A stochastic control problem for SNLPs

Ronnie Loeffen
University of Bath

19-4-2007

Overview

- Introduction/History
- Formulating the control problem
- Verification theorem/HJB-equation
- Value of the barrier strategy - brief review of scale functions
- Verifying optimality
- Examples

De Finetti's (1957) (discrete) model

$\{X_t : t = 0, 1, 2, \ldots\}$ is a random walk w.p. $p > 1/2$.

$\{L_t : t = 0, 1, \ldots\}$ represents the total paid out dividends at time t, where $L_0 = 0$ and $\Delta L_t := L_{t+1} - L_t$ should be chosen at time t.

Dividends can be paid out until the first time the surplus U becomes negative (ruin time), where

$$U_t = X_t - L_t, \quad t \in \{0, 1, 2, \ldots\}$$

L should be chosen such that the expected discounted dividends paid out up until ruin are maximized.

Optimal strategy is a barrier strategy, i.e. there exists a^* s.t. $L^*_{t+1} = (\sup_{0 \le s \le t} X_s - a^*) \vee 0$ or $\Delta L^*_t = (U_t - a^*) \vee 0$.

Economic survival games

$\Delta X_t := X_{t+1} - X_t \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\sum_{j=-\infty}^{j=\infty} j p_j > 0$, where $p_j = \mathbb{P}(\Delta X_t = j)$

Optimal strategy is a band strategy, i.e. there exists $2n+1$ integers $a_0 < b_1 < a_2 < b_2 \ldots a_{n-1} < b_n < a_n$, such that

$$\begin{array}{lll}
\Delta L_t^* = 0 & \text{if} & U_t \le a_0 \\
\Delta L_t^* = U_t - a_k & \text{if} & a_k < U_t \le b_k \\
\Delta L_t^* = 0 & \text{if} & b_k < U_t \le a_k \\
\Delta L_t^* = U_t - a_n & \text{if} & U_t > a_n
\end{array}$$

If $p_{-2} = p_{-3} = \ldots = 0$, then the optimal band strategy is a barrier strategy.

If $p_2 = p_3 = \ldots = 0$, then X can be used as a discrete approximation of the Cramér-Lundberg model,

$$X_s = cs - \sum_{i=1}^{N_s} C_i \quad s \in [0, \infty),$$

where $c > 0$ is the premium rate and C_i are iid positive r.v. representing the claims.

Continuous time models

Gerber (1969) 'proved' by using this approximation that a band strategy is optimal for the Cramér-Lundberg model. When the claims are exponential distributed, Gerber showed that a barrier strategy is optimal.

Radner & Shepp (1996), Jeanblanc-Shiryaev (1995), Asmussen & Taksar (1997) proved that a barrier strategy is optimal when X is a Brownian motion with drift.

Azcue & Muler (2005) reproved Gerber's results and gave an example for which the optimal strategy is not a barrier strategy.

Avram, Palmowski & Pistorius (2007) considered the case when X is a spectrally negative Lévy process.

- $X = \{X_t : t \geq 0\}$ is a spectrally negative Lévy process on $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$, with Lévy triplet $(\gamma, \sigma, \nu(\mathrm{d}x))$. X represents the risk process before dividends are deducted.

- π is a dividend/control strategy, i.e. $\pi = \{L_t^\pi : t \geq 0\}$ is a non-decreasing, caglad $\mathbb{F}$-adapted process which starts at zero. L_t^π represents the cumulative dividends paid up until time t.

- $U^\pi = \{U_t^\pi : t \geq 0\}$, where $U_t^\pi = X_t - L_t^\pi$, is the controlled risk process under the strategy π.

- $\sigma^\pi = \inf\{t > 0 : U_t^\pi < 0\}$ is the ruin time.

- $q > 0$ is the discount rate.

- v^π is the value function when using the strategy π, i.e.

$$v^\pi(x) = \mathbb{E}_x \left[\int_0^{\sigma^\pi} \mathrm{e}^{-qt} \mathrm{d} L_t^\pi \right],$$

- A strategy π is called admissible if ruin does not occur by a dividend payout. Let Π be the set of all admissible dividend policies.

- v_* is the value function of the optimal control problem, i.e.

$$v_*(x) = \sup_{\pi \in \Pi} v^\pi(x).$$

The control problem consists of finding the optimal value function v_* and an optimal policy $\pi_* \in \Pi$ such that

$$v^{\pi_*}(x) = v_*(x) \quad \text{for all } x \geq 0.$$

This is an example of a singular stochastic control problem.

Definition 1 *f belongs to the domain of the extended generator if there exists a function g with $\int_0^t |g(X_s)|\mathrm{d}s < \infty$ for each t, a.s. and*

$$f(X_t) - f(X_0) - \int_0^t g(X_s)\mathrm{d}s$$

is a local martingale.

Proposition 2 *Let Γ be the operator acting on smooth functions f, defined by*

$$\begin{aligned}\Gamma f(x) =& \gamma f'(x) + \frac{\sigma^2}{2} f''(x) + \int_{(-\infty,0)} [f(x+y) \\ & - f(x) - f'(x) y \mathbf{1}_{\{-1<y<0\}}]\nu(\mathrm{d}y).\end{aligned}$$

Then $f(X_t) - \int_0^t \Gamma f(X_s)\mathrm{d}s$ is a local martingale. Hence the domain of the extended generator contains $C^1(0,\infty)$ when X is of bounded variation and contains $C^2(0,\infty)$ when X is of unbounded variation.

Need to prove $f(X_t) - f(X_0) - \int_0^t \Gamma f(X_s) \mathrm{d}s$ is a local martingale, where

$$\Gamma f(x) = \gamma f'(x) + \frac{\sigma^2}{2} f''(x) + \int_{(-\infty,0)} [f(x+y) - f(x) - f'(x) y \mathbf{1}_{\{-1<y<0\}}] \nu(\mathrm{d}y).$$

From the Lévy-Itô decomposition it follows that we can write X as

$$X_t = \gamma t + \sigma B_t + X_t^{(2)} + X_t^{(3)},$$

where

$$X_t^{(2)} = \int_{[0,t]} \int_{y \leq -1} y N(\mathrm{d}s \times \mathrm{d}y)$$
$$X_t^{(3)} = \int_{[0,t]} \int_{-1<y<0} y N(\mathrm{d}s \times \mathrm{d}y) - t \int_{-1<y<0} y \nu(\mathrm{d}y).$$

Here N is the Poisson random measure with intensity $\mathrm{d}t \times \nu(\mathrm{d}y)$ corresponding to the jumps of X.

Verification theorem/HJB-equation

Theorem 3 (Verification theorem) *Suppose* $w : [0, \infty) \to [0, \infty)$ *is sufficiently smooth and extend* w *to the negative half-line by putting* $w(x) = 0$ *for* $x < 0$. *Suppose further that*

$$\max\{\Gamma w(x) - qw(x), 1 - w'(x)\} \leq 0. \quad \textit{(HJB-equation)}$$

Then $w(x) \geq v_*(x)$ *for all* $x \in \mathbb{R}$.

Barrier strategy

Let $\pi_a = \{L_t^a : t \geq 0\}$ denote the barrier strategy at level a, i.e. $L_0^a = 0$ and for $t > 0$

$$L_t^a := \left(\sup_{0 \leq s < t} X_s - a \right) \vee 0$$

Let U^a, σ^a and v_a be resp. the controlled risk process, ruin time and value function when using the dividend policy π_a.

Theorem 4 *Assume $W^{(q)} \in C^1(0,\infty)$. For $x \leq a$ we have*

$$v_a(x) = \frac{W^{(q)}(x)}{W^{(q)\prime}(a)}.$$

For $x > a$, $v_a(x) = x - a + \frac{W^{(q)}(a)}{W^{(q)\prime}(a)}$.

Define the optimal barrier level by

$$a^* = \sup \left\{ a \geq 0 : W^{(q)\prime}(a) \leq W^{(q)\prime}(x) \text{ for all } x \geq 0 \right\}.$$

Note that $a^* < \infty$, because $\lim_{x\to\infty} W^{(q)\prime}(x) = \infty$.

Review scale functions

For each $q \geq 0$ there exists a function $W^{(q)}: \mathbb{R} \to [0,\infty)$, such that $W^{(q)}(x) = 0$ for $x < 0$ and on $[0,\infty)$, $W^{(q)}$ is continuous, strictly increasing and

$$\int_0^\infty \mathrm{e}^{-\beta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\beta) - q} \quad \text{for } \beta > \Phi(q),$$

where $\psi(\beta) = \log \mathbb{E}\left(\mathrm{e}^{\beta X_1}\right)$ and $\Phi(q) = \sup\{\beta : \psi(\beta) = q\}$.

Further we have for $x \leq a$ and $q \geq 0$

$$\mathbb{E}_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+ < \tau_0^-\}}\right) = \frac{W^{(q)}(x)}{W^{(q)}(a)}.$$

Also we have $W^{(q)}(0) = 0$, $W^{(q)\prime}(0) > 0$ if X is of UBV and $W^{(q)}(0) = 1/c$, $W^{(q)\prime}(0) = (\nu(-\infty,0) + q)/c^2$ when X is of BV.

Theorem 5 *Suppose* $\nu(\mathrm{d}x) \ll \mathrm{d}x$ *or* $\sigma > 0$ *and*

$$W^{(q)\prime}(a) \leq W^{(q)\prime}(b) \quad \textit{for all } a^* \leq a \leq b.$$

Then the barrier strategy with barrier at a^* *is an optimal strategy and hence* $v_* = v_{a^*}$.

Proof By Chan & Kyprianou (2007) $W^{(q)}$ is sufficiently smooth when $\nu(\mathrm{d}x) \ll \mathrm{d}x$ or $\sigma > 0$. It follows that v_{a^*} is sufficiently smooth, since v_{a^*} is sufficiently smooth at a^* (smooth pasting).

Further, by definition of a^*, we have $v'_{a^*}(x) \geq 1$ for $x \leq a^*$. For $x > a^*$, we have $v'_{a^*}(x) = 1$.

Let $a > 0$. Since

$$\mathbb{E}_x \left(\mathrm{e}^{-q\tau_a^+} \mathbf{1}_{\{\tau_a^+ < \tau_0^-\}} \middle| \mathcal{F}_t \right) = \mathrm{e}^{-q(t\wedge\tau_0^-\wedge\tau_a^+)} \frac{W^{(q)}\left(X_{t\wedge\tau_0^-\wedge\tau_a^+}\right)}{W^{(q)}(a)},$$

it follows that $\mathrm{e}^{-q(t\wedge\tau_0^-\wedge\tau_a^+)} v_a(X_{t\wedge\tau_0^-\wedge\tau_a^+})$ is a $\mathbb{P}_x$-martingale. But by Itô's formula

$$\mathrm{e}^{-q(t\wedge\tau_0^-\wedge\tau_a^+)} v_{a^*}(X_{t\wedge\tau_0^-\wedge\tau_a^+}) - v(X_0) = \int_0^{t\wedge\tau_0^-\wedge\tau_a^+} \mathrm{e}^{-qs}[\Gamma v_a(X_s) - q v_a(X_s)]\mathrm{d}s + M_{t\wedge\tau_0^-\wedge\tau_a^+}.$$

This implies $\Gamma v_a(x) - q v_a(x) = 0$ for $0 \leq x \leq a$. In particular this holds for $a = a^*$.

Let $a^* \geq 0$. Suppose there exist $x > a^*$ such that $\Gamma v_{a^*}(x) - qv_{a^*}(x) > 0$. Then by the next lemma we have $(\Gamma - q)v_x(x) > 0$, which is a contradiction.

We have now shown that v_{a^*} satisfies the conditions of the verification theorem and hence $v_{a^*} \geq v_*$. Because π_{a^*} is an admissible strategy, we also have $v_{a^*} \leq v_*$. Hence $v_{a^*} = v_*$ and π_{a^*} is an optimal strategy. ■

Lemma 6 *Under the conditions of the above theorem, let $x \geq a^* \geq 0$. Then*

$$(\Gamma - q)v_{a^*}(x) \leq (\Gamma - q)v_x(x).$$

Proof Because $x \geq a^*$, we have $v_x'(x) = v_{a^*}'(x) = 1$ and therefore

$$(\Gamma - q)(v_{a^*} - v_x)(x) = -\frac{\sigma^2}{2}(v_x''(x) - v_{a^*}''(x))$$
$$\int_{-\infty}^{0}[(v_{a^*} - v_x)(x+z) - (v_{a^*} - v_x)(x)]\nu(\mathrm{d}z)$$
$$- q(v_{a^*} - v_x)(x).$$

Since $v_x''(x) \geq 0 = v_{a^*}''(x)$, $(v_{a^*}' - v_x')(y) \geq 0$ for $y \in [0, x]$ and $v_{a^*}(a^*) \geq v_x(a^*)$, the conclusion of the lemma follows. ■

Examples

Let $X_t = ct - \sum_{i=1}^{N_t} C_i$, where N is a poisson process with parameter λ and C_i are iid positive hyperexponential distributed random variables

$$P(C_1 > x) = \sum_{j=1}^{n} A_j \mathrm{e}^{-\alpha_j x}, \quad \alpha_j, A_j > 0, \sum_{j=1}^{n} A_j = 1.$$

The Laplace exponent is given by

$$\psi(u) = cu - \lambda + \lambda \sum_{j=1}^{n} A_j \frac{\alpha_j}{\alpha_j + u}.$$

It can be shown (by partial fraction expansion) that the scale function is given by

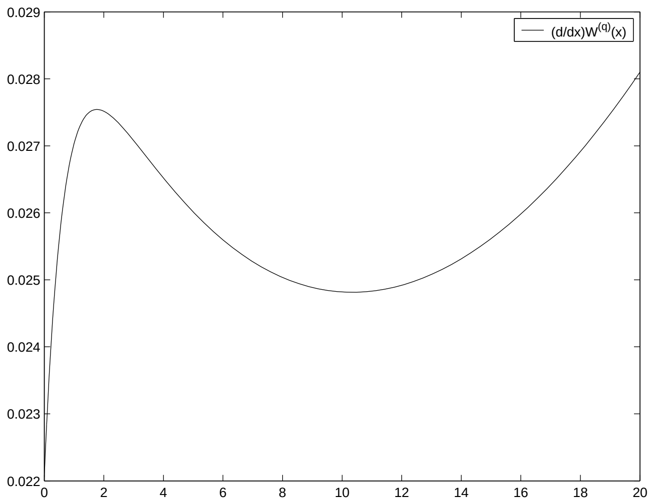
$$W^{(q)}(x) = \sum_{j=0}^{n} D_j \mathrm{e}^{\theta_j x},$$

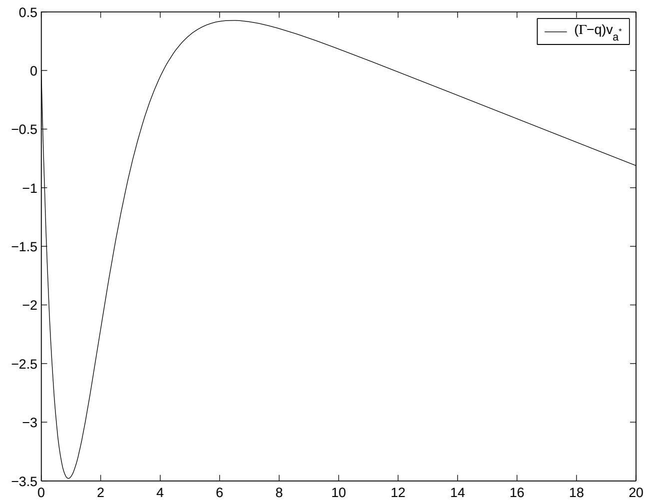
where $(\theta_j)_{j=0}^{n}$ are the roots of $\psi(u) = q$ and where $\theta_0, D_0 > 0$ and $\theta_j, D_j < 0$ for $j > 0$.

It follows that $W^{(q)'''}(x) = \sum_{j=0}^{n} D_j \theta_j^3 \mathrm{e}^{\theta_j x} > 0$ and hence a barrier strategy is optimal for the control problem.

Azcue-Muler example

Again let $X_t = ct - \sum_{i=1}^{N_t} C_i$, but now the claims have a Gamma$(2,\alpha)$-distribution. Let $c = 21.4$, $\lambda = 10, \alpha = 1$ and $q = 0.1$.





Same as the previous example, but now a Brownian motion is added in, i.e.

$$X_t = ct - \sum_{i=1}^{N_t} C_i + \sigma B_t.$$

In the first example $\sigma = 1.4$, in the second $\sigma = 2$.