Analytical properties of scale functions for spectrally negative Lévy processes

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[1] Joint work with Terence Chan and Mladen Savov

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- Then for $q \geq 0$ we may define the q-scale function $W^{(q)} : \mathbb{R} \to [0, \infty)$ by $W^{(q)}(x) = 0$ for $x < 0$ and on $(0, \infty)$ it is the unique right continuous function such that for $\beta > \Phi(q)$

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- Note, formally we need to prove that scale functions exist - they do!

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$$\int_0^\infty e^{-qt}\mathbb{P}_x(X_t \in dy,\ t < \tau_a^+ \wedge \tau_0^-)dt$$

$$= \left\{ \frac{W^{(q)}(x)W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y) \right\} dy.$$

where

$$\tau_a^+ = \inf\{t > 0 : X_t > a\} \text{ and } \tau_0^- = \inf\{t > 0 : X_t < 0\}.$$

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- Or another example is: for $a > 0$, $x \in [0, a]$,

$$\mathbb{E}_x(e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^- < \tau_a^+\}}) = Z^{(q)}(x) - W^{(q)}(x)\frac{Z^{(q)}(a)}{W^{(q)}(a)}$$

  where

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)dy.$$

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- Suppose that Γ is the infinitessimal generator of X. Then another way of expressing these martingale properties is by writing (in a loose sense)

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- As many known fluctuation identites turn out to be linear combinations of $W^{(q)}$ and $Z^{(q)}$, this suggest that potentially a small 'theory' would be possible in which these functions play the role of a fundamental basis of solutions to the equation

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- Before even pursuing that objective, one needs to ask whether $(\Gamma - q)W^{(q)}(x)$ makes sense mathematically.

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- Enough to answer the question for $q = 0$ and $\mathbb{E}(X_1) = \psi'(0+) \geq 0$ as all other cases can be reduced to this case via the relation

$$W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x)$$

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  - Excursion theory

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  - Renewal theory

The connection with excursion theory

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- It is known that

$$W(x) = W(a)\exp\left\{-\int_x^a \nu(\overline{\epsilon} > t)\,dt\right\}$$

where $\nu(\cdot)$ is the intensity measure associated with the Poisson point process of excursions and ϵ is the canonical excursion and $\overline{\epsilon}$ is its maximum value.

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- Hence

$$W'(x+) - W'(x-) = \nu(\overline{\epsilon} = x)$$

(which immediately implies that for unbounded variation processes $W \in C^1(0,\infty)$).

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- For processes with bounded variation paths (then necessarily $X_t = \delta t - S_t$ where $\delta > 0$ and S is a driftless subordinator):

$$\nu(\overline{\epsilon} > t) = \frac{1}{\delta}\Pi(-\infty, -t) + \frac{1}{\delta}\int_{[-t,0]} \Pi(dz)\left(1 - \frac{W(t+z)}{W(t)}\right)$$

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- In the presence of a Gaussian component ($\sigma \neq 0$):

$$\nu(\epsilon \text{ passes } x \text{ continuously}) = \frac{\sigma^2}{2}\left(\frac{W'(x)^2}{W(x)} - W''(x)\right)$$

showing that $W \in C^2(0,\infty)$.

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- An important observation which helps us understand what kind of answer we might expect comes from the Wiener-Hopf factorization:

$$\psi(\beta) = \beta\phi(\beta)$$

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- This means that W can be seen as the renewal function of a (killed) subordinator, say H, which has Laplace exponent ϕ, from which it can be calculated that its jump measure is given by $\Pi(-\infty, -x)dx$, its drift is given $\sigma^2/2$ and its killing rate $\mathbb{E}(X_1) > 0$. Here, renewal function means

$$W(dx) = \int_0^\infty \mathbb{P}(H_t \in dx)dt.$$

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- Then appealing to the formula

$$1 = \mathbb{P}(H_{\tau_x^+} = x) + \mathbb{P}(H_{\tau_x^+} > x)$$

using Kesten's classical result for the probability of continuous crossing for a subordinator and Kesten-Horowitz-Bertoin formula for overshoots of subordinators one derrives that

$$1 = \frac{\sigma^2}{2} W'(x) + \int_0^x W'(x-y)\{\overline{\overline{\Pi}}(y) + \mathbb{E}(X_1)\}dy$$

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- This can be seen as a renewal equation of the form $f = 1 + f * g$ for appropriate f and g. But only when $\sigma \neq 0$, otherwise it takes the form $f = f * g$.
- When X has bounded variation paths ($X_t = \delta t - S_t$) it a direct inverse of the Laplace transform for W also gives us

$$\delta W(x) = 1 + \int_0^x W(x-y)\Pi(-\infty, -y) dy$$

which is again a renewal function of the form $f = 1 + f * g$.

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- Suppose that $\sigma \neq 0$ and

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Then for $k = 0, 1, 2, \cdots$ $W \in C^{k+3}(0,\infty)$ if and only if $\overline{\Pi} \in C^{k}(0,\infty)$.

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- Suppose that X has paths of bounded variation and $-\overline{\Pi}$ has a derivative π such that $\pi(x) \leq Cx^{-1-\alpha}$ in the neighbourhood of the origin for some $\alpha < 1$ and $C > 0$. Then for $k = 0, 1, 2, \cdots$ $W \in C^{k+3}(0, \infty)$ if and only if $\overline{\Pi} \in C^k(0, \infty)$.

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