbf{1}_{\{x \geq b\}} \int_b^x \mathbb{W}^{(q)}(x-y)\, W^{(q)\prime}(y)\mathrm{d}y}{\varphi(q) \int_0^\infty \mathrm{e}^{-\varphi(q)y}\, W^{(q)\prime}(y+b)\mathrm{d}y},$$

where $\varphi(q)$ is the unique solution in $(0,\infty)$ to $\psi(\theta) - \delta\theta = 0$.

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- The refraction strategy at level $b^*$ is optimal amongst the absolutely continuous $\delta$-bounded strategies as soon as we assume that the common distribution of the claims is absolutely continuous with completely monotone density.[2]

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- There has been significant work recently in pushing forward methodology which allows one to develop either closed form or semi-explicit expressions for $W^{(q)}$. See the forthcoming review of the theory of scale functions in the springer Lecture Notes in Mathematics series "Lévy Matters": K., Rivero and Kuznetsov (2011).

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