The n-tuple laws

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[1] Based on 'Exact and asymptotic n-tuple laws at first and last passage' by K., Pardo and Rivero, to appear in Annals of Applied Probability

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- If X is a two-sided jumping strictly stable process with index $\alpha \in (0, 2)$ and positivity constant $\rho = \mathbb{P}(X_t \geq 0) \in (0, 1)$ then: For $x \in (0, b)$, $u \in [0, b - x]$, $v \in [u, b)$ and $y > 0$,

$$\mathbb{P}_x(b - \overline{X}_{\tau_b^+ -} \in du,\ b - X_{\tau_b^+ -} \in dv,\ X_{\tau_b^+} - b \in dy,\ \tau_b^+ < \tau_0^-)$$

$$= \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))} \times \frac{x^{\alpha(1-\rho)}(b-x-u)^{\alpha\rho-1}(v-u)^{\alpha(1-\rho)-1}(b-v)^{\alpha\rho}}{(b-u)^{\alpha}(y+v)^{\alpha+1}} du\, dv\, dy.$$

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- In this talk: address the previous bullet point by examining 'n-tuple laws' for Lévy processes and positive self-similar Markov processes.

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- Π will be the Lévy measure and for $x>0$, $\overline{\Pi}^+(x)=\Pi(x,\infty)$.
- Standard theory allows us to construct a local time at zero, say L, for the strong Markov Process $\overline{X}-X$. Then defining $H_t=X_{L_t^{-1}}$ (with the formality $H_\infty:=\infty$) gives us the ascending ladder height processes (L^{-1},H). The pair (L^{-1},H) is a (killed) bivariate subordinator with potential measure denoted by

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- The same construction can be done for $-X$ giving us the descending ladder height process $(\widehat{L}^{-1},\widehat{H})$ and associated potential measure $\widehat{V}(ds,dx)$.
- The ladder processes has (amongst other things) hidden information about the distribution of $\overline{X}_t$, τ_x^+ and

$$\overline{G}_t=\sup\{s<t : X_s=\overline{X}_s\}.$$

The quintuple law at first passage

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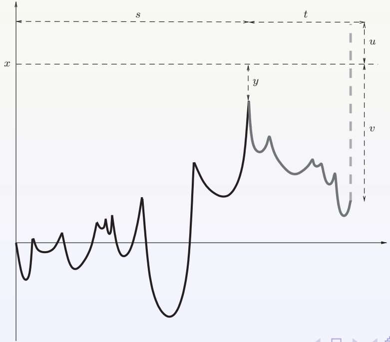
- **Theorem (Doney and K. 2006)**

  For each $x > 0$ *we have on* $u > 0$, $v \geq y$, $y \in [0, x]$, $s, t \geq 0$,

$$\mathbb{P}(\tau_x^+ - \overline{G}_{\tau_x^+ -} \in dt,\ \overline{G}_{\tau_x^+ -} \in ds,\ X_{\tau_x^+} - x \in du,\ x - X_{\tau_x^+ -} \in dv,\ x - \overline{X}_{\tau_x^+ -} \in dy)$$

$$= V(ds, x - dy)\widehat{V}(dt, dv - y)\Pi(du + v)$$

where the equality holds up to a multiplicative constant.



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$$\mathbb{P}_x^{\uparrow}(X_t \in dz) = \frac{\widehat{V}(z)}{\widehat{V}(x)} \mathbb{P}_x(X_t \in z; \tau_0^- > t)$$

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- The Tanaka-Doney pathwise construction of $(X, \mathbb{P}^{\uparrow})$ from $(X, \mathbb{P})$ replaces excursions of X from $\overline{X}$ by their time-reversed dual.
- Tanaka-Doney construction of $\mathbb{P}^{\uparrow}$ together with the quintuple law at first passage gives us a quintuple law at last passage.

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- Let

$$\underset{\rightarrow}{X}_t = \inf\{X_s : s \geq t\}$$

be the future infimum of X,

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is the right end point of the excursion of X from its future infimum straddling time t. Now define the last passage time

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- **Theorem**

Suppose that X is a Lévy process which does not drift to $-\infty$. For $s, t \geq 0$, $0 < y \leq x$, $w \geq u > 0$,

$$\mathbb{P}^{\uparrow}(\underrightarrow{D}_{U_x} - U_x \in dt,\ U_x \in ds,\ \underrightarrow{X}_{U_x} - x \in du,\ x - X_{U_x-} \in dy,\ X_{U_x} - x \in dw)$$
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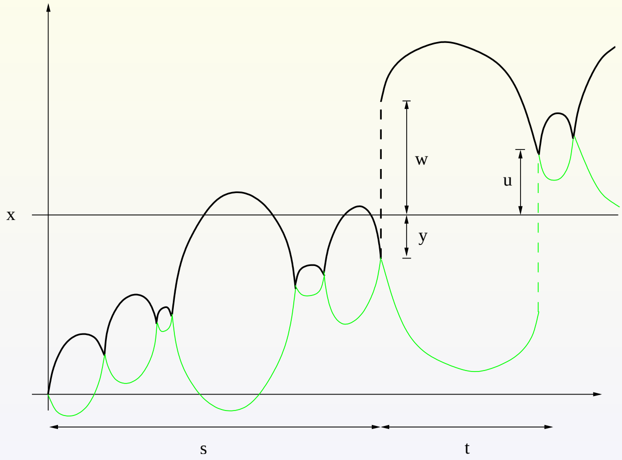

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- **Corollary**
  Suppose that X is a Lévy process which does not drift to $-\infty$. For $t > 0, x \geq z > 0, s > r > 0,\ 0 \leq v \leq z \wedge x,\ 0 < y \leq x - v,\ w \geq u > 0,$

$$\mathbb{P}_z^{\uparrow}(\underline{G}_\infty \in dr,\ \underline{X}_\infty \in dv,\ \underset{\rightarrow}{D}_{U_x} - U_x \in dt,$$
$$U_x \in ds,\ \underset{\rightarrow}{X}_{U_x} - x \in du,\ x - X_{U_x-} \in dy,\ X_{U_x} - x \in dw)$$
$$= \widehat{V}(z)^{-1}\,\widehat{V}(dr, z - dv)\,V(ds - r, x - v - dy)\,\widehat{V}(dt, w - du)\Pi(dw + y)$$

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Playing with an idea of Lamperti, Caballero and Chaumont

Suppose that X is a two-sided strictly stable process with index $\alpha \in (1,2)$ and positivity parameter $\rho = \mathbb{P}(X_t \geq 0) \in (0,1)$, then the following facts are known:

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- The process $(X, \mathbb{P}^{\uparrow}_x)$ is a positive self-similar Markov process with index α meaning for $k > 0$, the law of $(kX_{k^{-\alpha}t}, t \geq 0)$ under $\mathbb{P}^{\uparrow}_x$ is $\mathbb{P}^{\uparrow}_{kx}$ and that it respects the Lamperti representation

$$X_t = x \exp\{\xi_{\theta(tx^{-\alpha})}\}$$

where $\theta(t) = \inf\{s \geq 0 : \int_0^s \exp\{\alpha\xi_u\} du > t\}$. and ξ is a Lévy process.

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- Using the law of the global infimum of a conditioned Lévy process applied to $(X, \mathbb{P}_x^{\uparrow})$ one computes the law of the global infimum of ξ by the Lamperti-transformation and thereby obtains

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- One then uses Vigon's *equations amicales* to give us an expression for the jump measure of H: $\Pi_H(x,\infty) = \int_0^\infty \widehat{V}(du)\nu(u+x,\infty)$ from which it turns out to be easy to compute the potential

$$V(dx) = \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho)+1)} (1 - e^{-x})^{\alpha\rho-1} dx$$

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- The quintuple law at first passage for ξ marginalized to a triple law give us: For $y \in [0, x]$, $v \geq y$ and $u > 0$,

$$\mathbb{P}(\xi_{\tau_x^+} - x \in du,\, x - \xi_{\tau_x^+ -} \in dv,\, x - \overline{\xi}^{\uparrow}_{\tau_x^+ -} \in dy)$$
$$= \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))} (1 - e^{-x+y})^{\alpha\rho - 1} (1 - e^{-v+y})^{\alpha(1-\rho)-1}$$
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- Using the Lamperti representation this translates into a first passage problem for $(X, \mathbb{P}^{\uparrow}_x)$. Let $b > x > 0$. For $u \in [0, b-x]$, $v \in [u, b)$ and $y > 0$,

$$\mathbb{P}^{\uparrow}_x(b - \overline{X}_{\tau_b^+ -} \in du,\, b - X_{\tau_b^+ -} \in dv,\, X_{\tau_b^+} - b \in dy) = \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}$$
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- Recalling $\mathbb{P}^{\uparrow}_x(X_t \in dz) = (z/x)^{\alpha(1-\rho)}\mathbb{P}_x(X_t \in dz\,, t < \tau_0^-)$ we can use the last identity to deduce

$$\mathbb{P}_x(b - \overline{X}_{\tau_b^+ -} \in du,\ b - X_{\tau_b^+ -} \in dv,\ X_{\tau_b^+} - b \in dy,\ \tau_b^+ < \tau_0^-) = \frac{\sin(\pi\alpha\rho)}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}$$
$$\times \frac{x^{\alpha(1-\rho)}(b-x-u)^{\alpha\rho-1}(v-u)^{\alpha(1-\rho)-1}(b-v)^{\alpha\rho}}{(b-u)^{\alpha}(y+v)^{\alpha+1}}\,du\,dv\,dy.$$

Generating new identities playing X off against ξ

- Recalling $\mathbb{P}^{\uparrow}_x(X_t \in dz) = (z/x)^{\alpha(1-\rho)}\mathbb{P}_x(X_t \in dz\,, t < \tau_0^-)$ we can use the last identity to deduce

$$\mathbb{P}_x(b - \overline{X}_{\tau_b^+-} \in du,\ b - X_{\tau_b^+-} \in dv,\ X_{\tau_b^+} - b \in dy,\ \tau_b^+ < \tau_0^-) = \frac{\sin(\pi\alpha\rho)}{\pi}\,\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}$$
$$\times\ \frac{x^{\alpha(1-\rho)}(b-x-u)^{\alpha\rho-1}(v-u)^{\alpha(1-\rho)-1}(b-v)^{\alpha\rho}}{(b-u)^{\alpha}(y+v)^{\alpha+1}}\,du\,dv\,dy.$$

- We can bring this identity back through the Doob h-transforms relating $\mathbb{P}^{\uparrow}_x$ and $\mathbb{P}_x$ and through the Lamperti transformation to give (with obvious notation): For $\theta \in [0, b]$, $\theta \le \phi < b - u$ and $\eta > 0$

$$\mathbb{P}\Big(b - \overline{\xi}_{T_b^+-} \in d\theta,\ b - \xi_{T_b^+-} \in d\phi,\ \xi_{T_b^+} - b \in d\eta,\ T_b^+ < T_u^-\Big)$$
$$= \frac{\sin(\pi\alpha\rho)}{\pi}\,\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}\,e^{b}(1-e^{u})^{\alpha(1-\rho)}e^{-\theta-\phi}e^{(\alpha(1-\rho)+1)\eta}(e^{b-\theta}-1)^{\alpha\rho-1}(e^{-\theta}-e^{-\phi})^{\alpha}$$
$$\times\,(e^{b-\phi}-e^{u})^{\alpha\rho}(e^{b-\theta}-e^{u})^{-\alpha}(e^{\eta}-e^{-\phi})^{-\alpha-1}\,d\theta\,d\phi\,d\eta.$$