# First passage problems for stable processes

First exit from an interval $(0,a)$: $X$ is a 1-D stable two-sided-jumping process.

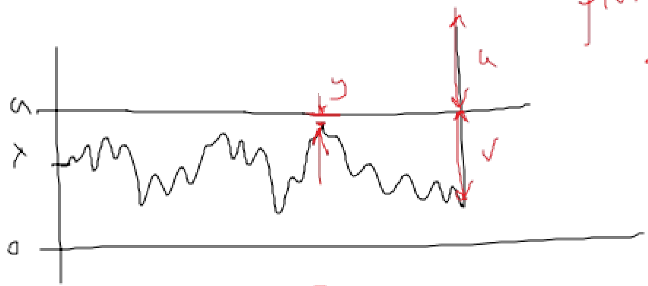
$$\left.\begin{aligned}\tau_a^+ &= \inf\{t>0 : X > a\}\\ \tau_0^- &= \inf\{t>0 : X<0\}\end{aligned}\right\}\quad \tau_a^+ \wedge \tau_0^-$$

**Theorem**: For $a, u>0$, $x\in[0,a]$, $y\in[0,a-x]$, $v\in[y,a]$

$$\mathbb{P}_x\left(X_{\tau_a^+}-a\in du,\ a-X_{\tau_a^+ -}\in dv,\ a-\overline{X}_{\tau_a^+ -}\in dy;\ \tau_a^+<\tau_0^-\right)$$

$$=\frac{\sin(\pi\alpha\rho)}{\pi}\,\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}\,\frac{x^{\alpha\hat\rho}(a-x-y)^{\alpha\rho-1}(v-y)^{\alpha\hat\rho-1}(a-v)^{\alpha\rho}}{(a-y)^{\alpha}(u+v)^{\alpha+1}}\,du\,dv\,dy$$

Note: $\mathbb{P}(\text{-----};\ \tau_a^+<\tau_0^-)$ can be seen as a similar probability at first passage of $X_t^* = X_t \mathbb{1}_{(t<\tau_0^-)}$ "$\hat{=}\ e^{\xi^*_{\varphi(t)}}$"




$v\in[0,a]$, $0\le y\le v$, $u>0$.

Proof: $\tau_b^{+,*} = \inf\{t \geq 0 : \xi^*_t > b\}$

$$X^*_t = x e^{\xi^*_{\varphi(x^{-\alpha}t)}} \mathbb{1}(t < \zeta)$$

$\mathbb{P}^*$ is law of $\xi^*$ issued from $0$.

$$\mathbb{P}_x\left( \frac{X_{\tau_a^+}}{a} - 1 > \frac{u}{a}, \ 1 - \frac{X_{\tau_a^+ -}}{a} > \frac{v}{a}, \ 1 - \frac{\overline{X}_{\tau_a^+ -}}{a} > \frac{y}{a}; \tau_a^+ < \tau_0^- \right)$$

$$= \mathbb{P}^*\left( \xi^*_{\tau^{+,*}_{\log a/x}} - \log\frac{a}{x} > \log\left(\frac{a+u}{\cancel{x}}\right), \ \log\frac{a}{x} - \xi^*_{\tau^{+,*}_{\log a/x}-} > \log\left(\frac{a-v}{a}\right), \right.$$

$$\left. \underline{\log a/x} - \overline{\xi}^*_{\tau^{+,*}_{\log a/x}} > \underline{-\log\left(\frac{a-y}{a}\right)}; \tau^{+,*}_{\log(a/x)} < \infty \right)$$

Recall for a general L.P. (with exp killing)

$$\mathbb{P}\left( \xi_{\tau_b^+} - b \in du, \ b - \xi_{\tau_b^+ -} \in dv, \ b - \overline{\xi}_{\tau_b^+ -} \in dy; \tau_b^+ < \infty \right)$$

(up to a constant!)

$$= U(b - \cancel{d}y)\hat{U}(dv - y)\Pi_\xi(du + v)$$

Recall $\Psi^*(z) = \boxed{\frac{\Gamma(\alpha - iz)}{\Gamma(\alpha\hat\rho - iz)}} \boxed{\frac{\Gamma(1+iz)}{\Gamma(1-\alpha\hat\rho + iz)}}$

$$\in LS(\beta, \gamma, \hat\gamma) = (1-\alpha\hat\rho, \alpha\rho, \alpha\hat\rho)$$

$$\frac{\Pi_{\xi^*}(dx)}{dx} = \frac{\Gamma(1+\alpha)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \frac{e^x}{(e^x - 1)^{1+\alpha}}$$

$$\frac{U^*(dx)}{dx} = \frac{1}{\Gamma(\alpha\rho)} e^{-\alpha\hat\rho}(1 - e^{-x})^{\alpha\rho - 1}$$

$$\frac{\hat{U}^*(dx)}{dx} = \frac{1}{\Gamma(\alpha\hat\rho)} e^{-(1-\alpha\hat\rho)x}(1 - e^{-x})^{\alpha\hat\rho - 1}$$

$$\mathbb{P}_x\left( X_{\tau_a^+} - a \in du, \ a - X_{\tau_a^+ -} \in dv, \ a - \overline{X}_{\tau_a^+ -} \in dy; \tau_a^+ < \tau_0^- \right)$$

$$= u^*\left(\log\left(\frac{a-u}{\cancel{x}}\right)\right) \hat{u}^*\left(\log\left(\frac{a-y}{a-v}\right)\right) \pi^*\left(\log\left(\frac{a+u}{a-v}\right)\right)$$

$$\times \frac{du\,dv\,dy}{(a-y)(a-v)(a+u)}$$

Warning (!): up to a mult. constant. (need to check integral over $u, v, y$ is one (or re-scale by the appropriate constant to make it one).

For the two sided probabilities:

$$\mathbb{P}_x\left(\tau_a^+ < \tau_0^-\right)$$

don't marginalise previous identity!

Use this identity:

$$\mathbb{P}_x\left(\tau_a^+ < \tau_0^-\right) = \mathbb{P}^*\left(\tau^{+,*}_{\log a/x} < \infty\right)$$

Can use a general identity for L.P.

$$\mathbb{P}\left(\tau_b^+ < \infty\right) = \kappa(0)\, U(b, \infty)$$

($\kappa(\cdot)$ is the exponent of ascending ladder process)

---

<u>Corollary:</u>

$$\mathbb{P}_x\left(\tau_a^+ < \tau_0^-\right) = \frac{\Gamma(\alpha)}{\Gamma(\alpha\rho)\,\Gamma(\alpha\hat{\rho})} \int_0^{x/a} t^{\alpha\hat{\rho}-1}(1-t)^{\alpha\rho-1}\, dt.$$

# First entrance into $[0,a]$ for a stable

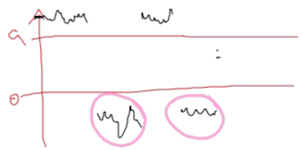
**Theorem**: $X$ is a two-sided jumping stable, let $x > a > 0$, $\alpha \in (0,1]$:

$$\mathbb{P}_x\left( X_{\tau^{[0,a]}} \in dy \,;\, \tau^{[0,a]} < \infty \right) / dy$$

$$= \frac{\sin \pi \alpha \hat{\rho}}{\pi}\, x^{\alpha\rho}\, y^{-\alpha\rho}\, (x-a)^{\alpha\hat{\rho}}\, (a-y)^{-\alpha\hat{\rho}}\, (x-y)^{-1}$$

where $\tau^{[0,a]} = \inf\{ t > 0 : X_t \in [0,a] \}$

Pf/ Trick: use censored stable processes:



$\vec{\tilde{X}}_t$ enters $[0,a]$ for the first time in exactly the same place that $X_t$ does.

$\vec{\tilde{X}}$ entering $[0,a]$ is the same position as

$x e^{\vec{\tilde{\xi}}_{\varphi(x^{-\alpha} t)}}$ entering $[0,a]$, same as

$\vec{\tilde{\xi}}$ entering $(-\infty, -\log(a/x))$ for the first time

From previous generic formula for first passage of a L.P. over threshold $b$, we need the char. exp. of $-\vec{\tilde{\xi}}$

i.e. $\vec{\tilde{\Psi}}(-z)$ ✓

and need the WHF of $\vec{\tilde{\Psi}}(-z)$ with potentials of ascending & descending ladder processes ✓

Do the math!

## Isotropic d-dimension stable proc.

1. $B_t$ be a d-dim standard BM.
$\Lambda_t$ to be a $\alpha/2$ - stable sub. $(\alpha \in (0,2))$
$(\perp\!\!\!\perp B!)$

$X_t = \sqrt{2} B_{\Lambda_t}$

$\theta \in \mathbb{R}^d$
$$\mathbb{E}[e^{i\theta \cdot X_t}] = \mathbb{E}[\mathbb{E}[e^{i\theta \cdot B_{\Lambda_t}} | \Lambda_t]]$$
$$= \mathbb{E}[e^{-\|\theta\|^2 \Lambda_t}]$$
$$= e^{-(\|\theta\|^2)^{\alpha/2} t}$$
$$= e^{-\|\theta\|^\alpha t}$$

2. Could also define it as a Lévy process in d-dim.

s.t. $\Pi(dx) = \Pi(dr, d\theta)$ , $r \in (0,\infty)$ , $\theta \in S_1^{d-1}$

$\underline{x \in \mathbb{R}^d}$ $x = (\|x\|, Arg(x))$ ; $Arg(x) = \frac{x}{\|x\|} \in S_1^{d-1}$ sphere radius 1 in d-1 dim.

$$\frac{1}{r^{1+\alpha}} dr \times d\sigma(\theta)$$

and $\sigma$ is a measure on $S_1^{d-1}$ s.t.
$$\mathbb{E}[e^{i\theta \cdot X_t}] = e^{-\|\theta\|^\alpha t}.$$

Fact: for any isotropic d-dim stable.

$X_t = (\|X_t\|, \mathrm{Arg}(X_t))$

Then $\|X_t\|$ is a pssMp.

In fact, the char. exp of the underlying L.p. $\xi^0$

$\left[\|X_t\| = \|x\| e^{\xi^0 \varphi(\|x\|^{-\alpha} t)} \quad (\text{where } X_0 = x)\right]$

is given by

$$\Psi^0(z) = C \cdot \frac{\Gamma\left(\frac{1}{2}(-iz+\alpha)\right)\Gamma\left(\frac{1}{2}(iz+d)\right)}{\Gamma\left(-\frac{1}{2}iz\right)\Gamma\left(\frac{1}{2}(iz+d-\alpha)\right)}$$

$$\boxed{\Psi(-i(\alpha-d)) = 0}$$

$$\in HG(\underbrace{\beta, \gamma, \hat{\beta}, \hat{\gamma}}_{\downarrow})$$

$$(1, \alpha/2, (d-\alpha)/2, \alpha/2)$$

$$\Psi^0(z - i(\alpha-d))$$

<u>Lemma</u>: $X$ is $d$-dim. iso-stable.
index $\alpha \in (0,2)$.

$$\|X_t\|^{\alpha-d} \mathbb{1}_{(t<T_0)} \quad . \quad \underbrace{T_0 = \inf\{t>0 : X_t = 0\}}$$

only relevant $d=1$, $\alpha>1$.

is a mg.

Moreover, the radial process, under the change of

$$\left.\frac{d\mathbb{P}^h_x}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{h(X_t)}{h(x)} \mathbb{1}_{(t<T_0)}, \qquad h(x) = \|x\|^{\alpha-d}$$

is a pssMp (again) but driven by L-p. $-\xi^\circ$.

<u>Pf</u>: Note $\|X_t\| = \|x\| e^{\xi^\circ \boxed{\varphi(\|x\|^{-\alpha} t)}}$ ← stopping time.

w.r.t. to $\xi^\circ$ the change of measure corresponds to:

exp. density $\boxed{e^{(\alpha-d)\xi^\circ_t}} + \cancel{\Psi^\circ(-i(\alpha-d))t}$

$$\mathbb{E}[e^{\lambda \xi^\circ_t}] = e^{-\Psi^\circ(-i\lambda)t}$$

Recall: if Laplace exponent exists at $\lambda \in \mathbb{R}$

$$\mathbb{E}[e^{\lambda \xi_t}] = e^{\psi(\lambda)t}, \text{ then } e^{\lambda \xi_t - \psi(\lambda)t}$$

is a mg.

can change measure $\left.\frac{d\mathbb{P}^\lambda}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\lambda \xi_t - \psi(\lambda)t}$

$\boxed{\text{Char exp. } \psi(\lambda) = -\Psi(-i\lambda)}$

$(\xi, \mathbb{P}^\lambda)$ is again a L-p. with new

char. exp. $\mathbb{E}^\lambda[e^{i\theta\xi_t}] = \mathbb{E}[e^{\lambda \xi_t - \psi(\lambda)t} e^{i\theta \xi_t}]$

$$= \exp\Big[-\underbrace{(\Psi(\theta - i\lambda) - \Psi(-i\lambda))}_{\Psi^\lambda(\theta)} t\Big]$$

Turning that argument around:

$$e^{(\alpha-d)\xi^0_t} \text{ is a Esscher transf.}$$

$\varphi(\|x\|^{-\alpha}t)$ is a stopping time

hence
$$\frac{\|x\|^{(\alpha-d)} e^{(\alpha-d)\xi^0_{\varphi(\|x\|^{-\alpha}t)}}}{\|x\|^{(\alpha-d)}} \mathbb{1}_{(t<\zeta)}$$
is a change of measure + a Mg.

$$= \frac{\|X_t\|^{\alpha-d}}{\|x\|^{\alpha-d}} \mathbb{1}(t<T_0)$$

Under $\mathbb{P}^h$, the char. exponent of $\xi^0$ changes to

$$\Psi^0(\theta - i(\alpha-d)) - \Psi^0(\cancel{-i(\alpha-d)})$$

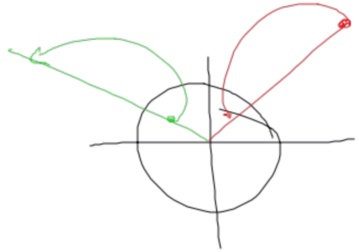
$$= \Psi^0(-z)$$

which is the char. exponent of $-\xi^0$

## Bogdan–Żak transform

$$Kx = \frac{x}{\|x\|^2} \qquad \text{(Kelvin transform)}$$

$$K(\|x\|, \mathrm{Arg}(x)) = \left(\frac{1}{\|x\|}, \mathrm{Arg}(x)\right)$$



**Theorem**: Suppose $X$ is $d$-dim iso. stable $(d \geq 2)$. Define

$$\eta(t) = \inf\left\{ s > 0 : \int_0^s \|X_u\|^{-2\alpha}\, du > t \right\}$$

$$\{KX_{\eta(t)} : t \geq 0\} \overset{\text{law}}{=} (X, \mathbb{P}^h)$$

**Exercise**: