Deep factorisation of the stable process

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Stable processes

**Definition**

A Lévy process $X$ is called (strictly) $\alpha$-stable if it satisfies the scaling property

$$(cX_{c^{-\alpha}t})_{t \geq 0} \overset{d}{=} X_{P_{c \times}}, \quad c > 0.$$ 

Necessarily $\alpha \in (0, 2]$. [$\alpha = 2 \rightarrow$ BM, exclude this.]

The quantity $\rho = P_0(X_t \geq 0)$ will frequently appear as will $\hat{\rho} = 1 - \rho$.

- The characteristic exponent $\Psi(\theta) := -t^{-1} \log \mathbb{E}(e^{i\theta X_t})$ satisfies
  $$\Psi(\theta) = |\theta|^\alpha (e^{\pi i \alpha (\frac{1}{2} - \rho)} 1_{(\theta > 0)} + e^{-\pi i \alpha (\frac{1}{2} - \rho)} 1_{(\theta < 0)}), \quad \theta \in \mathbb{R}.$$ 

- Assume jumps in both directions.
The Wiener–Hopf factorisation

For a given characteristic exponent of a Lévy process, $\Psi$, there exist unique Bernstein functions, $\kappa$ and $\hat{\kappa}$ such that, up to a multiplicative constant,

$$\Psi(\theta) = \hat{\kappa}(i\theta)\kappa(-i\theta), \quad \theta \in \mathbb{R}.$$

As Bernstein functions, $\kappa$ and $\hat{\kappa}$ can be seen as the Laplace exponents of (killed) subordinators.

The probabilistic significance of these subordinators, is that their range corresponds precisely to the range of the running maximum of $X$ and of $-X$ respectively.
The Wiener–Hopf factorisation

- Explicit Wiener-Hopf factorisations are extremely rare!
- For the stable processes we are interested in we have

\[ \kappa(\lambda) = \lambda^{\alpha \rho} \text{ and } \hat{\kappa}(\lambda) = \lambda^{\alpha \hat{\rho}}, \quad \lambda \geq 0 \]

where \( 0 < \alpha \rho, \alpha \hat{\rho} < 1 \).
- Hypergeometric Lévy processes are another recently discovered family of Lévy processes for which the factorisation are known explicitly: For appropriate parameters \((\beta, \gamma, \hat{\beta}, \hat{\gamma})\)

\[ \Psi(z) = \frac{\Gamma(1 - \beta + \gamma - iz)}{\Gamma(1 - \beta - iz)} \frac{\Gamma(\hat{\beta} + \hat{\gamma} + iz)}{\Gamma(\hat{\beta} + iz)}. \]
Deep factorisation of the stable process

- Another factorisation also exists, which is more ‘deeply’ embedded in the stable process.
- Based around the representation of the stable process as a real-valued self-similar Markov process (rssMp):

An $\mathbb{R}$-valued regular strong Markov process $(X_t : t \geq 0)$ with probabilities $P_x, x \in \mathbb{R}$, is a rssMp if, there is a stability index $\alpha > 0$ such that, for all $c > 0$ and $x \in \mathbb{R},$

$$(cX_{tc^{-\alpha}} : t \geq 0) \text{ under } P_x \text{ is } P_{cx}.$$
Markov additive processes (MAPs)

- $E$ is a finite state space
- $(J(t))_{t \geq 0}$ is a continuous-time, irreducible Markov chain on $E$
- process $(\xi, J)$ in $\mathbb{R} \times E$ is called a Markov additive process (MAP) with probabilities $P_{x,i}$, $x \in \mathbb{R}$, $i \in E$, if, for any $i \in E$, $s, t \geq 0$: Given $\{J(t) = i\}$,
  - $(\xi(t + s) - \xi(t), J(t + s)) \perp \{(\xi(u), J(u)) : u \leq t\}$,
  - $(\xi(t + s) - \xi(t), J(t + s)) \overset{d}{=} (\xi(s), J(s))$ with $(\xi(0), J(0)) = (0, i)$. 
Pathwise description of a MAP

The pair \((\xi, J)\) is a Markov additive process if and only if, for each \(i, j \in E\),

1. there exist a sequence of iid Lévy processes \((\xi^n_i)_{n \geq 0}\)
2. and a sequence of iid random variables \((U^n_{ij})_{n \geq 0}\), independent of the chain \(J\),
3. such that if \(T_0 = 0\) and \((T_n)_{n \geq 1}\) are the jump times of \(J\),

the process \(\xi\) has the representation

\[
\xi(t) = 1_{(n>0)}(\xi(T_{n-}) + U^n_{J(T_{n-}),J(T_n)}) + \xi^n_J(T_n)(t - T_n),
\]

for \(t \in [T_n, T_{n+1})\), \(n \geq 0\).
rssMps, MAPs, Lamperti-Kiu (Chaumont, Panti, Rivero)

- Take the statespace of the MAP to be \( E = \{1, -1\} \).
- Let
  \[
  X_t = xe^{\xi(\tau(t))} J(\tau(t)) \quad 0 \leq t < T_0,
  \]
  where
  \[
  \tau(t) = \inf \left\{ s > 0 : \int_0^s \exp(\alpha \xi(u)) du > t|x|^{-\alpha} \right\}
  \]
  and
  \[
  T_0 = |x|^{-\alpha} \int_0^\infty e^{\alpha \xi(u)} du.
  \]
- Then \( X_t \) is a real-valued self-similar Markov process in the sense that the law of \( (cX_{tc^{-\alpha}} : t \geq 0) \) under \( P_x \) is \( P_{cx} \).
- The converse (within a special class of rssMps) is also true.
Characteristics of a MAP

- Denote the transition rate matrix of the chain \( J \) by 
  \[ Q = (q_{ij})_{i,j \in E}. \]
- For each \( i \in E \), the Laplace exponent of the Lévy process \( \xi_i \) will be written \( \psi_i \) (when it exists).
- For each pair of \( i, j \in E \), define the Laplace transform 
  \[ G_{ij}(z) = \mathbb{E}(e^{zU_{ij}}) \]
  of the jump distribution \( U_{ij} \) (when it exists).
- Write \( G(z) \) for the \( N \times N \) matrix whose \((i,j)\)th element is \( G_{ij}(z) \).
- Let 
  \[ F(z) = \text{diag}(\psi_1(z), \ldots, \psi_N(z)) + Q \odot G(z), \]
  \( (\text{when it exists}) \), where \( \odot \) indicates elementwise multiplication.
- The matrix exponent of the MAP \((\xi, J)\) is given by 
  \[ \mathbb{E}_i(e^{z\xi(t)}; J(t) = j) = (e^{F(z)t})_{i,j}, \quad i, j \in E, \]
  \( (\text{when it exists}) \).
An $\alpha$-stable process is a rssMp

- An $\alpha$-stable process is a rssMp. Remarkably (thanks to work of Chaumont, Panti and Rivero) we can compute precisely its matrix exponent explicitly.
- Denote the underlying MAP $(\xi, J)$, we prefer to give the matrix exponent of $(\xi, J)$ as follows:

$$F(z) = \begin{bmatrix}
- \frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha \hat{\rho} - z)\Gamma(1 - \alpha \hat{\rho} + z)} & \frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha \hat{\rho})\Gamma(1 - \alpha \hat{\rho})} \\
\frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha \rho)\Gamma(1 - \alpha \rho)} & - \frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha \rho - z)\Gamma(1 - \alpha \rho + z)}
\end{bmatrix},$$

for $\text{Re}(z) \in (-1, \alpha)$. 
Ascending ladder MAP

- Observe the process \((\xi, J)\) only at times of increase of new maxima of \(\xi\). This gives a MAP, say \((H^+(t), J^+(t))_{t \geq 0}\), with the property that \(H\) is non-decreasing with the same range as the running maximum.

- Its exponent can be identified by \(-\kappa(-z)\), where

  \[
  \kappa(\lambda) = \text{diag}(\Phi_1(\lambda), \cdots, \Phi_N(\lambda)) - \Lambda \circ K(\lambda), \quad \lambda \geq 0.
  \]

- Here, for \(i = 1, \cdots, N\), \(\Phi_i\) are Bernstein functions (exponents of subordinators), \(\Lambda = (\Lambda_{i,j})_{i,j \in E}\) is the intensity matrix of \(J^+\) and \(K(\lambda)_{i,j} = \mathbf{E}[e^{-\lambda U^+_{i,j}}]\), where \(U^+_{i,j} \geq 0\) are the additional discontinuities added to the path of \(\xi\) each time the chain \(J^+\) switches from \(i\) to \(j\), and \(U^+_{i,i} := 0, \ i \in E\).
Theorem

For $\theta \in \mathbb{R}$, up to a multiplicative factor,

$$-F(i\theta) = \Delta^{-1}_\pi \hat{\kappa}(i\theta)^T \Delta_\pi \kappa(-i\theta),$$

where $\Delta_\pi = \text{diag}(\pi)$, $\pi$ is the stationary distribution of $Q$, $\hat{\kappa}$ plays the role of $\kappa$, but for the dual MAP to $(\xi, J)$.

The dual process, or time-reversed process is equal in law to the MAP with exponent

$$\hat{F}(z) = \Delta^{-1}_\pi F(-z)^T \Delta_\pi,$$
Define the family of Bernstein functions

\[ \kappa_{q+i,p+j}(\lambda) := \int_{0}^{\infty} (1 - e^{-\lambda x}) \frac{((q + i) \vee (p + j) - 1)}{(1 - e^{-x})q+i(1 + e^{-x})p+j} e^{-\alpha x} \, dx, \]

where \( q, p \in \{\alpha \rho, \alpha \hat{\rho}\} \) and \( i, j \in \{0, 1\} \) such that \( q + p = \alpha \) and \( i + j = 1 \).
Theorem

Fix $\alpha \in (0, 1]$. Up to a multiplicative constant, the ascending ladder MAP exponent, $\kappa$, is given by

$$
\begin{bmatrix}
\kappa_{\alpha \rho+1, \alpha \hat{\rho}}(\lambda) + \frac{\sin(\pi \alpha \hat{\rho})}{\sin(\pi \alpha \rho)} \kappa'_{\alpha \rho, \alpha \rho+1}(0+)
- \frac{\sin(\pi \alpha \hat{\rho})}{\sin(\pi \alpha \rho)} \kappa'_{\alpha \rho, \alpha \rho+1}(\lambda) \\
- \frac{\sin(\pi \alpha \rho)}{\sin(\pi \alpha \hat{\rho})} \kappa_{\alpha \rho, \alpha \hat{\rho}+1}(\lambda) \\
\end{bmatrix}
$$

Up to a multiplicative constant, the dual ascending ladder MAP exponent, $\hat{\kappa}$, is given by

$$
\begin{bmatrix}
\kappa_{\alpha \rho+1, \alpha \rho}(\lambda + 1 - \alpha) + \frac{\sin(\pi \alpha \rho)}{\sin(\pi \alpha \hat{\rho})} \kappa'_{\alpha \rho, \alpha \rho+1}(0+)
- \frac{\kappa_{\alpha \rho, \alpha \hat{\rho}+1}(\lambda + 1 - \alpha)}{\lambda + 1 - \alpha} \\
- \frac{\kappa_{\alpha \hat{\rho}, \alpha \rho+1}(\lambda + 1 - \alpha)}{\lambda + 1 - \alpha} \\
\end{bmatrix}
$$
\( \alpha \in (1, 2) \)

Define the family of Bernstein functions by

\[
\phi_{q+i,p+j}(\lambda) = \int_0^\infty \left(1 - e^{-\lambda u}\right)^q \frac{((q+i) \vee (p+j) - 1)}{(1 - e^{-u})^{q+i}(1 + e^{-u})^{p+j}} - \frac{(\alpha - 1)}{2(1 - e^{-u})^{q(1 + e^{-u})^p}} \right) e^{-u} du,
\]

for \( q, p \in \{\alpha \rho, \alpha \hat{\rho}\} \) and \( i, j \in \{0, 1\} \) such that \( q + p = \alpha \) and \( i + j = 1 \).
Deep Factorisation $\alpha \in (1, 2)$

**Theorem**

*Fix $\alpha \in (1, 2)$. Up to a multiplicative constant, the ascending ladder MAP exponent, $\kappa$, is given by*

$$
\begin{bmatrix}
\sin(\pi \alpha \rho) \phi_{\alpha \rho+1, \alpha \hat{\rho}}(\lambda + \alpha - 1) \\
+ \sin(\pi \alpha \rho) \phi'_{\alpha \hat{\rho}, \alpha \rho+1}(0+)
\end{bmatrix}
- \sin(\pi \alpha \hat{\rho}) \frac{\phi_{\alpha \hat{\rho}, \alpha \rho+1}(\lambda + \alpha - 1)}{\lambda + \alpha - 1}

- \sin(\pi \alpha \hat{\rho}, \alpha \rho+1)(0+)
$$

*for $\lambda \geq 0$.*

*Up to a multiplicative constant, the dual ascending ladder MAP exponent, $\hat{\kappa}$, is given by*

$$
\begin{bmatrix}
\sin(\pi \alpha \rho) \phi_{\alpha \rho+1, \alpha \rho}(\lambda) \\
+ \sin(\pi \alpha \rho) \phi'_{\alpha \rho, \alpha \rho+1}(0+)
\end{bmatrix}
- \sin(\pi \alpha \hat{\rho}) \frac{\phi_{\alpha \rho+1, \alpha \hat{\rho}}(\lambda)}{\lambda}

\sin(\pi \alpha \rho) \phi_{\alpha \rho+1, \alpha \rho}(\lambda) \\
+ \sin(\pi \alpha \rho) \phi'_{\alpha \rho, \alpha \rho+1}(0+)
$$

*for $\lambda \geq 0$.***
Recall that
\[
\kappa(\lambda) = \text{diag}(\Phi_1(\lambda), \Phi_{-1}(\lambda)) - \begin{bmatrix}
-\Lambda_{1,-1} & \Lambda_{1,-1} \int e^{-\lambda x} F_{1,-1}^+(dx) \\
\Lambda_{-1,1} \int e^{-\lambda x} F_{-1,1}^-(dx) & -\Lambda_{-1,1}
\end{bmatrix}
\]

In general, we can write
\[
\Phi_i(\lambda) = n_i(\zeta = \infty) + \int_0^\infty (1 - e^{-\lambda x}) n_i(\varepsilon_\zeta \in dx, J(\zeta) = i, \zeta < \infty),
\]

where \(\zeta = \inf\{s \geq 0 : \varepsilon(s) > 0\}\) for the canonical excursion \(\varepsilon\) of \(\xi\) from its maximum.
Lemma

Let $T_a = \inf\{t > 0 : \xi(t) > a\}$. Suppose that $\limsup_{t \to \infty} \xi(t) = \infty$ (i.e. the ladder height process $(H^+, J^+)$ does not experience killing). Then for $x > 0$ we have up to a constant

$$
\lim_{a \to \infty} P_{0,i}(\xi(T_a) - a \in dx, J(T_a) = 1)
$$

$$
= \left[ \pi_1 n_1(\varepsilon(\zeta) > x, J(\zeta) = 1, \zeta < \infty) + \pi_{-1} \Lambda_{-1,1}(1 - F_{-1,1}^+(x)) \right] dx.
$$

- $(\pi_{-1}, \pi_1)$ is easily derived by solving $\pi Q = 0$.
- We can work with the LHS in the above lemma e.g. via

$$
\lim_{a \to \infty} P_{0,1}(\xi(T_a) - a > u, J(T_a) = 1)
$$

$$
= \lim_{a \to \infty} \mathbb{P} e^{-a}(X_{\tau_1^+ \wedge \tau_{-1}^-} > e^u; \tau_1^+ < \tau_{-1}^-).
$$
The problem with applying the Markov additive renewal in the case that \( \alpha \in (1, 2) \) is that \((H^+, J^+)\) does experience killing.

It turns out that \( \det F(z) = 0 \) has a root at \( z = \alpha - 1 \). Moreover the exponent of a MAP (Esscher transform of \( F \))

\[
F^\circ(z) = \Delta_{\pi^\circ}^{-1} F(z + \alpha - 1) \Delta_{\pi^\circ},
\]

where \( \pi^\circ = (\sin(\pi \alpha \hat{\rho}), \sin(\pi \alpha \rho)) \) is the stationary distribution of \( F^\circ(0) \).

And \( \kappa^\circ(\lambda) = \Delta_{\pi^\circ}^{-1} \kappa(\lambda - \alpha + 1) \Delta_{\pi^\circ} \) does not experience killing.

However, in order to use Markov additive renewal theory to compute \( \kappa^\circ \), need to know something about the rssMp to which the MAP with exponent \( F^\circ \) corresponds.
Riesz-Bogdan-Zak transform

**Theorem (Riesz–Bogdan–Zak transform)**

Suppose that $X$ is a stable process as outlined in the introduction. Define

$$
\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} du > t\}, \quad t \geq 0.
$$

Then, for all $x \in \mathbb{R} \setminus \{0\}$, $(-1/X_\eta(t))_{t \geq 0}$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^0_{-1/x})$, where

$$
\frac{d\mathbb{P}^0_x}{d\mathbb{P}_x}\bigg|_{\mathcal{F}_t} = \left(\frac{\sin(\pi \alpha \rho) + \sin(\pi \alpha \hat{\rho}) - (\sin(\pi \alpha \rho) - \sin(\pi \alpha \hat{\rho}))\text{sgn}(X_t)}{\sin(\pi \alpha \rho) + \sin(\pi \alpha \hat{\rho}) - (\sin(\pi \alpha \rho) - \sin(\pi \alpha \hat{\rho}))\text{sgn}(x)}\right) \left|\frac{X_t}{x}\right|^\alpha \mathbf{1}_{(t < \tau \{0\})}
$$

and $\mathcal{F}_t := \sigma(X_s : s \leq t)$, $t \geq 0$. Moreover, the process $(X, \mathbb{P}^0_x)$, $x \in \mathbb{R} \setminus \{0\}$ is a self-similar Markov process with underlying MAP via the Lamperti-Kiu transform given by $F^0(z)$. 
Stable processes

Self-similar Markov processes and MAPs

MAP WHF

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Riesz-Bogdan-Zak

\((X, \mathbb{P})\)

Doob \(h\)-transform

\((X, \mathbb{P}^o)\)

\[\uparrow \hspace{1cm} \uparrow\]

Lamperti-Kiu

\[\downarrow \hspace{1cm} \downarrow\]

MAP: \(F(z)\)

\[\text{Esscher}\]

MAP: \(F^o(z)\)
Computing $\Phi_1^0(\lambda)$ from $\kappa^0(\lambda)$

If we write $\overline{X}_t = \sup_{s \leq t} X_s$ and $\underline{X}_t = \inf_{s \leq t} X_s$, $t \geq 0$, then we also have

$$\pi_1^0 n_1^0(\varepsilon(\zeta) > u, J(\zeta) = 1, \zeta < \infty)$$

$$= -\frac{d}{du} \lim_{x \to 0} P_x^0 \left( X_{\tau_1^+} > e^u, \overline{X}_{\tau_1^+-} > |X_{\tau_1^+-}|, \tau_1^+ < \tau_-^1 \right)$$

$$= -\lim_{x \to 0} \frac{d}{du} \int_0^1 P_x^0 \left( X_{\tau_1^+} > e^u, \overline{X}_{\tau_1^+-} \in dz, \tau_1^+ < \tau_-z \right)$$

$$= -\lim_{x \to 0} \int_0^1 \frac{d}{dy} \frac{d}{du} P_x^0 \left( X_{\tau_1^+} > e^u, \overline{X}_{\tau_1^+-} \leq y, \tau_1^+ < \tau_-z \right) \bigg|_{y=z} dz$$

$$= -\lim_{x \to 0} \int_0^1 \frac{d}{dy} \frac{d}{du} P_x^0 \left( X_{\tau_1^+} > e^u, \tau_1^+ < \tau_-z \right) \bigg|_{y=z} dz$$
Computing $\Phi^1_1(\lambda)$ from $\kappa^\circ(\lambda)$

For $0 < x < y < 1$ and $u > 0$, 

$$-\frac{d}{du} \lim_{x \to 0} \mathbb{P}_x^o \left( X_{\tau_y^+} > e^u, \tau_y^+ < \tau_z^- \right)$$

$$= \frac{d}{du} \lim_{x \to 0} \mathbb{P}_{-1/x}(X_{\tau(-1/y,1/z)} \in (-e^{-u}, 0))$$

$$= \frac{d}{du} \lim_{x \to 0} \hat{\mathbb{P}}_{1/x}(X_{\tau(-1/z,1/y)} \in (0, e^{-u}))$$

$$= \hat{p}_{\pm\infty} \left( \frac{2yz e^{-u} - z + y}{y + z} \right) \frac{2yz}{y + z} e^{-u},$$

where

$$\hat{p}_{\pm\infty}(y) = 2^{\alpha-1} \frac{\Gamma(2 - \alpha)}{\Gamma(1 - \alpha \hat{\rho})\Gamma(1 - \alpha \hat{\rho})}(1 + y)^{-\alpha \hat{\rho}}(1 - y)^{-\alpha \rho}$$

was computed recently in a paper by K. Pardo & Watson (2014).
Computing $\Phi_1^o(\lambda)$ from $\kappa^o(\lambda)$

Putting the pieces together, we have, up to a constant

$$
\Phi_1^o(\lambda) = \int_0^\infty (1 - e^{-\lambda x}) n_1^o(\varepsilon \zeta \in dx, J(\zeta) = 1, \zeta < \infty)
$$

$$
= \lambda \int_0^\infty e^{-\lambda x} n_1^o(\varepsilon \zeta > x, J(\zeta) = 1, \zeta < \infty)
$$

$$
= \phi_{\alpha \rho+1, \alpha \hat{\rho}}(\lambda)
$$
Thank you!