Stable process in a cone

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based on joint work with Victor Rivero and Weerapat Satitkanitkul (a.k.a. Pite)

Stable process

- For $d \geq 2$, let $X := (X_t : t \geq 0)$, with probabilities $\mathbb{P} = (\mathbb{P}_x, x \in \mathbb{R}^d)$, be a d-dimensional isotropic stable process of index $\alpha \in (0, 2)$.
- Equivalently, this means $(X, \mathbb{P})$ is a d-dimensional Lévy process with characteristic exponent (up to a multiplicative constant)

$$\Psi(\theta) = |\theta|^\alpha, \qquad \theta \in \mathbb{R}^d.$$

- Equivalently X is a Lévy process for which there is an $\alpha \in (0, 2)$ and which satisfies:

$$\text{under } \mathbb{P}_x, \text{ the law of } (cX_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P}_{cx},$$

for $c > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$.
- As a self-similar Markov process, X can be represented by the Lamperti-Kiu transformation

$$X_t = \mathrm{e}^{\xi_{\varphi(t)}} \Theta_{\varphi(t)}, \qquad t \geq 0,$$

where (ξ, Θ) is a Markov additive process on $\mathbb{R} \times \Omega$ with probabilities

$$\mathbf{P}_{\log|x|, \arg(x)}, \qquad x \in \mathbb{R}^d,$$

and

$$\varphi(t) = \inf\{s > 0 : \int_0^s \mathrm{e}^{\alpha \xi_u} \mathrm{d}u > t\}.$$

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HARMONIC FUNCTIONS ON THE CONE

- Lipchitz cone, $\Gamma = \{x \in \mathbb{R}^d : x \neq 0, \arg(x) \in \Omega\}$,
- Exit time from the cone i.e. $\kappa_\Gamma = \inf\{s > 0 : X_s \notin \Gamma\}$.
- Bañuelos and Bogdan (2004): There exists $M : \mathbb{R}^d \to \mathbb{R}$ such that
  - $M(x) = 0$ for all $x \notin \Gamma$.
  - M is locally bounded on $\mathbb{R}^d$
  - There is a $\beta = \beta(\Gamma, \alpha) \in (0, \alpha)$, such that

$$M(x) = |x|^\beta M(x/|x|) = |x|^\beta M(\arg(x)), \qquad x \neq 0.$$

  - Up to a multiplicative constant, M is the unique such that

$$M(x) = \mathbb{E}_x[M(X_{\tau_B})\mathbf{1}_{(\tau_B < \kappa_\Gamma)}], \qquad x \in \mathbb{R}^d,$$

    where B is any open bounded domain and $\tau_B = \inf\{t > 0 : X_t \notin B\}$.
- Bañuelos and Bogdan (2004) and Bogdan, Palmowski, Wang (2018): We have

$$\lim_{a \to 0} \sup_{x \in \Gamma,\, |t^{-1/\alpha} x| \leq a} \frac{\mathbb{P}_x(\kappa_\Gamma > t)}{M(x) t^{-\beta/\alpha}} = C,$$

where $C > 0$ is a constant.

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Theorem

(i) *For any* $t > 0$, *and* $x \in \Gamma$,

$$\mathbb{P}_x^{\triangleleft}(A) := \lim_{s\to\infty} \mathbb{P}_x\left(A \,|\kappa_\Gamma > t + s\right), \qquad A \in \mathcal{F}_t,$$

defines a family of conservative probabilities on the space of càdlàg paths such that

$$\left.\frac{\mathrm{d}\mathbb{P}_x^{\triangleleft}}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} := \mathbf{1}_{(t<\kappa_\Gamma)}\frac{M(X_t)}{M(x)}, \qquad t \geq 0, \text{ and } x \in \Gamma.$$

In particular, the right-hand side is a martingale.

(ii) *Let* $\mathbb{P}^{\triangleleft} := (\mathbb{P}_x^{\triangleleft}, x \in \Gamma)$. *The process* $(X, \mathbb{P}^{\triangleleft})$, *is a self-similar Markov process.*

It's just an Esscher transform.....

- As a self-similar Markov process, we can identify $(X_t \mathbf{1}_{(t<\kappa_\Gamma)}, t \geq 0)$ as a ssMp and hence has a representation on $\{t < \kappa_\Gamma\}$

$$X_t = \mathrm{e}^{\xi_{\varphi(t)}} \Theta_{\varphi(t)},$$

where $\varphi(t) = \inf\{s > 0 : \int_0^s \mathrm{e}^{\alpha \xi_u} \mathrm{d}u > t\}$ and (ξ, Θ) has conditional stationary and independent increments, i.e. is a MAP.

- Hence the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^{\triangleleft}_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} := \mathbf{1}_{(t<\kappa_\Gamma)} \frac{M(X_t)}{M(x)} = \mathbf{1}_{(\varphi(t)<\mathtt{k}^\Omega)} \mathrm{e}^{\beta(\xi_{\varphi(t)} - x)} \frac{M(\Theta_{\varphi(t)})}{M(\Theta_0)}$$

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ENTRANCE LAW

Let $p^\Gamma_t(x,y)$, $x,y\in\Gamma$, $t\geq 0$, be the semigroup of X killed on exiting the cone Γ.

Theorem (Bogdan, Palmowski, Wang (2018))

The following limit exits,

$$n_t(y) := \lim_{\Gamma\ni x\to 0}\frac{p^\Gamma_t(x,y)}{\mathbb{P}_x(\kappa_\Gamma>t)t^{\beta/\alpha}},\qquad x,y\in\Gamma, t>0, \tag{1}$$

and $(n_t(y)\mathrm{d}y, t>0)$, *serves as an entrance law to* $(X,\mathbb{P}^\Gamma)$, *in the sense that*

$$n_{t+s}(y)=\int_\Gamma n_t(x)p^\Gamma_s(x,y)\mathrm{d}x,\qquad y\in\Gamma, s,t\geq 0.$$

- Also easy to show that, in the sense of weak convergence,

$$\mathbb{P}^\triangleleft_0(X_t\in\mathrm{d}y):=\lim_{\Gamma\ni x\to 0}\frac{M(y)}{M(x)}\mathbb{P}_x(X_t\in\mathrm{d}y,\ t<\kappa_\Gamma)=CM(y)n_t(y)\mathrm{d}y.$$

- Can the process 'start from the apex of the cone' in a stronger sense?
- More precisely: Can we extend the Feller semigroup associated to $(\mathbb{P}^\triangleleft_x, x\in\Gamma)$ to a Feller semigroup on the state space $\Gamma\cup\{0\}$?

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Apex as an entrance point

We prove the stronger:

Theorem

The limit $\mathbb{P}_0^{\triangleleft} := \lim_{\Gamma \ni x \to 0} \mathbb{P}_x^{\triangleleft}$ *is well defined on the Skorokhod space, so that,* $(X, (\mathbb{P}_x^{\triangleleft}, x \in \Gamma \cup \{0\}))$ *is both Feller and self-similar which enters continuously at the origin, after which it never returns.*

Heuristic of proof: Duality

- The basic idea is to build an auxiliary process $(X, \mathbb{P}^{\triangleright})$, which continuously absorbs at the origin in such a way that when it is time reversed out of the origin, one sees $(X, \mathbb{P}^{\triangleleft})$. This 'duality' will play a central role in the analysis.
- The first step is thus to prove:

Theorem (Conditioning to continuously absorb a the apex)

For $A \in \mathcal{F}_t$, on the space of càdlàg paths with a cemetery state,

$$\mathbb{P}^{\triangleright}_x(A,\, t < k^{\{0\}}) := \lim_{a \to 0} \mathbb{P}_x(A,\, t < \kappa_\Gamma \wedge \tau_a^{\oplus} | \tau_a^{\oplus} < \kappa_\Gamma),$$

is well defined as a stochastic process which is continuously absorbed at the apex of Γ, where $k^{\{0\}} = \inf\{t > 0 : |X_t| = 0\}$ and $\tau_a^{\oplus} = \inf\{s > 0 : |X_s| < a\}$.
Moreover, for $A \in \mathcal{F}_t$,

$$\mathbb{P}^{\triangleright}_x(A,\, t < k^{\{0\}}) = \mathbb{E}_x\left[\mathbf{1}_{(A,\, t < \kappa_\Gamma)} \frac{H(X_t)}{H(x)}\right], \qquad t \geq 0,$$

where

$$H(x) = |x|^{\alpha - d} M(x/|x|^2) = |x|^{\alpha - \beta - d} M(\arg(x)),$$

thus making $(X, \mathbb{P}^{\triangleright})$ a ssMp.

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Theorem (Duality)

Under $\mathbb{P}_0^{\triangleleft}$, the time reversed process

$$\overleftarrow{X}_t := X_{(\mathtt{k}-t)-}, \qquad t \leq \mathtt{k},$$

is a homogenous strong Markov process whose transitions agree with those of $(X, \mathbb{P}_x^{\triangleright})$, $x \in \Gamma$, where $\mathtt{k}$ is an L-time of $(X, \mathbb{P}_x^{\triangleleft})$, $x \in \Gamma \cup \{0\}$.

- Note: this automatically gives the continuous emergence of $(X, \mathbb{P}^{\triangleleft})$ from the origin.
- As a parenthesis we can also prove a Riesz-Bogdan-Żak type result: With $Kx = x/|x|^2$, $x \in \mathbb{R}^d$, the process $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}_x^{\triangleleft}$, $x \in \Gamma$, is equal in law to $(X_t, t < \mathtt{k}^{\{0\}})$ under $\mathbb{P}_x^{\triangleright}$, $x \in \Gamma$, where

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A standard verification gives us the Skorokhod convergence:

Proposition (Verification of Skorokhod convergence)

Write

$$\tau_\varepsilon^\ominus = \inf\{t : |X_t| \geq \varepsilon\}, \qquad \varepsilon > 0.$$

Suppose that the following conditions hold:

(a) $\lim_{\varepsilon\to 0} \limsup_{\Gamma \ni z \to 0} \mathbb{E}_z^\triangleleft[\tau_\varepsilon^\ominus] = 0$

(b) $\lim_{\Gamma \ni z \to 0} \mathbb{P}_z^\triangleleft(X_{\tau_\varepsilon^\ominus} \in \cdot) =: \mu_\varepsilon(\cdot)$ *exists for all* $\varepsilon > 0$

(c) $\mathbb{P}_0^\triangleleft$*-almost surely,* $X_0 = 0$ *and* $X_t \neq 0$ *for all* $t > 0$

(d) $\mathbb{P}_0^\triangleleft((X_{\tau_\varepsilon^\ominus + t})_{t\geq 0} \in \cdot) = \int_\Gamma \mu_\varepsilon(\mathrm{d}y)\mathbb{P}_y^\triangleleft(\cdot)$ *for every* $\varepsilon > 0$

Then the mapping

$$\Gamma \cup \{0\} \ni z \mapsto \mathbb{P}_z^\triangleleft$$

is continuous in the weak topology on the Skorokhod space.

Where is the work?

- Conditioning to continuously absorb at the apex needs

$$\lim_{\Gamma \ni a \to 0} \frac{\mathbb{P}_x(\tau_a^{\oplus} < \kappa_\Gamma)}{H(x)a^{d+\beta-\alpha}} = C \in (0, \infty).$$

- We can convert the analysis in the above asymptotic into a question about the stability of $X_{\tau_a^{\oplus}}$ by working with yet another change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}_x^{\circ}}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} := \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}},$$

which conditions the stable process to continuously absorb at the origin (without taking account of the cone Γ). Note: This is an Esscher transform of the MAP underlying the stable process $(X, \mathbb{P})$.

- The Riesz-Bogdan-Żak transform states that:
The process $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}_x$, $x \neq 0$, is equal in law to $(X_t, \mathbb{P}^{\circ}_{Kx})$, where $\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}$, $t \geq 0$.

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- Through a sequences of changes of measures and RBZ-transformation with Kx, we can write:

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- Which in turn needs the stability (distributional convergence) as $a \to 0$ of the distribution of $(X_{\tau_1^{\ominus}}, \mathbb{P}^{\triangleleft}_x)$ as $\Gamma \ni x \to 0$.
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We need some MAP excursion theory

- The process $(X, \mathbb{P}^{\triangleleft})$ is a self-similar Markov process which observes the Lamperti-Kiu transform with underlying MAP (ξ, Θ), with probabilities

$$\mathbf{P}^{\triangleleft}_{x,\theta}, \quad x \in \mathbb{R}, \theta \in \mathbb{S}^{d-1}.$$

- For each $t > 0$, let $\bar{\xi}_t = \sup_{u \le t} \xi_u$. Maisonneuve's classical theory of exit systems in now implies that there exist an additive functional $(\ell_t, t \ge 0)$ carried by the set of times $\{t \ge 0 : (\bar{\xi}_t - \xi_t, \Theta_t) \in \{0\} \times \mathbb{S}^{d-1}\}$.
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Markov additive renewal theory to the rescue

- Classical work of Gerold Alsmeyer (and others before him) hold the key to the convergence of $(H^+_{T_b} - b, \Theta^+_{T_b})$ as $b \to \infty$
- Formally speaking, we can write

$$\mathbf{E}^{\triangleleft}_{0,\phi}[f(H^+_{T_b} - b, \Theta^+_{T_b})] = \int_0^b \int_\Omega U_\phi(\mathrm{d}z, \mathrm{d}\theta) \mathbb{N}^{\triangleleft}_\theta \Big(f(\epsilon(\zeta) - (b - z), \Theta^\epsilon(\zeta)); \epsilon(\zeta) > b - z \Big)$$

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for $z \geq 0$ and $\theta \in \Omega$, and $(\mathbb{N}^{\triangleleft}_\theta, \theta \in \Omega)$ is a family of excursion measures on the canonical space of MAP excursions $((\epsilon(t), \Theta^\epsilon(t)), t \leq \zeta)$.
- Just as with classical renewal theory we can take limits of the above convolution and expect

$$\lim_{b\to\infty} \mathbf{E}^{\triangleleft}_{0,\phi}[f(H^+_{T_b} - b, \Theta^+_{T_b})] = \int_0^\infty \int_\Omega \pi(\mathrm{d}\theta)\mathrm{d}r \mathbb{N}^{\triangleleft}_\theta \Big(f(\epsilon(\zeta) - r, \Theta^\epsilon(\zeta)); \epsilon(\zeta) > r \Big),$$

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How to get past π

- Generally speaking proving the existence of a stationary distribution on modulator of the ladder height process, Θ^+, of a MAP seems extremely difficult (and almost nothing seems to be known).
- How we got round it:

Theorem

There exists a probability measure, υ on Ω, which is invariant in the sense that

$$\mathbb{E}^{\triangleleft}_{\upsilon}\left[g(\arg(X_{\tau^{\ominus}_{\mathrm{e}}}))\right] := \int_{\Omega} \upsilon(\mathrm{d}\phi) g(\theta)$$

for bounded measurable $g : \Omega \to [0, \infty)$, such that, for all $x \in \Gamma$ and any continuous and bounded $f : \Gamma^2 \to [0, \infty)$,

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- This only appears to have transferred the problem of the existence of π to the existence of υ
- We are going to use the theory of Harris recurrence to check the condition that there exists a probability measure, $\rho(\cdot)$ on $\mathcal{B}(\Omega)$ (Borel sets in Ω) such that, for some $\lambda > 0$,
$$\mathbb{P}_{\theta}^{\triangleleft}(\Xi_1 \in E) \geq \lambda\rho(E), \text{ for all } \theta \in \Omega, E \in \mathcal{B}(\Omega),$$
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- The function

$$g(x;E) := \mathbb{E}_x\left[M(X_{\tau_1^\ominus})\mathbf{1}_{(\arg(X_{\tau_1^\ominus})\in E,\ \tau_1^\ominus<\kappa_\Gamma)}\right],$$

for $x \in \Gamma$ such that $|x| < 1$ is a regular harmonic function and note that, by scalings

$$g(\theta/\mathrm{e};E) = \mathbb{E}_\theta\left[M(X_{T_1})\mathbf{1}_{(\Xi_1\in E,\ T_1<\kappa_\Gamma)}\right], \qquad \theta \in \Omega.$$

- The function $M(x)$ is similarly harmonic.
- Hence, fix θ_0 with $|\theta_0| = 1$ so that $M(\theta_0/\mathrm{e}) = 1$ and then thanks to the Harnack inequality we have, for $x \in \Gamma$ such that $|x| < 1/2$,

$$C^{-1}M(x) \le \frac{g(x;E)}{g(\theta_0/\mathrm{e};E)} \le CM(x)$$

for a universal constant C which does not depend on E, x or x_0.

- Rearranging gives us for $x = \theta/\mathrm{e}$

$$\frac{g(\theta/\mathrm{e};E)}{M(\theta/\mathrm{e})} = \mathbb{P}^{\triangleleft}_\theta(\Xi_1 \in E) \ge C^{-1}g(\theta_0/\mathrm{e};E) =: \lambda\rho(E).$$

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Thank you!