

Entrance and exit at infinity for stable jump diffusions

Andreas Kyprianou (based on joint work with Leif Döring)

FELLER BOUNDARY CLASSIFICATION FOR DIFFUSIONS

- ▶ In his seminal work in the 1950s, William Feller classified one-dimensional diffusion processes on $-\infty \leq a < b \leq \infty$
- ▶ The four types of boundary points are:
 - regular, if it is both accessible and enterable;
 - exit, if it is accessible but not enterable;
 - entrance, if it is enterable but not accessible;
 - natural if it is neither accessible nor enterable.
- ▶ Feller's definitions and proofs are purely analytic, using Hille-Yosida theory to generate Feller semigroup of a process $(X_t, t \geq 0)$ from differential operators (diffusion generators)

$$\mathcal{A} := \kappa(x) \frac{d}{dx} + \frac{\sigma(x)^2}{2} \frac{d^2}{dx^2}$$

taking account of the different boundary conditions.

- ▶ A change of space via the so-called scale function (say s which makes $(s(X_t), t \geq 0)$ a martingale)

$$dZ_t = \tilde{\sigma}(Z_t) dB_t, \quad Z_0 = z \in \mathbb{R},$$

on a new interval (\tilde{a}, \tilde{b}) .

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THE CASE OF AN INFINITE BOUNDARY

- ▶ In the setting of the entire real line, i.e. $a = -\infty$ and $b = +\infty$, the notion of entrance (in applications also called **coming down from infinity**) and exit (explosion) becomes interesting
- ▶ Depending on the growth of σ at infinity the infinite boundary points can be of an entrance type. Feller's results for this scenario imply that $+\infty$ is an entrance boundary if and only if

$$\int^{+\infty} x \sigma(x)^{-2} dx < \infty,$$

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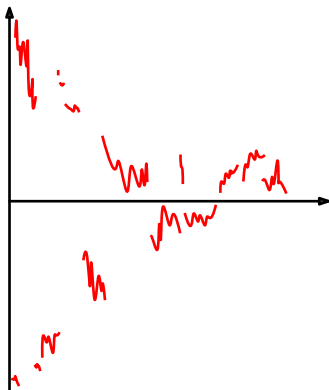
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STABLE JUMP-DIFFUSIONS

- ▶ Focus our study on so-called stable jump diffusions:

$$dZ_t = \sigma(Z_{t-}) dX_t, \quad Z_0 = z \in \mathbb{R}, t \geq 0.$$

- ▶ Interested in entrance from $\{+\infty\}$, $\{-\infty\}$ and $\pm\infty := \{+\infty\} \cup \{-\infty\}$



STABLE PROCESS

- ▶ A stable process lies in the intersection of the class of Lévy process (stationary and independent increments) and the class of self-similar Markov processes: **for all $c > 0$ and $x \in \mathbb{R}$,**

$(cX_{c^{-\alpha}t}, t \geq 0)$ under \mathbb{P}_x is equal in law to $(X_t, t \geq 0)$ under \mathbb{P}_{cx} ,

where $(\mathbb{P}_x, x \in \mathbb{R})$ are the probabilities of X and $\alpha \in (0, 2)$.

- ▶ Semigroup of X is entirely characterised by $\Psi(z) := -\log \mathbb{E}_0 [e^{izX_1}]$, satisfying

$$\Psi(z) = |z|^\alpha \left(e^{\pi i \alpha (\frac{1}{2} - \rho)} \mathbf{1}_{\{z > 0\}} + e^{-\pi i \alpha (\frac{1}{2} - \rho)} \mathbf{1}_{\{z < 0\}} \right), \quad z \in \mathbb{R}.$$

where $\rho = \mathbb{P}(X_1 > 0)$.

- ▶ The Lévy measure associated with Ψ :

$$\frac{\Pi(dx)}{dx} = \Gamma(1 + \alpha) \frac{\sin(\pi\alpha\rho)}{\pi} \frac{1}{x^{1+\alpha}} \mathbf{1}_{(x>0)} + \Gamma(1 + \alpha) \frac{\sin(\pi\alpha\hat{\rho})}{\pi} \frac{1}{|x|^{1+\alpha}} \mathbf{1}_{(x<0)},$$

where $\hat{\rho} := 1 - \rho$. In the case that $\alpha = 1$, we take $\rho = 1/2$, meaning that X corresponds to the Cauchy process.

Convention from now on: Anything with a $\hat{\cdot}$ is associated to the law of $-X$. E.g. $\hat{\mathbb{P}}_x$ is the law of $-X$ with $X_0 = -x$.

- ▶ If X has only upwards (resp. downwards) jumps we say X is spectrally positive (resp. negative). If X has jumps in both directions we say X is two-sided. A spectrally positive (resp. negative) stable process with $\alpha < 1$ is necessarily increasing (resp. decreasing).

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Proposition (Zanzotto (2002), Döring & K. (2018))

Suppose that σ is strictly positive. Then there is a unique (possibly exploding) weak solution Z to the SDE

$$dZ_t = \sigma(Z_{t-}) dX_t, \quad Z_0 = z \in \mathbb{R}, t \geq 0.$$

and Z can be expressed as time-change under \mathbb{P}_z via

$$Z_t := X_{\tau_t}, \quad t < T,$$

where

$$\tau_t = \inf \left\{ s > 0 : \int_0^s \sigma(X_s)^{-\alpha} ds > t \right\}$$

and the finite or infinite explosion time is $T = \int_0^\infty \sigma(X_s)^{-\alpha} ds$.

The law of the unique solution Z will be denoted by $\mathbb{P}_z, z \in \mathbb{R}$.

Technical point: when $\alpha \in (1, 2)$, the origin is a recurrent point, hence as $\sigma > 0$, $T = \infty$.

However, when $\alpha \in (1, 2)$, $k := \inf\{t > 0 : Z_t = 0\}$ is almost surely finite (irrespective of Z_0).

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ENTRANCE AT INFINITY

Definition

We say that $\pm\infty$ is a (continuous) entrance point for a Feller process Y on \mathbb{R} with transition semigroup \mathcal{P} (with probabilities $\mathbb{P}_x, x \in \mathbb{R}$) if

- (i) the point $\pm\infty$ is not accessible,
- (ii) the semigroup \mathcal{P} can be extended to a Feller semigroup $\overline{\mathcal{P}}$ on $C_b(\overline{\mathbb{R}})$,
- (iii) there is continuous entrance in the sense that

$$\mathbb{P}_{\pm\infty} \left(\lim_{t \downarrow 0} |Y_t| = \infty, \limsup_{t \downarrow 0} Y_t = +\infty, \liminf_{t \downarrow 0} Y_t = -\infty \right) = 1$$

Analogously, we define entrance from $-\infty$ as extension to $C_b(\mathbb{R})$ and entrance from $+\infty$ as extension to $C_b(\overline{\mathbb{R}}) = C(\overline{\mathbb{R}})$.

ENTRANCE AT INFINITY

Theorem (Döring & K. (2018))

Suppose that σ is uniformly bounded away from the origin and let

$$I^{\sigma,\alpha}(A) = \int_A \sigma(x)^{-\alpha} |x|^{\alpha-1} dx \quad \text{and} \quad I^{\sigma,1} = \int_{\mathbb{R}} \sigma(x)^{-1} \log |x| dx.$$

Then the following table exhaustively summarizes entrance points at infinity of

$$dZ_t = \sigma(Z_{t-}) dX_t, \quad Z_0 = z \in \mathbb{R}, t \geq 0.$$

Necessary and sufficient conditions for entrance from infinite boundary points				
α	Jumps	$+\infty$	$-\infty$	$\pm\infty$
< 1	only \downarrow	\times	\times	\times
	only \uparrow	\times	\times	\times
	\uparrow & \downarrow	\times	\times	\times
$= 1$	\uparrow & \downarrow	\times	\times	\checkmark iff $I^{\sigma,1} < \infty$
> 1	only \downarrow	\times	\checkmark iff $I^{\sigma,\alpha}(\mathbb{R}_-) < \infty$	\times
	only \uparrow	\checkmark iff $I^{\sigma,\alpha}(\mathbb{R}_+) < \infty$	\times	\times
	\uparrow & \downarrow	\times	\times	\checkmark iff $I^{\sigma,\alpha}(\mathbb{R}) < \infty$
$= 2$	none	\checkmark iff $I^{\sigma,2}(\mathbb{R}_+) < \infty$	\checkmark iff $I^{\sigma,2}(\mathbb{R}_-) < \infty$	\times

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$= 2$	none	\checkmark iff $I^{\sigma, 2}(\mathbb{R}_+) < \infty$	\checkmark iff $I^{\sigma, 2}(\mathbb{R}_-) < \infty$	\times

Henceforth concentrate on the case of two-sided jumps.

RIESZ–BOGDAN–ŽAK TRANSFORM

Convention from now on: Anything with a $\hat{\cdot}$ is associated to the law of $-X$. E.g. $\hat{\mathbb{P}}_x$ is the law of $-X$ with $X_0 = -x$.

Theorem (Bogdan & Žak (2010), K. (2016))

Suppose that X is a stable process with two-sided jumps. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} du > t\}, \quad t \geq 0.$$

Then, for all $x \in \mathbb{R} \setminus \{0\}$,

$$\frac{1}{X_{\eta(t)}}, \quad t \geq 0$$

under $\hat{\mathbb{P}}_x$ a self-similar Markov process equal in law to $(X, \mathbb{P}_{1/x}^{\circ})$, where

$$\frac{d\mathbb{P}_x^{\circ}}{d\mathbb{P}_x} \Big|_{\mathcal{F}_t} = \frac{h(X_t)}{h(x)} \mathbf{1}_{(t < \tau^{\{0\}})}$$

$$h(z) = (\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\operatorname{sgn}(z)) |z|^{\alpha-1}$$

and $\mathcal{F}_t := \sigma(X_s : s \leq t)$, $t \geq 0$.

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STABLE CONDITIONED TO AVOID THE ORIGIN

- ▶ Recalling that $\alpha \in (1, 2)$, $|x|^{\alpha-1}$ as a Doob h -function, rewards paths that are far from the origin ($|x| \gg 1$) and punishes paths that stray too close to the origin ($|x| \ll 1$).
- ▶ In fact it has been shown [Chaumont, Panti & Rivero (2013), Kuznetsov, K., Pardo, Watson (2014)] that (X, \mathbb{P}_y°) , $y \neq 0$, can be identified by the limit

$$\mathbb{P}_y^\circ(A) = \lim_{s \rightarrow \infty} \mathbb{P}_y(A \mid T_0 > t + s),$$

for $A \in \mathcal{F}_t$ and $T_0 = \inf\{t > 0 : X_t = 0\}$.

- ▶ (WARNING! Ultra specialist information): As X is a point recurrent process, there exists an excursion measure $n(\cdot)$ for the Poisson random field of excursions from the origin, from which one can construct (up to a constant)

$$\mathbb{P}_0^\circ(X_t^\circ \in dz) := h(z)n(X_t \in dz, t < \zeta)$$

consistently with \mathbb{P}_y° , $y \neq 0$, where ζ is the excursion lifetime and

$$h(z) = (\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\text{sgn}(z)) |z|^{\alpha-1}$$

- ▶ (Executive summary of last point): The limit

$$\mathbb{P}_0^\circ := \lim_{|y| \rightarrow 0} \mathbb{P}_y^\circ$$

is well defined in the sense of Skorohod convergence.

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- ▶ Recalling that $\alpha \in (1, 2)$, $|x|^{\alpha-1}$ as a Doob h -function, rewards paths that are far from the origin ($|x| \gg 1$) and punishes paths that stray too close to the origin ($|x| \ll 1$).
- ▶ In fact it has been shown [Chaumont, Panti & Rivero (2013), Kuznetsov, K., Pardo, Watson (2014)] that (X, \mathbb{P}_y°) , $y \neq 0$, can be identified by the limit

$$\mathbb{P}_y^\circ(A) = \lim_{s \rightarrow \infty} \mathbb{P}_y(A \mid T_0 > t + s),$$

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TIME CHANGE AND RIESZ-BOGDAN-ŻAK

Remember there is a unique weak solution Z to the SDE

$$dZ_t = \sigma(Z_{t-}) dX_t, \quad Z_0 = z \in \mathbb{R}, t \geq 0.$$

and Z can be expressed as time-change under \mathbb{P}_z via $Z_t := X_{\tau_t}, t < T$, where

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Proposition (Döring & K. (2018))

Set

$$\beta(x) = \sigma(1/x)^{-\alpha} |x|^{-2\alpha}, \quad x \in \mathbb{R} \setminus \{0\}.$$

Define the time-space transformation

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SUFFICIENCY (HEURISTIC)

Writing $G_{\hat{X}^\circ}(x, dy)$ for the resolvent of \hat{X}° and $G_{\hat{X}^\dagger}(x, dy)$ for the resolvent of X killed on first hitting the origin,

$$\begin{aligned} & \hat{\mathbb{E}}_x^\circ \left[\int_0^\infty \beta(\hat{X}_u^\circ) du \right] \\ &= \int_{\mathbb{R}} G_{\hat{X}^\circ}(x, dy) \sigma(1/y)^{-\alpha} |y|^{-2\alpha} \\ &= \int_{\mathbb{R}} G_{\hat{X}^\dagger}(x, dy) \frac{\hat{h}(y)}{\hat{h}(x)} \sigma(1/y)^{-\alpha} |y|^{-2\alpha} \\ &\approx \int_{\mathbb{R}} \left(|y|^{\alpha-1} s(y) - |y-x|^{\alpha-1} s(y-x) + |x|^{\alpha-1} s(-x) \right) \frac{|y|^{\alpha-1}}{|x|^{\alpha-1}} \sigma(1/y)^{-\alpha} |y|^{-2\alpha}, \end{aligned}$$

which is finite if

$$\int_{\mathbb{R}} \sigma(x)^{-\alpha} |x|^{\alpha-1} dx < \infty.$$

Note, for a Markov process Y , with probabilities P_x , $x \in E$,

$$G_Y(x, dy) = \int_0^\infty P_x(Y_t \in dy) dt, \quad x, y \in E.$$

HUNT-NAGASAWA DUALITY

Proposition (Döring & K. (2018))

Suppose that \hat{X}° has probabilities $\hat{\mathbb{P}}_x^\circ$, $x \in \mathbb{R}$. Define $\hat{Z}_t^\circ = \hat{X}_{\iota_t}^\circ$, $t \geq 0$, where the time-change ι is given by

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Recall that Z has the law of the unique weak solution to the SDE and Z^\dagger is the same process killed on first hitting 0.

If $\pm\infty$ is an entrance point for Z , then the time reversed process $Z_{(k-t)^-}^\dagger$, $t \leq k$, under $\mathbb{P}_{\pm\infty}$ is a time-homogenous Markov process with transition semigroup which agrees with that of \hat{Z}° , where k is any almost surely finite last passage time for Z^\dagger (e.g. $k = \inf\{t > 0 : Z_t^\dagger = 0\}$).

Remark on proof: Important step is to prove weak duality:

$$p_{Z^\dagger}(t, y, dz)\mu(dy) = p_{\hat{Z}^\circ}(t, z, dy)\mu(dz)$$

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$$\mu(dy) = \int_{\mathbb{R}} \nu(dx) G_{\hat{Z}^\circ}(x, dy) = \sigma(x)^{-\alpha} h(x) dx$$

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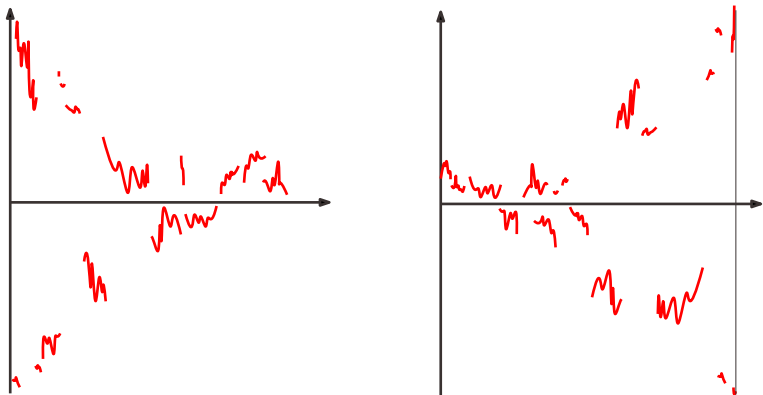
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The time reversed process $Z_{(k-t)-}^{\dagger}, t \leq k$, under $P_{\pm\infty}$ is a time-homogenous Markov process with transition semigroup which agrees with that of \hat{Z}° , where k is any almost surely finite last passage time for Z^{\dagger} (e.g. $k = \inf\{t > 0 : Z_t^{\dagger} = 0\}$)

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then necessarily $\int_{\mathbb{R}} \sigma(x)^{-\alpha} |x|^{\alpha-1} dx < \infty$.

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- ▶ A bit of work

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for any $x \in \mathbb{R}$.

NECESSITY (HEURISTIC)

- ▶ We want to show that if $\pm\infty$ is an entrance point for

$$dZ_t = \sigma(Z_{t-}) dX_t, \quad Z_0 = z \in \mathbb{R}, t \geq 0,$$

then necessarily $\int_{\mathbb{R}} \sigma(x)^{-\alpha} |x|^{\alpha-1} dx < \infty$.

- ▶ If $\pm\infty$ is an entrance point, then Z can be seen as a **Feller process on the compact space $\overline{\mathbb{R}}$** .
- ▶ Gettoor's equivalent definitions of transience:
 - ▶ On the one hand, last exit from any compact set is a.s. finite
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DIFFICULTIES IN OTHER REGIMES

- ▶ Two sided jumps
 - ▶ $\alpha \leq 1$ Cannot hit the origin, so cannot time reverse from the origin or condition to avoid the origin
 - ▶ $\alpha = 1$ Can time reverse from first entry into strip $(-1, 1)$
 - ▶ $\alpha < 1$ Can do the same as $\alpha = 1$ but cannot control the time change to explosion
- ▶ One sided jumps
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EXPLOSION (EXIT AT INFINITY)

Theorem (Döring & K. (2018))

Suppose that $\sigma > 0$ and let

$$I^{\sigma, \alpha}(A) = \int_A \sigma(x)^{-\alpha} |x|^{\alpha-1} dx.$$

Then the following table exhaustively summarises finite time explosion for

$$dZ_t = \sigma(Z_{t-}) dX_t, \quad Z_0 = z \in \mathbb{R}, t \geq 0.$$

Necessary and sufficient conditions for exit at infinite boundary points				
α	Jumps	$+\infty$	$-\infty$	$\pm\infty$
< 1	only \downarrow	\times	\checkmark iff $I^{\sigma, \alpha}(\mathbb{R}_-) < \infty$	\times
	only \uparrow	\checkmark iff $I^{\sigma, \alpha}(\mathbb{R}_+) < \infty$	\times	\times
	\uparrow & \downarrow	\times	\times	\checkmark iff $I^{\sigma, \alpha}(\mathbb{R}) < \infty$
$= 1$	\uparrow & \downarrow	\times	\times	\times
> 1	only \downarrow	\times	\times	\times
	only \uparrow	\times	\times	\times
	\uparrow & \downarrow	\times	\times	\times
$= 2$	none	\times	\times	\times

Thank you!