

Spatial branching in random media

[To appear in Ann. Inst. H. Poincaré]

János Engländer

University of California, Santa Barbara

The purpose of this talk

To give you an impression about how many **interesting** problems arise for

spatial branching processes

in

random (Poissonian) media

— by discussing a number of different models in mathematics and in math biology (population models).

Motivation.

- Survival asymptotics for a single Brownian particle among **Poissonian obstacles**
 - Back in 1975: Wiener-sausage asymptotics of **Donsker and Varadhan**
 - later **Sznitman** and others
- **'Catalytic branching'**
 - for superprocesses: **Dawson, Fleischmann**, etc.
 - for discrete particle systems: **Gärtner-den Hollander, Kesten-Sidoravicius**, etc.

Two related papers:

- [E, Stoch. Process. Appl. 2000]

Model of a [branching process in a random environment](#) introduced. Hard obstacles and instantaneous killing of the branching process once any particle hits the trap configuration K

- [E, den Hollander, Markov Process. Related Fields, 2003]

[Survival asymptotics for branching Brownian motion in a Poissonian trap field](#)

Both models are [annealed](#).

Model of second paper:

BBM on \mathbb{R}^d with branching rate β in a Poissonian field of spherical traps

Intensity: $d\nu/dx \sim \ell/|x|^{d-1}$, $|x| \rightarrow \infty$.

The process starts with a single particle at the origin.

Main results:

The **annealed** probability that *none* of the particles hits a trap until time t :

$$\exp[-I(\ell, \beta, d)t + o(t)], \quad t \rightarrow \infty$$

$I(\ell, \beta, d)$ is computed in terms of a variational problem.

Rate constant exhibits a **crossover at** a critical value $\ell_{cr} = \ell_{cr}(\beta, d)$.

Optimal survival strategy is obtained. Conditional on survival until time t :

For $\ell < \ell_{cr}$, a ball of radius $\sqrt{2\beta}t$ around the origin is emptied, the system stays inside this ball and branches at rate β . For $\ell > \ell_{cr}$, on the other hand, the system

- $d = 1$: suppresses the branching until time t , empties a ball of radius $o(t)$ around the origin and stays inside this ball;

- $d \geq 2$: suppresses the branching until time η^*t ;

empties a ball of radius $\sqrt{2\beta}(1-\eta^*)t$ around a point at distance c^*t from the origin;

during the remaining time $(1-\eta^*)t$ branches at rate β .

Here, $0 < \eta^* < 1$ and $c^* > 0$ are the minimizers of the variational problem for $I(\ell, \beta, d)$.

New model: **Mild obstacles.**

Mechanism is even ‘milder’ than ‘soft killing’:

Only **inhibits** temporarily **the reproduction** of the individual particle but *does not* eliminate particle.

Model.

Let ω be a Poisson point process (PPP) on \mathbb{R}^d with intensity $\nu > 0 \sim \mathbf{P}$.

$a > 0$ and $\beta_2 > \beta_1 > 0$ fixed.

$$K = K_\omega := \bigcup_{x_i \in \text{supp}(\omega)} \bar{B}(x_i, a).$$

K : Mild obstacle configuration attached to ω .

K^c : 'Swiss cheese'

Given ω , we define P^ω as the law of the (strictly dyadic) BBM on \mathbb{R}^d , $d \geq 1$ with spatially dependent branching rate

$$\beta := \beta_1 \mathbf{1}_K + \beta_2 \mathbf{1}_{K^c}.$$

The process Z under P^ω is called a **BBM with mild Poissonian obstacles**.

Total mass process: $|Z|$.

Questions:

1. Growth of the **global**/**local** population size?
How much will the absence of branching in K slow the global reproduction down?
Change the exponent β_2 ?
2. What are the **large deviations**?
[E.g. $P(\text{atypically small population})=?$]
3. How about the spatial spread? How will the $\sqrt{2\beta_2}$ speed reduce?

Questions can be asked in 2 diff. ways: **annealed** and **quenched** sense.

Related models in biology

- (i) **Migration with unfertile areas** (Population dynamics):

Population moves in space and reproduces by binary splitting, but randomly located reproduction-suppressing areas modify the growth.

- (ii) **Fecundity selection** (Genetics):

Reproduction and mutation. Certain randomly distributed genetic types have low fitness: even though they can be obtained by mutation, they themselves do not reproduce easily, unless mutation transforms them to different genetic types.

['Space' = space of genetic types rather than physical space.]

Questions: (local and global) growth rate of the population? Once one knows the global population size, the model can be normalized by the global population size, giving a population of unit mass; then the question becomes the *shape* of the population.

- **Population dynamics setting:** Is there a preferred spatial location for the process to populate?
- **Genetic setting:** Existence of a certain kind of genetic type that is preferred in the long run that lowers the risk of low of fecundity caused by mutating into less fit genetics types?

Genealogical structure— exciting problem!

E.g. it seems quite possible that for large times the ‘bulk’ of the population **consists of descendants of a single particle** that

- decided to travel far enough (resp. to mutate many times)
- reached a less hostile environment (resp. in high fitness genetic type area), where she and her descendants can reproduce freely.

Related phenomenon in marine systems: **hypoxic patches** form in estuaries because of stratification of the water.

⇒ The patches affect different organisms in different ways but are detrimental to some of them. They appear and disappear in an effectively stochastic way.

‘**Source-sink theory**’: some patches of habitat are good for a species (and growth rate is positive) whereas other patches are poor (and growth rate smaller, or is zero or negative). Individuals can **move** between patches **randomly** or according to more detailed biological rules for behavior.

Systems with **periodic local disturbances** like e.g.

- **FORESTS** where trees sometimes fall creating gaps (which have various effects on different species but may harm some)
- **AREAS OF GRASS** or brush which are subject to occasional fires — burned areas can be expected to be less suitable habitats for at least some organisms.

Back to math: **Expected global growth**

Theorem 1 *On a set of full \mathbf{P} -measure, and as $t \rightarrow \infty$,*

$$E^\omega |Z_t| = \exp \left[\beta_2 t - c(d, \nu) \frac{t}{(\log t)^{2/d}} (1 + o(1)) \right],$$

(quenched asymptotics), and

$$(\mathbf{E} \otimes E^\omega) |Z_t| = \exp \left[\beta_2 t - \hat{c}(d, \nu) t^{d/(d+2)} (1 + o(1)) \right],$$

(annealed asymptotics).

Here, e.g.

$$\hat{c}(d, \nu) := -\nu^{2/(d+2)} \left(\frac{d+2}{2} \right) \left(\frac{2\gamma_d}{d} \right)^{d/(d+2)},$$

and γ_d is the lowest eigenvalue of $-\frac{1}{2}\Delta$ for the d -dimensional sphere of unit volume with Dirichlet boundary condition.

WHY?

W : d -dim Brownian motion $\sim \{\mathbb{P}_x, x \in \mathbb{R}^d\}$.

A basic fact:

Lemma 1 (Expectation is Brownian funct.)

Fix ω . Recalling $\beta := \beta_1 \mathbf{1}_K + \beta_2 \mathbf{1}_{K^c}$, we have

$$E^\omega |Z_t| = \mathbb{E} \exp \left[\int_0^t \beta(W_s) ds \right]. \quad (1)$$

Proof. Well known ('first moment formula'): $E_x^\omega |Z_t| = (T_t \mathbf{1})(x)$, where $u(x, t) := (T_t \mathbf{1})(x)$ is the minimal solution of the parabolic problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \left(\frac{1}{2} \Delta + \beta \right) u \text{ on } \mathbb{R}^d \times (0, \infty), \\ u(\cdot, 0) &= 1, \\ u &\geq 0, \end{aligned} \quad (2)$$

being equivalent (by Feynman-Kac) to (1). ■

How about *random environment*?

Write $\beta := \beta_1 \mathbf{1}_K + \beta_2 \mathbf{1}_{K^c} = \beta_2 - (\beta_2 - \beta_1) \mathbf{1}_K$, recall that

$$E^\omega |Z_t| = \mathbb{E} \exp \left[\int_0^t \beta(W_s) \, ds \right]$$

and rewrite it with *negative* exponent:

$$E^\omega |Z_t| = e^{\beta_2 t} \mathbb{E} \exp \left[- \int_0^t (\beta_2 - \beta_1) \mathbf{1}_K(W_s) \, ds \right].$$

The expectation on RHS = survival probability among ‘soft obstacles’ with height $\beta_2 - \beta_1$, except: **we do not sum up the shape functions** on the overlapping balls. That is, we have

$$\mathbf{1}_K(W_s) = \mathbf{1}_{\bigcup_{x_i \in \text{supp}(\omega)} \bar{B}(x_i, a)}(W_s)$$

instead

$$\sum_{x_i \in \text{supp}(\omega)} \mathbf{1}_{\bar{B}(x_i, a)}(W_s).$$

Quenched asymptotics of global growth

“Average growth rate”:

$$r_t = r_t(\omega) := \frac{\log |Z_t(\omega)|}{t}.$$

Replace $|Z_t(\omega)|$ by expectation:

$$\bar{Z}_t := E^\omega |Z_t(\omega)|,$$

and define

$$\hat{r}_t = \hat{r}_t(\omega) := \frac{\log \bar{Z}_t(\omega)}{t}.$$

Recall from Theorem 1, that on a set of full \mathbf{P} -measure,

$$\lim_{t \rightarrow \infty} (\log t)^{2/d} (\hat{r}_t - \beta_2) = -c(d, \nu). \quad (3)$$

Analogous statement holds for r_t :

Theorem 2 (First Main Result) *On a set of full \mathbb{P} -measure,*

$$\lim_{t \rightarrow \infty} (\log t)^{2/d} (r_t - \beta_2) = -c(d, \nu). \quad (4)$$

in P^ω -probability.

That is, loosely speaking,

$$r_t \approx \beta_2 - c(d, \nu) (\log t)^{-2/d}.$$

Second Main Result: An upper estimate on the spatial spread. The order of the correction term is *larger* than the $\mathcal{O}(\log t)$ term in a result of Bramson, namely it is $\mathcal{O}\left(\frac{t}{(\log t)^{2/d}}\right)$.

We showed that, β_1 plays no role and, loosely speaking, at time t the spread of the process

$$\leq t\sqrt{2\beta_2} - c(d, \nu) \sqrt{\frac{\beta_2}{2}} \cdot \frac{t}{(\log t)^{2/d}}.$$

Method of proof for first result: “bootstrap” .

- **First step:** Let $0 < \delta < \beta_2$. Then on a set of full \mathbb{P} -measure

$$\lim_{t \rightarrow \infty} P^\omega(|Z_t| \geq e^{\delta t}) = 1.$$

- **Second step:** one particle makes it into a “clearing”
 - of a specific radius
 - with its center at a specific distance.

[Picture]

She and her descendants reproduce there without constraint.

Time scales:

1. $S(t)$: Producing many particles — they are “not too far away”
2. $T(t)$: \exists particle going to clearing
3. $U(t) := t - S(t) - T(t)$: free branching inside clearing

$S(t)$ and $T(t)$ are calibrated in a careful way; in particular, as $t \rightarrow \infty$,

$$S(t) = o(T(t)), \quad T(t) = o(U(t)) \quad \text{and} \quad U(t) = \mathcal{O}(t).$$

Problem 2 (Shape of branching tree) How does the discrete probability measure valued process

$$\tilde{Z}_t(\cdot) := \frac{Z_t(\cdot)}{|Z_t|}$$

look like? Is it true that \exists Unique dominant branch?

Problem 3 (More general branching) What happens when dyadic branching is replaced by a *general* one?

E.g. **critical branching** — taking expectation now does not provide a clue:

$$E^\omega |Z_t(\omega)| = 1, \quad \forall t > 0, \quad \forall \omega \in \Omega.$$

Let $\beta_1 = 0$. It is still true (nontrivial) that

$$P^\omega(\text{extinction}) = 1, \quad a.e. \omega \in \Omega.$$

Q: What is the order of the tail $P(\tau_{\text{ext}} > t)$?

THE END

THANKS FOR
YOUR ATTENTION!