Survival of homogenous fragmentation processes with killing

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Abstract

We consider a homogenous fragmentation process with killing at an exponential barrier. With the help of two families of martingales we analyse the growth of the largest fragment for parameter values that allow for survival. In this respect the present paper is also concerned with the probability of extinction of the killed process.

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1 Introduction and main results

This paper considers the growth of the largest fragment in a homogenous fragmentation process in which there is an additional killing mechanism. We therefore begin our exposition by briefly reviewing what is meant by a homogenous fragmentation process, thereby introducing some notation.

1.1 Homogenous fragmentation processes

Let $\mathcal{P}$ be the space of partitions of the natural numbers. Here a partition of $\mathbb{N}$ is a sequence $\pi = (\pi_1, \pi_2, \cdots)$ of disjoint sets, called blocks, such that $\bigcup_{i\in\mathbb{N}} \pi_i = \mathbb{N}$. The blocks of a partition are enumerated in the increasing order of their least element, that is to say $\min \pi_i \le \min \pi_j$ when $i \le j$ (with the convention that $\min \emptyset = \infty$). Now consider the measure μ on $\mathcal{P}$, given by

$$\mu(d\pi) = \int_{\mathcal{S}} \varrho_{\mathbf{s}}(d\pi)\nu(d\mathbf{s}),$$

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where $\varrho_{\mathbf{s}}$ is the law of Kingman's paint-box based on $\mathbf{s} \in \mathcal{S}$ where

$$\mathcal{S} := \left\{ \mathbf{s} = (s_1, s_2, \cdots) : s_1 \geq s_2 \geq \cdots \geq 0, \sum_{i=1}^{\infty} s_i \leq 1 \right\},$$

and the so-called *dislocation measure* ν is a measure on $\mathcal{S}$ such that

$$\nu(\{(a, 0, \cdots)\}) = 0 \tag{1}$$

for all $a \in [0, 1]$ and

$$\int_{\mathcal{S}} (1 - s_1)\nu(d\mathbf{s}) < \infty. \tag{2}$$

It is known that μ is an exchangeable partition measure, meaning that it is invariant under the action of finite permutations on $\mathcal{P}$. It is also known (cf. Chapter 3 of Bertoin [5]) that it is possible to construct a fragmentation process on the space of partitions $\mathcal{P}$ with the help of a Poisson point process on $\mathcal{P} \times \mathbb{N}$, $\{(\pi(t), k(t)) : t \geq 0\}$, which has intensity measure $\mu \otimes \sharp$. The aforementioned $\mathcal{P}$-valued fragmentation process is a Markov process which we denote by $\Pi = \{\Pi(t) : t \geq 0\}$, where $\Pi(t) = (\Pi_1(t), \Pi_2(t), \cdots) \in \mathcal{P}$ is such that at all times $t \geq 0$ for which an atom $(\pi(t), k(t))$ occurs in $(\mathcal{P} \backslash (\mathbb{N}, \emptyset, \ldots)) \times \mathbb{N}$, $\Pi(t)$ is obtained from $\Pi(t-)$ by partitioning the $k(t)$-th block into the sub-blocks $(\Pi_{k(t)}(t-) \cap \pi_j(t) : j = 1, 2, \cdots)$. When ν is a finite measure each block experiences an exponential holding time before it fragments.

Thanks to the properties of the exchangeable partition measure μ it can be shown that for each $t \geq 0$ the distribution of $\Pi(t)$ is exchangeable and moreover, blocks of $\Pi(t)$ have asymptotic frequencies in the sense that for each $i \in \mathbb{N}$,

$$|\Pi_i(t)| := \lim_{n \to \infty} \frac{1}{n} \sharp\{\Pi_i(t) \cap \{1, \cdots, n\}\}$$

exists almost surely.

We denote the countable random jump times of Π by $\mathcal{I} \subseteq \mathbb{R}_0^+$. Further, let $\mathcal{F} := (\mathcal{F}_t)_{t \in \mathbb{R}_0^+}$ denote the filtration generated by Π. In addition, let $\mathcal{G} := (\mathcal{G}_t)_{t \in \mathbb{R}_0^+}$ be the sub–filtration generated by the asymptotic frequencies of Π and let $\mathcal{F}^1 := (\mathcal{F}_t^1)_{t \in \mathbb{R}_0^+}$ denote the filtration generated by $(\Pi_1(t))_{t \in \mathbb{R}_0^+}$.

If we define $\xi_t = -\log |\Pi_1(t)|$ for $t \geq 0$, then it is easily deduced from the Poissonian construction of the fragmentation process that $\{\xi_t : t \geq 0\}$ is a (possibly killed) subordinator. Indeed, it is well known that its Laplace exponent Φ, given by

$$\Phi(p) := -\log \mathbb{E}(e^{-p\xi_1}),$$

can be characterised over an appropriate domain of p through the dislocation measure ν as follows. Define the constant

$$\underline{p} = \inf \left\{ p \in \mathbb{R} : \int_{\mathcal{S}} \left| 1 - \sum_{i=1}^{\infty} s_i^{1+p} \right| \nu(d\mathbf{s}) < \infty \right\}$$

which is necessarily in $(-1, 0]$. Then

$$\Phi(p) = \int_{\mathcal{S}} \left(1 - \sum_{i=1}^{\infty} s_i^{1+p} \right) \nu(d\mathbf{s})$$

for all $p \geq \underline{p}$. The tagged fragment Π_1, and in particular its Laplace exponent Φ, can be used to extract information about the growth and spatial evolution of the fragmentation process. A case in point concerns the asymptotic rate of decay of the largest block

$$\lambda_1(t) := \sup_{n\in\mathbb{N}} |\Pi_n(t)|,\ t \geq 0.$$

To this end, note that Φ is strictly increasing, concave and differentiable. The equation

$$(p+1)\Phi'(p) = \Phi(p)$$

for $p > \underline{p}$ is known to have a unique solution in $(0,\infty)$ which we shall denote by $\overline{p}$ (cf. Lemma 1 in [4]). Moreover, $(p+1)\Phi'(p) - \Phi(p) > 0$ when $p \in (\underline{p}, \overline{p})$. This implies that the function

$$p \mapsto c_p := \frac{\Phi(p)}{p+1}$$

reaches its unique maximum on $(\underline{p}, \infty)$ at $\overline{p}$. This maximal value turns out to characterise the asymptotic rate of decay of the largest block, as shown in the following proposition that is lifted from Bertoin [5].

Proposition 1 (Corollary 1.4 in [5])**.** *We have*

$$\lim_{t\to\infty} \frac{-\log \lambda_1(t)}{t} = c_{\overline{p}}$$

$\mathbb{P}$*–almost surely.*

1.2 Killed homogenous fragmentation processes

Now let $c > 0$ and $x \in \mathbb{R}_0^+$. We want to introduce killing of Π upon hitting the space–time barrier

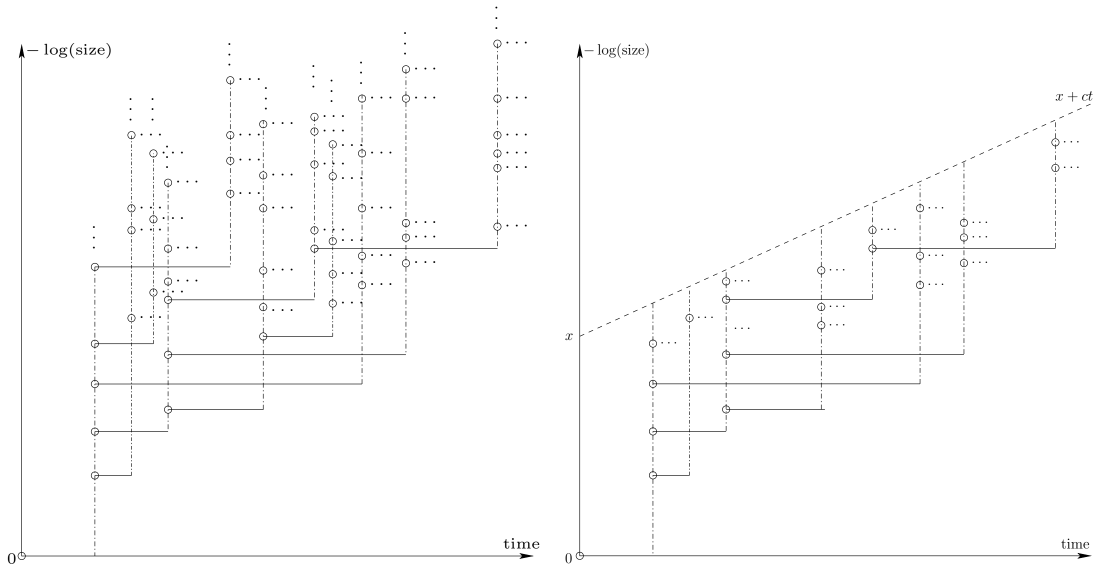
$$\left\{(y,t) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : y < e^{-(x+ct)}\right\}$$

as follows. A block $\Pi_n(t)$ is killed at the moment of its creation $t \in \mathcal{I}$ if $|\Pi_n(t)| < e^{-(x+ct)}$, see Figure 1. Here, killing a block means that it is sent to a *cemetery state*, which we shall identify by $(\emptyset, \ldots)$.

Suppose that for $t \geq 0$ we define $\mathcal{N}_t^x$ to be the index set of the blocks $(\Pi_n(t))_{n\in\mathbb{N}}$ that are not yet killed by time t. It is important to note that $N_t^x := \mathrm{card}(\mathcal{N}_t^x)$ is finite for each t. Indeed, as $\sum_{n\in\mathbb{N}} |\Pi_n(t)| \leq 1$ we infer that $|\Pi_n(t)| \geq e^{-(x+ct)}$ for at most e^{x+ct}–many $n \in \mathbb{N}$. That is

$$N_t^x \leq e^{x+ct}$$

for all $t \in \mathbb{R}_0^+$. Denote by $\Pi^x := (\Pi^x(t) : t \geq 0)$, where $\Pi^x(t) = (\Pi_n(t))_{n\in\mathcal{N}_t^x}$, the resulting killed fragmentation process and note that Π^x is not necessarily $\mathcal{P}$-valued. If we re-assign the indices of the blocks in $(\Pi_n(t))_{n\in\mathcal{N}_t^x}$ so that the ordering is given by the least element of each block relative to the set of natural numbers found in $\Pi^x(t)$ rather than the whole of $\mathbb{N}$, then we shall notationally identify $\Pi^x(t)$ as $(\Pi_n^x(t))_{n\in\mathbb{N}}$, where necessarily we have $\Pi_n^x(t) = \emptyset$ for $n > N_t^x$.



(a) Realisation of a fragmentation process without killing.

(b) Realisation of the killed fragmentation process corresponding to (a).

Figure 1: Realisation of a fragmentation process with finite dislocation measure without killing, in (a), and with killing, in (b).

For each $n \in \mathbb{N}$ the block of Π^x containing $n \in \mathbb{N}$ has a killing time that may be finite or infinite. Note that the killed fragmentation process Π^x also depends on the constant $c > 0$. However, in order to keep the notation as simple as possible we do not include the parameter c in the notation as this constant does not change within the results or proofs of this paper.

In this paper we shall answer the question whether it is possible that the supremum over all the aforementioned respective individual killing times, which is henceforth denoted by ζ^x, is finite. We say that Π^x becomes *extinct* if $\{\zeta^x < \infty\}$. Our first main result in this respect is the following.

Theorem 2. *For all* $c \le c_{\bar{p}}$ *we have* $\mathbb{P}(\zeta^x < \infty) = 1$ *for any* $x \in \mathbb{R}_0^+$. *If* $c > c_{\bar{p}}$, *then* $x \mapsto \mathbb{P}(\zeta^x < \infty)$ *is a nonincreasing,* $(0,1)$*–valued function on* $\mathbb{R}_0^+$.

In the case that extinction does not occur with probability 1, we shall give two qualitative results concerning the growth of the process on survival. The first result shows that the total number of fragments in the surviving process explodes.

Theorem 3. *Let* $c > c_{\bar{p}}$. *Then we have that*

$$\limsup_{t \to \infty} N_t^x = \infty$$

holds true $\mathbb{P}(\cdot | \zeta^x = \infty)$*–a.s. for any* $x \in \mathbb{R}_0^+$.

The second result shows that the asymptotic exponential rate of decay of the largest fragment,

$$\lambda_1^x(t) := \sup_{n \in \mathbb{N}} |\Pi_n^x(t)|, \; t \ge 0,$$

is the same as when the killing scheme is not in effect, cf. Proposition 1.

Theorem 4. *Let $c > c_{\bar{p}}$ and $x \in \mathbb{R}_0^+$. Then we have*

$$\lim_{t\to\infty} \frac{-\log \lambda_1^x(t)}{t} = c_{\bar{p}}$$

$\mathbb{P}(\cdot|\zeta^x = \infty)$*–almost surely.*

What lies fundamentally behind the proofs of our main results is a detailed study of the interaction between two classes of martingales.

The outline of this paper is as follows. In the next section we provide some general notions that are used in the subsequent parts of the present paper and in particular we employ the connection between fragmentations and Lévy processes. Section 3 is concerned with the proof of Theorem 2 and in Section 3 we provide the proof of Theorem 3 . Then, in Section 5, we introduce a multiplicative process and examine when this process is a martingale. The object under consideration in Section 6 is an additive process which also turns out to be a martingale and whose limit we study with regard to strict positivity. In the final section of this paper we prove Theorem 4.

2 Preliminaries

Let $B_n(t)$, $t \in \mathbb{R}_0^+$, denote the block in $\Pi(t)$ that contains the element $n \in \mathbb{N}$ and recall that under $\mathbb{P}$ the process $\xi_n = (-\log |B_n(t)|)_{t\in\mathbb{R}_0^+}$ is a (possibly killed) subordinator (with cemetery state $+\infty$).

Definition 5. *For every $n \in \mathbb{N}$ let the process $X_n := (X_n(t))_{t\in\mathbb{R}_0^+}$ be defined by*

$$X_n(t) := ct - \xi_n(t)$$

for all $t \in \mathbb{R}_0^+$.

Notice that under $\mathbb{P}$ the dynamics of the process X_n are those of a (possibly killed) spectrally negative Lévy process of bounded variation (with cemetery state $-\infty$). Moreover, the jump times of X_n, henceforth denoted by $\mathcal{I}_n \subseteq \mathbb{R}_0^+$, are the set of dislocation times of $(B_n(t))_{t\in\mathbb{R}_0^+}$. That is, X_n jumps exactly when the subordinator ξ_n jumps. For any $n \in \mathbb{N}$ and $x \in \mathbb{R}_0^+$ consider the following $\mathcal{F}$–stopping times:

$$\tau_{n,x}^+ := \inf\{t \in \mathbb{R}_0^+ : X_n(t) > x\} \qquad \text{as well as} \qquad \tau_{n,x}^- := \inf\{t \in \mathbb{R}_0^+ : X_n(t) < -x\}.$$

For any $p \in (\underline{p}, \infty)$ consider the change of measure given by

$$\left.\frac{\mathrm{d}\mathbb{P}^{(p)}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\Phi(p)t - p\xi(t)} = e^{pX_1(t) - \psi(p)t}, \tag{3}$$

where

$$\psi(p) = \Phi(p) - cp = \frac{1}{t}\log \mathbb{E}(e^{pX_1(t)})$$

is the Laplace exponent of X_1. Moreover, considering the projection of (3) onto the sub–filtration $\mathcal{G}$ results in

$$\left.\frac{\mathrm{d}\mathbb{P}^{(p)}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_t} = M_t(p) := \sum_{n\in\mathbb{N}} |\Pi_n(t)|^{1+p} e^{\Phi(p)t},$$

for all $p \in (\underline{p}, \infty)$ and $t \in \mathbb{R}_0^+$.

Remark 6. In view of Theorem 2 of [4] we have that restricted, to the σ–algebra $\mathcal{G}_\infty := \bigcup_{t\in\mathbb{R}_0^+} \mathcal{G}_t$, the measures $\mathbb{P}^{(p)}$ and $\mathbb{P}$ are equivalent for any $p \in (\underline{p}, \bar{p})$. Moreover, since $M(p)$ is a uniformly integrable unit–mean martingale, we infer that $\mathbb{P}^{(p)}$ is a probability measure on $\mathcal{G}_\infty$. $\diamondsuit$

Corollary 3.10 in [11] shows that under the measure $\mathbb{P}^{(p)}$ the process X_1 is again a spectrally negative Lévy process such that

$$\frac{1}{t}\log \mathbb{E}^{(p)}(e^{\lambda X_1(t)}) =: \psi_p(\lambda) = \psi(\lambda + p) - \psi(p) = c\lambda - \Phi(\lambda + p) + \Phi(p) \tag{4}$$

for all $\lambda > \underline{p} - p$. Let W_p be the scale function of the spectrally negative Lévy process X_1 under $\mathbb{P}^{(p)}$. That is to say, W_p is the unique non-decreasing continuous function on $(0, \infty)$ that is defined through the Laplace transform

$$\int_0^\infty e^{-\lambda x} W_p(x)\mathrm{d}x = \frac{1}{\psi_p(\lambda)},$$

for all sufficiently large λ. For convenience we shall write W in place of W_0.

A fundamental identity involving the scale function W_p that we shall appeal to later is the following result taken from Theorem 8.1, equation (8.7), in [11].

$$\mathbb{P}^{(p)}(\tau_{1,x}^- = \infty) = (\psi_p'(0+) \vee 0) W_p(x), \tag{5}$$

for all $x \geq 0$. Another important fact that we shall also make use of concerns the value of W_p at zero. Indeed, thanks to the fact that X_1 has paths of bounded variation, it turns out that for all $p \geq 0$, $W_p(0+) = 1/c$. See for example Lemma 8.6 in [11].

An important role in what follows will be played by X_n killed upon hitting $(-\infty, -x)$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}_0^+$. For $t \in \mathbb{R}_0^+$ set

$$X_n^x(t) := (X_n(t) + x)\mathbb{1}_{\{\tau_{n,x}^- > t\}} = (x + ct + \log|B_n(t)|)\,\mathbb{1}_{\{\tau_{n,x}^- > t\}}.$$

Next observe that

$$\nu(\mathbf{s} \in \mathcal{S} : s_1 \in (0, x]) < \infty$$

for every $x \in (0, 1)$ as otherwise

$$\int_{\mathcal{S}} (1 - s_1)\nu(\mathrm{d}\mathbf{s}) \geq \int_{\{\mathbf{s}\in\mathcal{S}:s_1\in(0,x]\}} (1 - s_1)\nu(\mathrm{d}\mathbf{s}) \geq (1 - x)\nu(\mathbf{s} \in \mathcal{S} : s_1 \in (0, x]) = \infty,$$

which contradicts (1). Further, note that there exists some $x \in (0, 1)$ such that

$$\nu(\mathbf{s} \in \mathcal{S} : s_1 \in (0, x]) > 0, \tag{6}$$

as otherwise $\nu(\mathbf{s} \in \mathcal{S} : s_1 = 1) = \nu(\mathcal{S})$ which contradicts (2).

With the above in hand, recall that $\mathcal{I}_1 \subseteq \mathbb{R}_0^+$ consists of the jump times of the block containing 1 in the (unkilled) fragmentation process Π and for any $x \in (0,1)$ set

$$\tau(x) := \inf \{t \in \mathcal{I}_1 : |\pi_1(t)| \leq x\} . \tag{7}$$

For all $x \in (0,1)$ with $\nu(\mathbf{s} \in \mathcal{S} : s_1 \in (0,x]) > 0$, Proposition 2 in Section 0.5 of [2] implies that under $\mathbb{P}$ the stopping time $\tau(x)$ is exponentially distributed. In particular, for any such $x \in (0,1)$ the infimum in (7) is actually a minimum and $\tau(x) \in (0,\infty)$ $\mathbb{P}$–almost surely.

3 Properties of the extinction probability

In this section we prove Theorem 2 by dealing with its statements for the two cases $c \in (0, c_{\bar{p}}]$ and $c > c_{\bar{p}}$ as two separate lemmas. The first lemma below deals with the easier, but less interesting, case that $c \in (0, c_{\bar{p}}]$.

Lemma 7. *Let $c \in (0, c_{\bar{p}}]$. Then $\mathbb{P}(\zeta^x < \infty) = 1$ for all $x \in \mathbb{R}_0^+$.*

Proof. Let $p \geq \bar{p}$ be such that $c_p = c$. Note that such a $p \geq \bar{p}$ does indeed exist, since according to Lemma 1 in [4] the mapping $p \mapsto \Phi(p)/(1+p)$ is continuous and decreasing to 0 as $p \to \infty$. It was shown in Theorem 4 in [7] (cf. also Theorem 1 in [6]) that $M_t(p) \to 0$ $\mathbb{P}$–a.s. as $t \to \infty$. Since $M_t(p) \geq e^{\Phi(p)t} \lambda_1^{1+p}(t)$ for all $t \in \mathbb{R}_0^+$, we thus deduce that

$$\Phi(p)t + \log\left(\lambda_1(t)^{1+p}\right) \to -\infty$$

as $t \to \infty$. Since $c < c_{\bar{p}}$ and $p \geq \bar{p} > \underline{p} > -1$, this implies that

$$(ct + \log(\lambda_1(t))) \to -\infty$$

as $t \to \infty$ and hence $\mathbb{P}(\zeta^x < \infty) = 1$ for all $x \in \mathbb{R}_0^+$. □

Notice that the statement of the previous lemma is obvious for $c \in (0, c_{\bar{p}})$ as the asymptotic speed of the largest fragment in the non–killed setting is given by $c_{\bar{p}}$, see Proposition 1, and thus the fragmentation process eventually crosses the killing line almost surely. However, for the critical value $c = c_{\bar{p}}$ this argument does not work as one needs to rule out the possibility that the largest fragment could approach the killing line without intersecting it.

All the following results in this chapter deal with the more interesting case that $c > c_{\bar{p}}$.

Lemma 8. *Let $c > c_{\bar{p}}$. Then*

$$\mathbb{P}(\zeta^x < \infty) \in (0,1)$$

for all $x \in \mathbb{R}_0^+$.

Proof. The proof is divided into two parts. The first part shows that $\mathbb{P}(\zeta^x < \infty) < 1$ and the second part proves that $\mathbb{P}(\zeta^x < \infty) > 0$ for all $x \in \mathbb{R}_0^+$.

Part I. Choose some $p \in (\underline{p}, \bar{p})$ such that $c > \Phi'(p)$, and note that $\psi_p'(0+) = \psi'(p) = c - \Phi'(p) > 0$. Hence, we deduce from (5) that

$$\mathbb{P}^{(p)}(\tau_{1,0}^- < \infty) = 1 - \psi_p'(0+)W_p(0) = 1 - \frac{\psi_p'(0+)}{c} \in (0,1). \tag{8}$$

By means of the nondecreasingness of $\mathbb{P}(\zeta^{(\cdot)} = \infty)$, equation (8) implies that

$$\mathbb{P}^{(p)}(\zeta^x = \infty) \geq \mathbb{P}^{(p)}(\tau_{1,0}^- = \infty) = \frac{\psi_p'(0+)}{c} \in (0,1)$$

for all $x \in \mathbb{R}_0^+$. According to Remark 6 this results in

$$\mathbb{P}(\zeta^x = \infty) > 0, \qquad \text{i.e.} \qquad \mathbb{P}(\zeta^x < \infty) < 1.$$

Part II. In order to show that $\mathbb{P}(\zeta^x < \infty) > 0$ for every $x \in \mathbb{R}_0^+$ we fix some $a > x$ and some $y_0 \in (1/2 \vee (1 - e^{-a}), 1)$ satisfying

$$q := \nu(\mathbf{s} \in \mathcal{S} : s_1 \leq y_0) \in (0, \infty).$$

(Recall from the discussion in Section 2 that this is possible). Note that $\mathbb{P}(\zeta^x < \infty) > 0$ trivially holds if ν is finite as in that case the probability of extinction on the first dislocation is positive. We therefore assume without loss of generality that ν is infinite. In particular, this yields that $\nu(\mathbf{s} \in \mathcal{S} : s_1 > y_0) = \infty$. Consider the fragmentation processes $\widetilde{\Pi}$ obtained from the restricted dislocation measure

$$\widetilde{\nu} := \nu|_{\{\mathbf{s} \in \mathcal{S} : s_1 > y_0\}}.$$

By means of Proposition 2 in Section 0.5 of [2] we have that $\tau(y_0) = \inf\{t \in \mathcal{I}_1 : |\pi_1(t)| \leq y_0\}$ is exponentially distributed with parameter q and, moreover, is independent of $\widetilde{\Pi}$. Indeed, the processes $\widetilde{\Pi}$ and Π agree in law on the time interval $[0, \tau(y_0))$. Note that until time $\tau(y_0)$ every dislocation of $\widetilde{\Pi}$ produces a fragment of largest relative size to its parent block which is at least y_0 in proportion. It is also straightforward to check that until this time, the second largest block produced at each dislocation will be no larger than a proportion e^{-a} of its parent. Since $(X_1(u))_{u \in [0, \tau(y_0))}$ is determined by $\widetilde{\Pi}$ stopped at $\tau(y_0)$ it follows that $(X_1(u))_{u \in [0, \tau(y_0))}$ is also independent of $\tau(y_0)$ and has the law of a Lévy process that will necessarily experience jumps of size no larger than $-\log y_0$. Let $\widetilde{X}_1^x$ be a spectrally negative Lévy process which is independent of $\tau(y_0)$ and whose paths are equal in law on the time interval $[0, \tau(y_0))$ to those of $(x + X_1(u))_{u \in [0, \tau(y_0))}$. Now define

$$R^{(q)}(a, x, \mathrm{d}y) = \int_0^\infty e^{-qt} \mathrm{d}t \cdot \mathbb{P}\left(\widetilde{X}_1^x(t) \in \mathrm{d}y, \sup_{s \leq t} \widetilde{X}_1^x(s) < a, \inf_{s \leq t} \widetilde{X}_1^x(s) > 0\right)$$

for $y \in [0, a]$. Theorem 8.7 in [11] shows that $R^{(q)}(a, x, \mathrm{d}y)$ is absolutely continuous with strictly positive Lebesgue density in the neighbourhood of the origin (this is at least immediately obvious for $y \in (0, x)$ by inspecting the expression for the resolvent in the aforementioned theorem). A little thought in light of the remarks in the previous paragraph reveals that, on the event $\{\sup_{s \leq \tau(y_0)} \widetilde{X}_s^x \leq$

a, $\inf_{s\leq t}\widetilde{X}_1^x(s)>0\}$, the process $(\widetilde{X}_1^x(u))_{u\in[0,\tau(y_0))}$ describes (on the negative-logarithmic scale and relative to the killing barrier) the only surviving block in the process Π^x over the time horizon $[0,\tau(y_0))$.

Using these facts, as well as the observation that $\tau(y_0)$ is almost surely not a jump time for $\widetilde{X}_1^x$, we now have the estimate

$$\begin{aligned}\mathbb{P}(\zeta^x<\infty) &\geq \mathbb{P}\left(\widetilde{X}_1^x(\tau(y_0))\in[0,-\log(y_0)),\ \sup_{s\leq\tau(y_0)}\widetilde{X}_s^x<a,\ \inf_{s\leq\tau(y_0)}\widetilde{X}_1^x(s)>0\right)\\ &= qR^{(q)}\left(a,x,[0,-\log(y_0))\right)>0\end{aligned}$$

as required. □

4 Explosion of mass on survival

In this section we provide the proof of Theorem 3. To this end, we shall use the following auxiliary lemma which states that for any $n\in\mathbb{N}$ there exists a stopping time such that with positive probability there are at least n blocks alive at that stopping time. More precisely, we have the following result.

Lemma 9. *Let $c>c_{\bar{p}}$. Then for any $n\in\mathbb{N}$ there exists a $t>0$ such that*

$$\mathbb{P}\left(N_t^0\geq n\right)>0. \tag{9}$$

Proof. In the first part of the proof we show that the probability of the event $\{N_t^0\geq 2\}$ is positive for some $t\in\mathbb{R}_0^+$ and in the second part we use this in conjunction with an induction argument to prove the assertion.

Part I. Since, by (1), in the unkilled fragmentation process there are at least two blocks at each jump time, it follows from (2) that there exists some $y_0\in(1/2,1)$ such that

$$\nu(\mathbf{s}\in\mathcal{S}:s_2\geq 1-y_0)>0.$$

Indeed, assume $\nu(\mathbf{s}\in\mathcal{S}:s_2\geq a)=0$ for all $a\in(0,1)$. Then $\nu(\mathbf{s}\in\mathcal{S}:s_2\neq 0)=0$, which contradicts (1). Furthermore, in the light of (6) we assume that

$$q=\nu(\mathbf{s}\in\mathcal{S}:s_1\in(0,y_0])>0.$$

We have

$$\nu(\mathbf{s}\in\mathcal{S}:s_2\geq 1-y_0)\leq\nu(\mathbf{s}\in\mathcal{S}:s_1\leq y_0)<\infty,$$

and thus Proposition 2 in Section 0.5 of [2] shows that

$$\mathbb{P}\left(|\pi(\tau(y_0))|_2^{\downarrow}\geq 1-y_0\right)=\frac{\nu(\mathbf{s}\in\mathcal{S}:s_2\geq 1-y_0)}{\nu(\mathbf{s}\in\mathcal{S}:s_1\leq y_0)}>0, \tag{10}$$

where $|\pi(\cdot)|^{\downarrow}$ denotes the decreasingly ordered vector of the asymptotic frequencies of $\pi(\cdot)$. As in the proof of Lemma 8 consider the fragmentation process $\widetilde{\Pi}$ obtained from the restricted dislocation measure

$$\widetilde{\nu} := \nu|_{\{\mathbf{s}\in\mathcal{S}:s_1>y_0\}}.$$

Moreover, recall that the stopping time $\tau(y_0) = \inf\{t \in \mathcal{I}_1 : |\pi_1(t)| \le y_0\}$ is exponentially distributed with parameter $q \in \mathbb{R}^+$ and is independent of $\widetilde{\Pi}$. Accordingly, recall the definition of the process $\widetilde{X}_1^x$ and the resolvent $R^{(q)}(a, x, \mathrm{d}y)$, for $0 \le x, y \le a$, from the proof of Lemma 8. We want to work with the resolvent

$$R^{(q)}(x, \mathrm{d}y) = \lim_{a\to\infty} R^{(q)}(a, x, \mathrm{d}y),$$

which is also known to have a strictly positive density; cf Corollary 8.8 of [11]. We have, again recalling that $\tau(y_0)$ is almost surely not a jump time for $\widetilde{X}_1^0$, that

$$\begin{aligned}
&\mathbb{P}\left(N^0_{\tau(y_0)} \ge 2\right)\\
&= \mathbb{P}\left(\widetilde{X}_1^0(\tau(y_0)-) \in (-\log(1-y_0), \infty),\ |\pi(\tau(y_0))|_2^{\downarrow} \ge 1-y_0\right)\\
&= qR^{(q)}\left(0, (-\log(1-y_0), \infty)\right)\mathbb{P}\left(|\pi(\tau(y_0))|_2^{\downarrow} \ge 1-y_0\right) > 0
\end{aligned} \tag{11}$$

Given that $\tau(y_0)$ is exponentially distributed, it is now a standard argument to deduce that there must exist a $t > 0$ such that

$$\mathbb{P}\left(N_t^0 \ge 2\right) > 0. \tag{12}$$

<u>Part II</u>. We prove (9) by resorting to the principle of mathematical induction. To this end, let $n \in \mathbb{N}$, fix some $u_0 > 0$ such that (12) holds and, as the induction hypothesis, assume that

$$\mathbb{P}(N^0_{nu_0} \ge n+1) > 0. \tag{13}$$

To provide an estimate for $\mathbb{P}(N^0_{(n+1)u_0} \ge n+2)$ note that the event $\{N^0_{(n+1)u_0} \ge n+2\}$ contains the event that $N^0_{nu_0} \ge n+1$ and subsequently n of the blocks alive at time nu_0 survive for a further u_0 units of time, whilst one of the blocks at time nu_0 succeeds in fragmenting further to produce at least two further particles u_0 units of time later. A lower bound on the probability of the latter event that makes use of the fragmentation property and the monotonicity in x of $\mathbb{P}(N^x_{nu_0} \ge n+1)$ and $P\left(\zeta^x > u_0\right)$, produces the estimate,

$$\mathbb{P}\left(N^0_{(n+1)u_0} \ge n+2\right) \ge \mathbb{P}\left(N^0_{nu_0} \ge n+1\right)\mathbb{P}\left(N^0_{u_0} \ge 2\right)\mathbb{P}\left(\zeta^0 > u_0\right)^n > 0.$$

Coupled with (12), which closes the argument by induction, the proof of the lemma is complete. □

Having established the previous lemma we are now in a position to tackle the proof of Theorem 3.

Proof of Theorem 3. By Lemma 9, fix some $k \in \mathbb{N}$ as well as $t_0 > 0$ such that $\mathbb{P}\left(N^x_{t_0} \ge k\right) > 0$ and for every $n \in \mathbb{N}$ define

$$E_n := \left\{\omega \in \Omega : N^0_{nt_0}(\omega) \ge k\right\}.$$

By means of the fragmentation property and the monotonicity in x of $\mathbb{P}\left(N_{t_0}^x \geq k\right)$

$$\mathbb{P}\left(E_n \left| \mathcal{F}_{(n-1)t_0}\right.\right) \geq \mathbb{P}(N_{t_0}^0 \geq k) > 0. \tag{14}$$

As a consequence of (14) we obtain that

$$\sum_{n\in\mathbb{N}} \mathbb{P}\left(E_n \middle| \mathcal{F}_{(n-1)t_0}\right) = \infty \tag{15}$$

$\mathbb{P}$–a.s. on $\{\zeta^x = \infty\}$ for any $x \in \mathbb{R}_0^+$.

Since E_n is $\mathcal{F}_{nt_0}$–measurable, we can apply the extended Borel–Cantelli lemma (see e.g. Corollary (3.2) in Chapter 4 of [9] or Corollary 5.29 in [8]) to deduce that

$$\{E_n \text{ happens infinitely often}\} = \left\{\sum_{n\in\mathbb{N}} \mathbb{P}\left(E_n \middle| \mathcal{F}_{(n-1)t_0}\right) = \infty\right\},$$

and thus (15) shows that on the event $\{\zeta^x = \infty\}$, $x \in \mathbb{R}_0^+$, the event E_n happens for infinitely many $n \in \mathbb{N}$. Consequently, we infer by monotonicity in x of N_t^x that

$$\mathbb{P}\left(\limsup_{t\to\infty} N_t^x \geq k \middle| \zeta^x = \infty\right) = 1,$$

which proves the assertion on account of the fact that k may be taken arbitrarily large. □

5 Multiplicative martingales

Like many different types of spatial branching processes, the probability of extinction of our killed fragmentation process turns out to be intimately related to certain product martingales which we now introduce.

More specifically, the object under consideration in the present section is the stochastic process defined as follows. For any function $f : \mathbb{R} \to [0,1]$ and $x \in \mathbb{R}_0^+$ let $Z^{x,f} := \{Z_t^{x,f} : t \geq 0\}$ be given by

$$Z_t^{x,f} = \prod_{n\in\mathcal{N}_t^x} f\left(x + ct + \log|\Pi_n(t)|\right),\ t \geq 0.$$

We are interested in understanding which functions f make the above process a martingale. In that case we refer to $Z^{x,f}$ as a *multiplicative martingale*. The following theorem shows that within the class of nonincreasing functions which are valued zero at ∞, there is a unique choice of f.

Theorem 10. *Let $c > c_{\bar{p}}$ and let $f : \mathbb{R} \to [0,1]$ be a monotone function. Then the following two statements are equivalent.*

(i) For any $x \in \mathbb{R}_0^+$ the process $Z^{x,f}$ is a martingale with respect to the filtration $\mathcal{F}$ and

$$\lim_{x\to\infty} f(x) = 0.$$

(ii) For all $x \in \mathbb{R}_0^+$ *:*

$$f(x) = \mathbb{P}(\zeta^x < \infty).$$

In order to prove Theorem 10 we shall use the following lemma.

Lemma 11. *Let* $c > c_{\bar{p}}$ *and* $x \in \mathbb{R}_0^+$*. Define*

$$R_1^x(t) = x + ct + \log \lambda_1(t).$$

Then we have

$$\limsup_{t \to \infty} R_1^x(t) = \infty$$

$\mathbb{P}(\cdot|\zeta^x = \infty)$*–almost surely.*

Proof. Let $z > x$ and set

$$\Gamma_z^x := \left\{\omega \in \Omega : \inf\{t \in \mathbb{R}_0^+ : X_n^x(t)(\omega) \notin [0, z)\} = \infty \,\forall\, n \in \mathbb{N}\right\}.$$

Theorem 12 in Section VI.3 of [2] shows that the probability that a spectrally negative Lévy process never leaves the interval $(0, z)$ when started in its interior is zero. Consequently, we have that

$$\tau_{n,x}^- < \tau_{n,z-x}^+ = \infty \quad \text{on} \quad \Gamma_z^x.$$

For each $n \in \mathbb{N}$ set

$$\sigma_n := \inf\{t \in \mathbb{R}_0^+ : N_t^x \geq n\}$$

and note that Theorem 3 implies that σ_n is a $\mathbb{P}$–a.s. finite stopping time. Let $\widetilde{\mathcal{N}}_t^x = \{n \in \mathbb{N} : X_t^n(t) > 0\}$ and note that the cardinality of $\widetilde{\mathcal{N}}_t^x$ is equal to N_t^x. By means of Lemma 8.6 of [11], we thus infer from the strong fragmentation property and equation (8.8) of Theorem 8.1 of [11] that

$$\begin{aligned}
\mathbb{P}(\Gamma_z^x|\mathcal{F}_{\sigma_n}) &\leq \prod_{k \in \widetilde{\mathcal{N}}_{\sigma_n}^x} \mathbb{P}(\Gamma_z^y)|_{y = X_k^x(\sigma_n)} \\
&\leq \prod_{k \in \widetilde{\mathcal{N}}_{\sigma_n}^x} \mathbb{P}(\tau_{k,y}^- < \tau_{k,z-y}^+)\Big|_{y = X_k^x(\sigma_n)} \\
&\leq \prod_{k \in \widetilde{\mathcal{N}}_{\sigma_n}^x} \left(1 - \frac{W(X_k^x(\sigma_k))}{W(z)}\right) \\
&\leq \left(1 - \frac{1}{cW(z)}\right)^{N_{\sigma_n}^x} \\
&\leq \left(1 - \frac{1}{cW(z)}\right)^n
\end{aligned}$$

$\mathbb{P}$–a.s. on $\{\zeta^x = \infty\}$ for any $n \in \mathbb{N}$. Therefore, since $\{R_1^x(s) < z \,\forall\, s \in \mathbb{R}_0^+\} = \Gamma_z^x$,we have

$$\mathbb{P}\left(\left\{\sup_{s \in \mathbb{R}_0^+} R_1^x(s) < z\right\} \cap \{\zeta^x = \infty\}\right) = \mathbb{P}\left(\Gamma_z^x \cap \{\zeta^x = \infty\}\right)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \mathbb{E}\left(\mathbb{P}\left(\Gamma_z^x \cap \{\zeta^x = \infty\} \middle| \mathcal{F}_{\sigma_n}\right)\right) \\
&= \mathbb{E}\left(\lim_{n\to\infty} \mathbb{P}\left(\Gamma_z^x \cap \{\zeta^x = \infty\} \middle| \mathcal{F}_{\sigma_n}\right)\right) \\
&= 0.
\end{aligned}
$$

Because $R_1^x(s) \le x + cs$ for all $s \in \mathbb{R}_0^+$, we thus deduce that

$$
\mathbb{P}\left(\left\{\sup\left\{R_1^x(s) : s \ge \frac{z-x}{c}\right\} < z\right\} \cap \{\zeta^x = \infty\}\right) = 0.
$$

Consequently, resorting to the Dominated Converge Theorem and recalling that $z > x$ was chosen arbitrarily we conclude that

$$
\begin{aligned}
\mathbb{P}\left(\limsup_{s\to\infty} R_1^x(s) = \infty \middle| \zeta^x = \infty\right) &= \lim_{z\to\infty} \mathbb{P}\left(\sup\left\{R_1^x(s) : s \ge \frac{z-x}{c}\right\} \ge z \middle| \zeta^x = \infty\right) \\
&= 1,
\end{aligned}
$$

which proves the assertion. □

Let us now tackle the proof of Theorem 10.

Proof of Theorem 10. The proof is guided by a similar result in Harris et al. [10] for branching Brownian motion. We divide the proof into two parts. The first part proves the uniqueness of monotone functions f satisfying $\lim_{y\to\infty} f(y) = 0$ for which $Z^{x,f}$ is a martingale. Part II of the proof shows that the probability of extinction constitutes a function that makes $Z^{x,f}$ a martingale.

Part I. By the martingale convergence theorem we have that $Z^{x,f}$ being a nonnegative martingale implies that $Z_\infty^{x,f} := \lim_{t\to\infty} Z_t^{x,f}$ exists $\mathbb{P}$–almost surely. Since the empty product equals 1 it is immediately clear that

$$
Z_\infty^{x,f} = 1 \tag{16}
$$

holds $\mathbb{P}$–a.s. on $\{\zeta^x < \infty\}$. Moreover, according to Lemma 11 we have that $\limsup_{t\to\infty} R_1^x(t) = \infty$ $\mathbb{P}$–a.s. on $\{\zeta^x = \infty\}$. Since $\lim_{y\to\infty} f(y) = 0$, we thus deduce that

$$
0 \le Z_\infty^{x,f} \le \liminf_{t\to\infty} f(R_1^x(t)) = 0 \tag{17}
$$

$\mathbb{P}$–a.s. on $\{\zeta^x = \infty\}$. Hence, in view of (16) and (17) we infer that

$$
Z_\infty^{x,f} = \mathbb{1}_{\{\zeta^x < \infty\}} \tag{18}
$$

holds true $\mathbb{P}$–almost surely. As a consequence of $Z^{x,f}$ being a bounded, and hence uniformly integrable, martingale we conclude from (18) that

$$
f(x) = \mathbb{E}(Z_0^{x,f}) = \mathbb{E}(Z_\infty^{x,f}) = \mathbb{P}(\zeta^x < \infty).
$$

Part II. Let $g : \mathbb{R} \to [0,1]$ be given by $g(x) = \mathbb{P}(\zeta^x < \infty)$. Since g is monotone and bounded, the limit $g(+\infty) := \lim_{x\to\infty} g(x)$ exists in $[0,1]$. Furthermore, for any $t \in \mathbb{R}_0^+$ we have $\mathcal{N}_t^x \to \mathbb{N}$

$\mathbb{P}$–a.s. as $x \to \infty$, that is $\lim_{x\to\infty} \mathbb{1}_{\mathcal{N}_t^x}(n) = 1$ $\mathbb{P}$–a.s. for every $n \in \mathbb{N}$. In addition, we have that $X_n^x(t) \to \infty$ $\mathbb{P}$–a.s. for any $n \in \mathbb{N}$ and $t \in \mathbb{R}_0^+$ as $x \to \infty$. Resorting to the fragmentation property of Π we deduce that

$$g(x) = \mathbb{E}\left(\mathbb{P}\left(\zeta^x < \infty \middle| \mathcal{F}_t\right)\right) = \mathbb{E}\left(\prod_{n\in\mathcal{N}_t^x} g\left(x + ct + \log|\Pi_n(t)|\right)\right) = \mathbb{E}\left(Z_t^{x,g}\right) \tag{19}$$

holds for all $t \in \mathbb{R}_0^+$. By means of the fragmentation property we thus have that

$$\mathbb{E}\left(Z_{t+s}^{x,g} \middle| \mathcal{F}_t\right) = \prod_{n\in\mathcal{N}_t^x} g(x + ct + \log|\Pi_n(t)|) = Z_t^{x,g}$$

$\mathbb{P}$–almost surely. Hence, $Z^{x,g}$ is a $\mathbb{P}$–martingale. Moreover, by the Dominated Converge Theorem, we deduce from (19) that

$$\begin{aligned} g(+\infty) &= \lim_{x\to\infty} \mathbb{E}\left(\prod_{n\in\mathcal{N}_t^x} g(x + ct + \log|\Pi_n(t)|)\right) \\ &= \mathbb{E}\left(\lim_{y\to\infty} \prod_{n\in\mathcal{N}_t^y} \lim_{x\to\infty} g(x)\right) \\ &= \mathbb{E}\left(\lim_{y\to\infty} \lim_{x\to\infty} g(x)^{N_t^y}\right). \end{aligned}$$

Consequently, $g(+\infty) \in \{0, 1\}$. Since g is decreasing and $g(x) \in (0, 1)$ for all $x \in \mathbb{R}_0^+$, this forces us to choose $g(+\infty) = 0$. □

6 Additive martingales

In this section we deal with an additive stochastic process $M^x(p) := (M_t^x(p))_{t\in\mathbb{R}_0^+}$, $p \in (\underline{p}, \infty)$, that for $c > c_{\bar{p}}$ and $x \in \mathbb{R}_0^+$, is given by

$$M_t^x(p) = \sum_{n\in\mathcal{N}_t^x} W_p\left(x + ct + \log|\Pi_n(t)|\right) e^{\Phi(p)t} |\Pi_n(t)|^{1+p}.$$

The main result of this section is the following theorem.

Theorem 12. *Let $c > c_{\bar{p}}$ and let $p \in (\underline{p}, \bar{p})$ be such that $c > \Phi'(p)$. Then the process $M^x(p)$ is a nonnegative $\mathcal{F}$–martingale with $\mathbb{P}$–a.s. limit $M_\infty^x(p)$. Moreover, this martingale limit satisfies*

$$\mathbb{P}\left(\{M_\infty^x(p) = 0\} \triangle \{\zeta^x < \infty\}\right) = 0,$$

where $\triangle$ denotes the symmetric difference.

The following lemma is a version of the so-called *many-to-one identity.*

Lemma 13. *We have*

$$\mathbb{E}\left(\sum_{n\in\mathbb{N}}|B_n(t)|f(\{|B_n(s)|:s\le t\})\right)=\mathbb{E}\left(f(\{|\Pi_1(s)|:s\le t\})\right)$$

for every $t\in\mathbb{R}_0^+$ *and* $f:\mathrm{RCLL}([0,t],[0,1])\to\mathbb{R}_0^+$*, where* RCLL *denotes the space of càdlàg functions.*

Proof. The proof follows directly as a consequence of the fact that $\Pi_1(t)$ has the law of a size-biased pick from $\Pi(t)$. See for example Lemma 2 of Berestycki et al. [1] □

The next lemma establishes the first assertion of Theorem 12 in that it shows that under $\mathbb{P}$ the process $M^x(p)$ is a martingale for suitable c and p.

Lemma 14. *Let* $c>c_{\bar{p}}$ *and let* $p\in(\underline{p},\bar{p})$ *be such that* $c>\Phi'(p)$*. Further, let* $x\in\mathbb{R}_0^+$*. Then the process* $M^x(p)$ *is a* $\mathbb{P}$*–martingale with respect to the filtration* $\mathcal{F}$*.*

Proof. Let us first show that for any $t\in\mathbb{R}_0^+$ the process $(W_p(X_1^x(s))\mathbb{1}_{\{s<\tau_{1,x}^-\}})_{s\in\mathbb{R}_0^+}$ is a $\mathbb{P}^{(p)}$–martingale with respect to $\mathcal{F}$. It is a straightforward exercise using (4) to show that $\psi_p'(0+)=c-\Phi'(p)>0$. By the Markov property of X_1 under $\mathbb{P}^{(p)}$ we thus infer from (5) that

$$\begin{aligned}\mathbb{E}^{(p)}\left(\mathbb{1}_{\{\tau_{1,x}^-=\infty\}}\middle|\mathcal{F}_s\right) &= \mathbb{P}^{(p)}\left(\tau_{1,y}^-=\infty\right)\Big|_{y=x+X_1(s)}\mathbb{1}_{\{s<\tau_{1,x}^-\}}\\ &= \psi_p'(0+)W_p(x+X_1(s))\mathbb{1}_{\{s<\tau_{1,x}^-\}}\end{aligned}\tag{20}$$

holds $\mathbb{P}^{(p)}$–a.s. for any $s\in\mathbb{R}_0^+$. Note that the left–hand side of (20) defines a closed $\mathbb{P}^{(p)}$–martingale. Further, observe that $x+X_1(s)=X_1^x(s)$ on the event $\{s<\tau_{1,x}^-\}$.

By means of Lemma 13 we deduce that

$$\begin{aligned}\mathbb{E}\left(M_t^x(p)\right) &= e^{\Phi(p)s}\mathbb{E}\left(\sum_{n\in\mathcal{N}_t^x}W_p\left(x+ct+\log|\Pi_n(t)|\right)e^{\Phi(p)t}|\Pi_n(t)|^{1+p}\right)\\ &= e^{\Phi(p)s}\mathbb{E}\left(\sum_{n\in\mathbb{N}}W_p\left(x+ct+\log|B_n(t)|\right)e^{\Phi(p)t}|B_n(t)|^{1+p}\mathbb{1}_{\{t<\tau_{n,x}^-\}}\right)\\ &= \mathbb{E}\left(W_p(X_1^x(t))\mathbb{1}_{\{t<\tau_{1,x}^-\}}e^{\Phi(p)t-p\xi(t)}\right)\\ &= \mathbb{E}^{(p)}\left(W_p(X_1^x(t))\mathbb{1}_{\{t<\tau_{1,x}^-\}}\right)\\ &= W_p(x)\end{aligned}\tag{21}$$

for all $t\in\mathbb{R}_0^+$, where the final equality is a consequence of the above–mentioned martingale property of $(W_p(X_1^x(s))\mathbb{1}_{\{s<\tau_{1,x}^-\}})_{s\in\mathbb{R}_0^+}$. In view of (21) we infer from the fragmentation property of Π that

$$\mathbb{E}\left(M_{t+s}^x(p)\middle|\mathcal{F}_t\right)=\sum_{n\in\mathcal{N}_t^x}e^{\Phi(p)t}|\Pi_n(t)|^{1+p}\mathbb{E}\left(M^{(n)}\middle|\mathcal{F}_t\right)$$

$$= \sum_{n \in \mathcal{N}_t^x} e^{\Phi(p)t} |\Pi_n(t)|^{1+p} W_p(x + ct + \log|\Pi_n(t)|)$$
$$= M_t^x(p)$$

$\mathbb{P}$–a.s. for all $s, t \in \mathbb{R}_0^+$, where conditional on $\mathcal{F}_t$ the $M^{(n)}$ are independent and satisfy

$$\mathbb{P}\left(M^{(n)} \in \cdot \,\middle|\, \mathcal{F}_t\right) = \mathbb{P}\left(M_s^y(p) \in \cdot\right)\big|_{y = X_n^x(t)}$$

$\mathbb{P}$–almost surely. □

Let us now turn to proof of Theorem 12. The main ingredient in the proof of Theorem 12 turns out to be Theorem 10 which deals with the product martingale $Z^{x,f}$.

Proof of Theorem 12. According to Lemma 14 we have that $M^x(p)$ is a nonnegative martingale and by the Martingale Convergence Theorem it follows that $M_\infty^x(p) := \lim_{t\to\infty} M_t^x(p)$ exists $\mathbb{P}$–almost surely. It remains to show that the symmetric difference $\{M_\infty^x(p) = 0\} \triangle \{\zeta^x < \infty\}$ is a $\mathbb{P}$–null set. Define the function $g_p : \mathbb{R}_0^+ \to [0, 1]$ given by

$$g_p(x) = \mathbb{P}(M_\infty^x(p) = 0)$$

for any $x \in \mathbb{R}_0^+$. Resorting to the fragmentation property we deduce that

$$\mathbb{P}(M_\infty^x(p) = 0 | \mathcal{F}_t) = \prod_{n \in \mathcal{N}_t^x} g_p\left(x + ct + \log|\Pi_n(t)|\right) = Z_t^{x,g_p} \tag{22}$$

holds $\mathbb{P}$-almost surely for all $t \in \mathbb{R}_0^+$. Therefore, Z^{x,g_p} is a $\mathbb{P}$–martingale. Note also that, thanks to the fact that both $\mathcal{N}_t^x$ and $W_p(x)$ are monotone increasing in x, for all $\epsilon > 0$, $M_\infty^{x+\epsilon}(p) \geq M_\infty^x(p)$ and hence $g_p(\cdot)$ is a monotone function. It follows that $g_p(+\infty)$ exists in $[0, 1]$ and moreover, by taking expectations and then limits as $x \to \infty$ in (22), we have

$$g_p(+\infty) = \mathbb{E}\left(\prod_{n \in \mathbb{N}} g_p(+\infty)\right).$$

It follows that $g_p(+\infty) = 0$ or 1. Taking account of (5) we have that $M_\infty^x(p) \leq M_\infty(p)/\psi_p'(0+)$. Hence, since $M(p)$ is an L^q-convergent martingale for some $q > 1$, it follows that $M^x(p)$ is too. (Note that the necessary computations to show that $M(p)$ is an L^q-convergent martingale can be found in the proof of Theorem 2 of [4] for the case that ν is conservative and these computations can also go through almost verbatim if not with obvious minor modifications to cover the case that ν is dissipative). Coupled with the stochastic monotonicity of $M_\infty^x(p)$ in x, it follows that necessarily $g_p(+\infty) = 0$.

We may now apply Theorem 10 and infer that $g_p(x) = \mathbb{P}(\zeta^x < \infty)$. Since $\{\zeta^x < \infty\} \subseteq \{M_\infty^x(p) = 0\}$ for each $x > 0$ this implies that

$$\mathbb{P}\left(\{\zeta^x < \infty\} \triangle \{M_\infty^x(p) = 0\}\right) = 0$$

for every $x > 0$ as required. □

7 Exponential decay rate of the largest fragment

The final section of this paper is devoted to the proof of Theorem 4. That is, in this section we deal with the asymptotic behaviour of the largest fragment in the killed fragmentation process.

Proof of Theorem 4. Our approach is based on the method of proof for Corollary 1.4 in [5] and makes use of the martingale $M^x(p)$ that we considered in the previous section.

For the time being, let $p \in (\underline{p}, \infty)$. In view of the earlier noted fact that $W_p(x) \geq c^{-1}$ for all $x \in \mathbb{R}_0^+$ we deduce that

$$\begin{aligned} c^{-1} e^{\Phi(p)t} (\lambda_1^x(t))^{1+p} &\leq c^{-1} e^{\Phi(p)t} \sum_{n \in \mathcal{N}_t^x} |\Pi_n(t)|_n^{1+p} \\ &\leq e^{\Phi(p)t} \sum_{n \in \mathcal{N}_t^x} W_p\left(x + ct + \log|\Pi_n(t)|\right) |\Pi_n(t)|^{1+p} \\ &= M_t^x(p) \end{aligned} \tag{23}$$

Recalling that $M_\infty^x(p)$ is well defined as a limit, we deduce from (23) that

$$\limsup_{t \to \infty} \frac{1}{t} \log(\lambda_1^x(t)) \leq -\frac{\Phi(p)}{1+p},$$

and thus, since $p \in (\underline{p}, \infty)$ was chosen arbitrarily,

$$\limsup_{t \to \infty} \frac{1}{t} \log(\lambda_1^x(t)) \leq -\frac{\Phi(\bar{p})}{1+\bar{p}} = -\Phi'(\bar{p}), \tag{24}$$

$\mathbb{P}$–a.s., where the final equality follows from the definition of $\bar{p}$.

In order to show the converse let $p \in (\underline{p}, \bar{p})$ as well as $\epsilon \in (0, p - \underline{p})$ and observe that

$$\begin{aligned} M_t^x(p) &= \sum_{n \in \mathcal{N}_t^x} W_p\left(x + ct + \log|\Pi_n(t)|\right) e^{\Phi(p)t} |\Pi_n(t)|^{1+p} \\ &\leq e^{(\Phi(p) - \Phi(p-\epsilon))t} [\lambda_1^x(t)]^\epsilon e^{\Phi(p-\epsilon)t} \sum_{n \in \mathcal{N}_t^x} W_{p-\epsilon}\left(x + ct + \log|\Pi_n(t)|\right) |\Pi_n(t)|^{1+p-\epsilon} \\ &= e^{(\Phi(p) - \Phi(p-\epsilon))t} [\lambda_1^x(t)]^\epsilon M_t^x(p - \epsilon). \end{aligned} \tag{25}$$

According to Theorem 12 we have that both $M_\infty^x(p-\epsilon)$ and $M_\infty^x(p)$ are $(0, \infty)$-valued $\mathbb{P}(\cdot|\zeta^x = \infty)$–almost surely. Consequently, taking the logarithm and taking the limit inferior as $t \to \infty$ we thus deduce from (25) that

$$\liminf_{t \to \infty} \frac{1}{t} \log(\lambda_1^x(t)) \geq -\frac{\Phi(p) - \Phi(p-\epsilon)}{\epsilon}$$

$\mathbb{P}(\cdot|\zeta^x = \infty)$–almost surely. Therefore, we have

$$\liminf_{t \to \infty} \frac{1}{t} \log(\lambda_1^x(t)) \geq -\lim_{\varepsilon \to 0} \frac{\Phi(p) - \Phi(p-\varepsilon)}{\varepsilon} = -\Phi'(p) \tag{26}$$

$\mathbb{P}(\cdot|\zeta^x = \infty)$–almost surely. Letting $p \to \bar{p}$ and resorting to the convexity of Φ, which ensures the continuity of Φ', (26) results in

$$\liminf_{t\to\infty} \frac{1}{t} \log(\lambda_1^x(t)) \geq -\Phi'(\bar{p}) \tag{27}$$

$\mathbb{P}(\cdot|\zeta^x = \infty)$–almost surely.

Recalling that $c_{\bar{p}} = \Phi'(\bar{p})$, (24) and (27) imply the assertion of the theorem. □

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