Random walks on percolation clusters

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Random motion in random media

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Random motion in random media

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- Mathematicians are just developing the tools to start thinking about such questions!
Percolation

- Introduced by Broadbent and Hammersley (1957).
- Euclidean lattice \( \mathbb{Z}^d \), edges (bonds) \( E_d \).
- Fix \( p \in [0, 1] \). For \( x \sim y \), let \( \mu_{xy} \) be independent random variables with \( \mathbb{P}(\mu_{xy} = 1) = p \), \( \mathbb{P}(\mu_{xy} = 0) = 1 - p \).
- The bonds (edges) such that \( \mu_{xy} = 1 \) are called open bonds. Let \( \mathcal{O} \) be the set of open bonds.
- The connected components of the graph \( (\mathbb{Z}^d, \mathcal{O}) \) are called (open) clusters.
- There exists \( p_c \in (0, 1) \) such that, a.s.,
  - if \( p < p_c \), all clusters are finite,
  - if \( p > p_c \), then there exists a unique infinite cluster, \( C_\infty \).
  - if \( p = p_c \), no infinite cluster for \( d = 2, d \geq 19 \), believed \( \forall d \).
$p = 0.2$
$p = 0.2$, largest cluster marked
$p = 0.2$
$p = 0.4$
\( p = 0.4 \)
\[ p = 0.5 \]
$p = 0.5$
Random walks on percolation clusters – p. 4

\[ p = 0.5 \]
$p = 0.6$
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$p = 0.8$
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Graphs

Let $\Gamma = (G, E_\Gamma)$ be an infinite connected locally finite graph. Define weights or conductances $\mu_{xy} = \mu_{yx}$ for $x \sim y$. We allow $\mu_{xx} > 0$. Let $\mu_{xy} = 0$ if $x \not\sim y$. Set

$$\mu_x = \sum_y \mu_{xy},$$

and extend $\mu$ to a measure on $G$. The volume of $B(x, r)$, a ball in the graph of radius $r$ at $x$, is

$$V(x, r) = \sum_{y \in B(x, r)} \mu_y.$$

Discrete Laplacian:

$$\Delta f(x) = \frac{1}{\mu_x} \sum_y \mu_{xy} (f(y) - f(x)). \quad (1)$$
Random walks

- Continuous time random walk $Y = (Y_t, t \in [0, \infty))$ on $(\Gamma, \mu)$. If $Y_t = x$, then the probability of a jump to $y \sim x$ in $(t, t + \delta]$ is $\approx \delta \mu_{xy}/\mu_x$.

- Let $q_t(x, y)$ be the transition density of $Y$ (w.r.t. $\mu$), i.e.

$$\mathbb{P}^x(Y_t = y) = q_t(x, y)\mu_x.$$  

Then $q_t(x, y) = q_t(y, x)$ satisfies the discrete heat equation (time continuous, space discrete)

$$\frac{\partial}{\partial t}u(x, t) = \Delta u(x, t).$$
The discrete random walk \( X = (X_n, n = 0, 1, \ldots) \) on \((\Gamma, \mu)\). If \( X_n = x \), then the probability of a jump to \( y \sim x \) is \( \mu_{xy}/\mu_x \).

Let \( p_n(x, y) \) be the transition density of \( Y \) (w.r.t. \( \mu \)), i.e.

\[
P^x(X_n = y) = p_n(x, y)\mu_x.
\]

Then \( p_n(x, y) = p_n(y, x) \) satisfies the discrete heat equation (discrete time and space):

\[
u(x, n + 1) - u(x, n) = \Delta u(x, n).
\]

Note that to deal with bipartite graphs we use \( \hat{p}_n(x, y) = p_{n+1}(x, y) + p_n(x, y) \).
We work in the supercritical case $p > p_c$. Fix a percolation configuration $\omega$. Let $G = C_\infty(\omega)$, $E$ be the open bonds in $C_\infty(\omega)$. This defines an (infinite, connected) weighted graph. Let $Y_t$ be the continuous time random walk on $(C_\infty(\omega), \mu(\omega))$. Its transition density is

$$q_t^\omega(x, y) \mu_y(\omega) = P_\omega^x(Y_t = y).$$

The discrete version was called the ‘ant in the labyrinth’ by De Gennes 1976.

We can consider myopic ants - for which $\mu_{xx} = 0$ and blind ants for which $\mu_{xx} = 2d - \sum_{y \neq x} \mu_{xy}$.

Grimmett, Kesten, Zhang, 1993: $Y$ is transient iff $d \geq 3$. 
Problems for the random walk $Y$ on $\mathcal{C}_\infty$:
(1) Gaussian bounds (GB) on $q_t^\omega(x, y)$.
(2) Central limit theorem/ Invariance principle for $Y$.
(3) A local limit theorem for $Y$.

CLT for $\mathbb{Z}^d$. Let $Y_t^{(n)} = n^{-1} Y_{n^2 t}$. Then

$$
\mathbb{P}^0(Y_t^{(n)} \in U) \to \int_U (2\pi C_d t)^{-d/2} \exp\left(-\frac{|x|^2}{2C_d t}\right) dx
$$

Invariance principle for $\mathbb{Z}^d$ (Donsker 1951):

$$(Y_t^{(n)}, t \geq 0) \Rightarrow (C_d^{1/2} W_t, t \geq 0)$$

where $W$ is Brownian motion.
The critical case

In the critical case there is no infinite cluster with probability 1 (at least for \(d = 2, d \geq 19\)). In this case we must define an ‘Incipient infinite cluster’ (IIC). This critical cluster should have fractal structure. For \(d = 2\) it can be described via an SLE.

- Kesten (1986): random walk on the IIC in \(d = 2\) is subdiffusive.
- Barlow & Kumagai (2006): random walk on the IIC on a tree (‘\(d = \infty\)’) has sub-Gaussian heat kernel estimates. Croydon (2006), the scaling limit is Brownian motion on the continuum random tree.
- Barlow, Jarai, Kumagai and Slade (2007), random walk on high dimensional spreadout oriented percolation.
Two types of invariance principle

- $Y$ is a RW in a random environment $\mathcal{C}_\infty(\omega)$. Let $\mathbb{P}$ be the probability measure for the percolation configuration $\mathcal{C}_\infty$. Let $\mathbb{P}^x_\omega$ be the probability measure for $Y$ on $\mathcal{C}_\infty(\omega)$ starting at $x \in \mathcal{C}_\infty$.

- *Quenched or almost sure.* The Invariance principle for $Y$ holds (w.r.t. $\mathbb{P}^x_\omega$) for a set of environments $\omega$ with $\mathbb{P}$ probability 1.

- *Averaged, or ‘annealed’.* The Invariance principle for $Y$ holds w.r.t. $\mathbb{P} \times \mathbb{P}^x_\omega$.

- De Masi, Ferrari, Goldstein, Wick 1989: The averaged invariance principle holds for processes in stationary ergodic random environments. In particular, this holds for $Y$ on $\mathcal{C}_\infty$. 
Delmotte’s theorem

Theorem (T. Delmotte, 1999). Let \((\Gamma, \mu)\) be a weighted graph. (Assume \(\Gamma\) locally finite, \(\mu_{xy} \in [C^{-1}, C]\) whenever \(x \sim y\).) The following are equivalent:

(a) Solutions of the heat equation on \(G\) satisfy a Parabolic Harnack inequality (PHI).

(b) \((\Gamma, \mu)\) satisfies volume doubling (VD) and a Poincare inequality (PI).

(c) \(q_t(x, y)\) satisfies Gaussian bounds:

\[
\frac{c_1 e^{-c_2 d(x,y)^2 / t}}{V(x, t^{1/2})} \leq q_t(x, y) \leq \frac{c_3 e^{-c_4 d(x,y)^2 / t}}{V(x, t^{1/2})},
\]

if \(t \geq \max(1, |x - y|)\).
Poincare inequality for graphs

Let \( B = B(x, r) \), \( f : B \to \mathbb{R} \). Then

\[
\sum_{x \in B} (f(x) - \bar{f})^2 \mu_x \leq C_P r^2 \sum_{x, y \in B} (f(y) - f(x))^2 \mu_{xy}
\]

\[= C_P r^2 \mathcal{E}_B(f, f).\]

As usual \( \bar{f} \) is the real number which minimises the LHS.
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An example of a graph for which the PI fails is two copies of $\mathbb{Z}^d$ ($d \geq 3$) connected at their origins. If $f = 1$ on one copy, $f = -1$ on the other and $B = B(x, r)$ then LHS $\approx r^d$ while the RHS $\approx r^2$. 
Bounds on $q_t$

- The natural idea is to try to apply Delmotte’s theorem.
- However, neither VD nor PI hold for $C_\infty$. The reason is that if we look far enough we can find arbitrarily large ‘bad regions’.
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- However, neither VD nor PI hold for $C_\infty$. The reason is that if we look far enough we can find arbitrarily large ‘bad regions’.
Obtaining Gaussian bounds for $C_\infty$

For the on-diagonal bound isoperimetric or Nash inequality ideas lead to (Mathieu and Remy (2004))

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\sup_y q_t^\omega(x, y) \leq ct^{-d/2},
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Obtaining Gaussian bounds for $C_\infty$

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$$\sup_y q^\omega_t(x, y) \leq ct^{-d/2},$$

for $t \geq S_x(\omega)$.

- The next, and hardest, step in controlling $q_t(x, y)$ is to obtain ‘off-diagonal’ bounds, i.e.

$$q^\omega_t(x, y) \leq \theta(t, |x - y|),$$

where $\theta(t, r) \to 0$ as $r \to \infty$. 
Gaussian bounds

**Theorem 1.** (Barlow, 2004) Let $p > p_c$. For each $x \in \mathbb{Z}^d$ there exist r.v. $S_x$ with $\mathbb{P}_p(S_x \geq n) \leq c \exp(-n^{\varepsilon_d})$ and (non-random) constants $c_i = c_i(d, p)$ such that the transition density of $Y$ satisfies,

$$
\frac{c_1}{t^{d/2}} e^{-c_2|x-y|_1^2/t} \leq q_t^\omega(x, y) \leq \frac{c_3}{t^{d/2}} e^{-c_4|x-y|_1^2/t},
$$

for $x, y \in \mathcal{C}_\infty(\omega)$, $t \geq \max(S_x(\omega), 1)$.

**Note.** The randomness of the environment is taken care of by the $S_x(\omega)$, which will be small for most points, and large for the rare ‘bad points’. The same bounds hold for the discrete transition density.
**Quenched invariance principles**

**Theorem 2.** (Sidoravicius and Sznitman, 2004 \(d \geq 4\), Berger and Biskup, 2005, Mathieu and Piatnitski, 2005). A quenched or a.s. invariance principle holds for \(Y\).

The BB, MP papers used the *corrector*. This is a (random) function \(\chi(\omega, x) : C_\infty(\omega) \to \mathbb{R}^d\) such that \(h(x) = x + \chi(x)\) is harmonic.

This implies that if \(q^{(n, \omega)}_t(x, y) = n^d q^\omega_{n^2 t}(\lfloor nx \rfloor, \lfloor ny \rfloor)\), then for \(f \in C_K(\mathbb{R}^d)\), with \(\mathbb{P}\)-probability 1,

\[
\int q^{(n, \omega)}_t(x, y) f(y) dy \to \int k_t(x, y) f(y) dy
\]

where, \(k_t(x, y) = (2\pi D)^{-d/2} \exp(-|x - y|^2 / 2Dt)\), \(D > 0\).
The Gaussian bounds for $q_\omega^t(x, y)$ lead to a Parabolic Harnack inequality. This gives Hölder continuity of $q_\omega^t(x, y)$, and will allow us to replace the integrals by pointwise expressions.

We say a Ball $B(x, R)$ in the graph is good if it has a PI and a C such that $\mu(B(x, R)) \geq CR^d$. It is very good if all balls of a reasonable size in $B(x, R)$ are good. We prove our PHI for very good balls.
Parabolic Harnack inequality

Let

\[ Q(x, R, T) = [0, T] \times B(x, R), \]

and

\[ Q_-(x, R, T) = \left[ \frac{1}{4}T, \frac{1}{2}T \right] \times B(x, \frac{1}{2}R), \]

\[ Q_+(x, R, T) = \left[ \frac{3}{4}T, T \right) \times B(x, \frac{1}{2}R). \]

We say that a function \( u(n, x) \) is *caloric* on \( Q \) if \( u \) is defined on \( \overline{Q} = ([0, T] \cap \mathbb{Z}) \times \overline{B}(x, R) \), and

\[ u(n + 1, y) - u(n, y) = \Delta u(n, y) \text{ for } 0 \leq n \leq T - 1, \ y \in B(x, R). \]
We say the parabolic Harnack inequality (PHI) holds with constant $C_H$ for $Q = Q(x, R, T)$ if whenever $u = u(n, x)$ is non-negative and caloric on $Q$, then

$$
\sup_{(n,x) \in Q_-} \hat{u}(n, x) \leq C_H \inf_{(n,x) \in Q_+} \hat{u}(n, x).
$$

(1)
The PHI

We assume that the conductivities $\mu_{xy}$ are bounded away from 0 and $\infty$ ($\mu_{xx}$ can be 0) and $\mu(B(x, r)) \leq cr^d$ for $r > 1$. Bounds on the heat kernel can be used to establish the PHI via a balayage argument.

**Theorem 3.**

Let $x_0 \in G$. Suppose that $R \geq 16$ and $B(x_0, R)$ is very good. Let $x_1 \in B(x_0, R/3)$, and $R_1 \log R_1 = R$. Then there exists a constant $C_H$ such that the PHI (in both discrete and continuous time settings) holds with constant $C_H$ for $Q(x_1, R_1, R_1^2)$.

By applying this PHI on a nested set of cubes we can control the oscillation in caloric functions.
Hölder Continuity

Let \( x_0 \in G \). Suppose the PHI (with constant \( C_H \)) holds for \( Q(x_0, R, R^2) \) for \( R \geq s(x_0) \). Let \( \theta = \log(2C_H/(2C_H - 1))/\log 2 \), and

\[
\rho(x_0, x, y) = s(x_0) \lor d(x_0, x) \lor d(x_0, y).
\]

Let \( r_0 \geq s(x_0) \), \( t_0 = r_0^2 \), and suppose that \( u = u(n, x) \) is caloric in \( Q = Q(x_0, r_0, r_0^2) \). Let \( x_1, x_2 \in B(x_0, \frac{1}{2}r_0) \), and \( t_0 - \rho(x_0, x_1, x_2)^2 \leq n_1, n_2 \leq t_0 - 1 \). Then

\[
|\hat{u}(n_1, x_1) - \hat{u}(n_2, x_2)| \leq c \left( \frac{\rho(x_0, x_1, x_2)}{t_0^{1/2}} \right)^\theta \sup_{Q_+} |\hat{u}|.
\]
Let \( k_t^{(D)}(x) \) be the Gaussian heat kernel in \( \mathbb{R}^d \) with diffusion constant \( D > 0 \) and let \( X_t^{(n)} = n^{-1/2} X_{\lfloor nt \rfloor} \). For \( x \in \mathbb{R}^d \), set

\[
H(x, r) = x + [-r, r]^d, \quad \Lambda(x, r) = H(x, r) \cap G. \tag{2}
\]

In general \( \Lambda(x, r) \) will not be connected. Let

\[
\Lambda_n(x, r) = \Lambda(xn^{1/2}, rn^{1/2}).
\]

For \( x \in \mathbb{R}^d \) let \( g_n(x) \) be a closest point in \( G \) to \( n^{1/2}x \), in the \( \| \cdot \|_\infty \) norm.
Assumption 1 There exists a constant $\delta > 0$, and positive constants $D, C_H, C_i, a_G$ such that the following hold.

(a) (CLT for $X$). For any $y \in \mathbb{R}^d, r > 0$,

$$P^0(X_t^{(n)} \in H(y, r)) \to \int_{H(y,r)} k_t^{(D)}(y)dy. \quad (3)$$

(b) There is an upper heat kernel bound

$$p_k(0, y) \leq C_2 k^{-d/2}, \quad \forall y \in G, k \geq C_3.$$  

(c) For each $y \in G$ there exists $s(y) < \infty$ such that the PHI holds with constant $C_H$ for $Q(y, R, R^2)$ for $R \geq s(y)$. 
(d) For any $r > 0$

$$\frac{\mu(\Lambda_n(x, r))}{(2n^{1/2}r)^d} \to a G \quad \text{as } n \to \infty.$$  \hspace{1cm} (4)

(e) For each $r > 0$ there exists $n_0$ such that, for $n \geq n_0$,

$$|x' - y'|_\infty \leq d(x', y') \leq (C_1|x' - y'|_\infty) \lor n^{1/2 - \delta},$$

for all $x', y' \in \Lambda_n(x, r)$.

(f) $n^{-1/2 + \delta} s(g_n(x)) \to 0$ as $n \to \infty$. 
Theorem 4

Let $x \in \mathbb{R}^d$ and $t > 0$. Suppose Assumption 1 holds. Then

$$\lim_{n \to \infty} n^{d/2} \hat{p}_{nt}(0, g_n(x)) = 2a_g^{-1} k_t^{(D)}(x).$$

Proof idea: Let $\Lambda_n = \Lambda_n(x, \kappa) = \Lambda(n^{1/2} x, n^{1/2} \kappa)$ and recall $X_t^{(n)} = n^{-1/2} X_{\lfloor nt \rfloor}$. Let

$$J(n) = P^0\left(X_t^{(n)} \in \Lambda(x, \kappa)\right) + P^0\left(X_{t+1/n}^{(n)} \in \Lambda(x, \kappa)\right) - 2 \int_{\Lambda(x, \kappa)} k_t(y) dy.$$
Then

\[ J(n) = \sum_{z \in \Lambda_n} \left( \hat{p}_{nt}(0, z) - \hat{p}_{nt}(0, g_n(x)) \right) \mu_z \]

\[ + \mu(\Lambda_n)\hat{p}_{nt}(0, g_n(x)) - \mu(\Lambda_n)n^{-d/2}a^{-1}G_2k_t(x) \]

\[ + 2k_t(x) \left( \mu(\Lambda_n)n^{-d/2}a^{-1} - 2^d \kappa^d \right) \]

\[ + 2 \int_{H(x,\kappa)} (k_t(x) - k_t(y)) dy \]

We want the second term and deal with the rest by our assumptions.
Uniform version

Assumption 2
(a) For any compact \( I \subset (0, \infty) \), the CLT in Assumption 1 (a) holds uniformly for \( t \in I \).
(b) There exist \( C_i \) such that

\[
\hat{p}_k(0, x) \leq C_2 k^{-d/2} \exp\left(-C_4 d(0, x)^2 / k\right), \quad \text{for } k \geq C_3 \text{ and } x \in \mathcal{G}.
\]

(c) We have a PHI as in Assumption 1 (c).
(d) Let \( h(r) \) be the size of the biggest ‘hole’ in \( \Lambda(0, r) \). More precisely, \( h(r) \) is the supremum of the \( r' \) such that \( \Lambda(y, r') = \emptyset \) for some \( y \in H(0, r) \). Then \( \lim_{r \to \infty} h(r)/r = 0 \).
(e) There exist constants \( \delta, C_1, C_H \) such that for each \( x \in \mathbb{Q}^d \) Assumption 1 (d), (e) and (f) hold.
We now state a uniform version of our local limit result.

**Theorem 5**

Let $T_1 > 0$. Suppose Assumption 2 holds. Then

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \sup_{t \geq T_1} |n^{d/2} \hat{p}_{nt}(0, g_n(x)) - 2a_1^{-1} k_t^{(D)}(x)| = 0. \quad (6)$$
Application to Percolation

With some work we can show that for supercritical percolation clusters the assumptions of Theorem 5 hold.

**Theorem 6**

Let $T_1 > 0$. Then there exist constants $a, D$ such that $\mathbb{P}_0$-a.s.,

$$
\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \sup_{t \geq T_1} |n^{d/2} \hat{p}^\omega_{nt}(0, g_n^\omega(x)) - 2a^{-1} k_t^{(D)}(x)| = 0. \quad (7)
$$

This result holds for both blind and myopic ants as well as continuous time walks on $C_\infty$.

The earlier theorems can be used to prove local limit theorems for random walks in a bounded random conductance model.