WHF : $\boxed{\begin{aligned} &\mathbb{E}\left[e^{i\theta\xi_t}\right] = e^{-\Psi(\theta)t} \\ &\Psi(\theta) = \kappa(-i\theta)\hat{\kappa}(i\theta) \end{aligned}}$

even when ξ is killed at an indep. exp time.

killing $\Psi(0) > 0$.

$$\mathbb{E}\left[e^{-\lambda H_t}\right] = e^{-\kappa(\lambda)t} \quad : \lambda \geq 0$$

(similar for $\hat{\kappa}$)

in general

$$\kappa(\lambda) = q + \delta\lambda + \int_{(0,\infty)} \left(1 - e^{-\lambda x}\right)\nu(dx)$$

"creeping" $\iff$ $\delta > 0$.

$$\{\xi_{\tau_x^+} = x\} \qquad \tau_x^+ = \inf\{t > 0 : \xi_t > x\}$$

$$\| $$

$$\{x \in \text{Range } H\}$$

Theory of philanthropy

- Lets call two subordinators **friends** if they "belong" together in a WHF.

- Define a **philanthropist** to be a subordinator with Lévy measure which is abs. cts wrt Leb. and has non-increasing density.

Theorem: Any two philanthropists can be friends!

In fact Vigon showed that the resulting Ψ has the following characteristics:

Philanthropist 1: $\kappa(\lambda) = q + \delta\lambda + \int_{(0,\infty)} (1-e^{-\lambda x})\,\nu(dx)$

Philanthropist 2: $\hat{\kappa}(\lambda) = \hat{q} + \hat{\delta}\lambda + \int_{(0,\infty)} (1-e^{-\lambda x})\,\hat{\nu}(dx)$

- $\Psi(0) = q\hat{q}$.
- Triplet of Ψ is (a, σ, Π): then

$$\boxed{a = q\hat{\delta} - \hat{q}\delta} \quad ?$$

$$\sigma = 2\delta\hat{\delta}$$

for Π: $\nu(dx) = v(x)\,dx$, $\hat{\nu}(dx) = \hat{v}(x)\,dx$

$V(x) = \int_x^\infty v(y)\,dy$, $\hat{V}(x) = \int_x^\infty \hat{v}(y)\,dy$

$$\Pi(x,\infty) = \int_0^\infty \hat{V}(u)\,v(x+u)\,du + \hat{\delta}\,v(x) + \hat{q}V(x)$$

$$\Pi(-\infty,-x) = \int_0^\infty V(u)\,\hat{v}(x+u)\,du + \delta\,\hat{v}(x) + q\hat{V}(x)$$

For a given (with killing) $\boxed{\text{subordinator}}$ H with Laplace exponent κ introduce the potential measure (renewal measure)

$$U(dx) = \mathbb{E}\int_0^\infty \mathbb{1}(H_t \in dx)\,dt$$

$$= \int_0^\infty \mathbb{P}(H_t \in dx)\,dt$$

$$\int_{[0,\infty)} e^{-\lambda x}\,U(dx) = \int_0^\infty \mathbb{E}\left[e^{-\lambda H_t}\right] dt$$

$$= 1/\kappa(\lambda)$$

Theorem: Suppose H is a subordinator

Then for $a, u \geq 0$ and $y \in [0, a]$

$$\mathbb{P}\left(H_{T_a^+} - a \in du,\ a - H_{T_a^+ -} \in dy;\ T_a^+ < \infty\right)$$

$$= U(a - dy)\, \nu(y + du)$$

$$= u(a-y)\nu(y+u)\, dy\, du$$

if U and ν have densities

$$T_a^+ = \inf\{t > 0 : H_t > a\}$$

$$\mathbb{E}\left[f\left(H_{T_a^+} - a\right) g\left(a - H_{T_a^+ -}\right);\ T_a^+ < \infty\right]$$

$$\int_{y=0}^{a}\int_{u=0}^{\infty} f(u) g(y)\, U(a - dy)\, \nu(y + du)$$

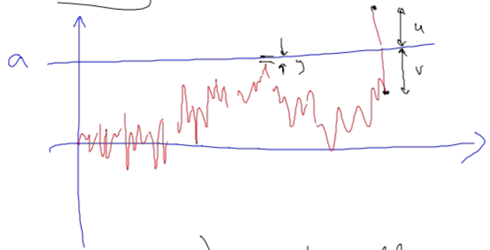
<u>Theorem</u> Suppose ξ is a L.P.
but not a compound Poisson and not with monotone paths
Then for $a>0$, $u>0$, $v\geq y$, $y\in[0,a]$
(up to a multi. constant)

$$\mathbb{P}\left(\xi_{\tau_a^+}-a\in du,\ a-\xi_{\tau_a^+-}\in dv,\ \boxed{a-\overline{\xi}_{\tau_a^+-}\in dy};\ \tau_a^+<\infty\right)$$

$$=U(a-dy)\,\hat{U}(dv-y)\,\Pi(du+v)$$

where $\tau_a^+=\inf\{t>0:\xi_t>a\}$

$U,\hat{U}$ are potentials of ascending & descending ladder height processes.



<u>Theorem (creeping)</u> (up to mult. constant.)

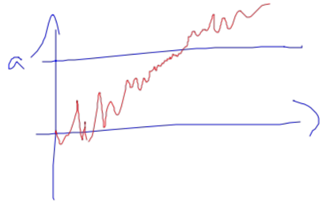
$$\mathbb{P}\left(\xi_{\tau_a^+}=a,\ a-\xi_{\tau_a^+-}\in dv,\ a-\overline{\xi}_{\tau_a^+-}\in dy;\ \tau_a^+<\infty\right)$$

$$=\boxed{\delta}\,u(a)\,\delta_0(dv)\,\delta_0(dy)$$

drift term in $\mathcal{H}$

where necessarily $U(dx)=u(x)\,dx$ if $\delta>0$.

[if $\delta=0$ then we understand RHS $\equiv 0$]



Consider a two-sided jumping stable process

$\kappa(\lambda) \propto \lambda^{\alpha\rho}$ $\quad \hat{\kappa}(\lambda) \propto \lambda^{\alpha\hat{\rho}}$

$$\boxed{\int_{[0,\infty)} e^{-\lambda x}\, U(dx) = \frac{1}{\lambda^{\alpha\rho}}}$$

$U(dx) \propto x^{\alpha\rho - 1}\, dx$ [Hint: think about gamma density functions]

Hence, for α-stable process X:

$$\mathbb{P}\left(X_{\tau_a^+} - a \in du,\ a - X_{\tau_a^+ -} \in dv,\ a - \overline{X}_{\tau_a^+ -} \in dy\right)$$

$$\boxed{c}\cdot \frac{(a-y)^{\alpha\rho - 1}(v-y)^{\alpha\hat{\rho}-1}}{(v+u)^{1+\alpha}}\, dy\cdot dv\cdot du$$

$$\to \frac{\sin \alpha\rho\pi}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}$$

$\kappa(\lambda) \xrightarrow{\lambda\to\infty} \delta$

$\lambda^{\alpha\rho}/\lambda \to 0$

[Note: reflection formula for Γ-functions:
$\Gamma(z)\Gamma(1-z) = \pi / \sin(z\pi)$]

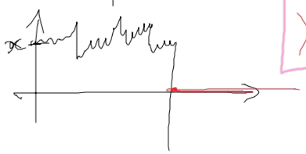
Hint: Use Beta function

$$\int_0^1 z^{a-1}(1-z)^{b-1}\, dz = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

Exercise: Show that $\left(\frac{X_{\tau_a^+} - a}{a},\ \frac{a - X_{\tau_a^+ -}}{a},\ \frac{a - \overline{X}_{\tau_a^+ -}}{a}\right)$

is invariant in dist^n (doesn't depend on a)

Back to pssMp:



$$\boxed{X^*_t = x e^{\xi^*_{\varphi(x^{-\alpha}t)}}}$$

$$\int_0^{\varphi(x^{-\alpha}t)} e^{\alpha \xi^*_s}\, ds = x^{-\alpha} t.$$

Q: What is ξ^*?

Refined Q: what is $\Psi^*(\theta) = -\log E[e^{i\theta \xi^*_1}]$?

We know $\underset{q^*}{\underset{\|}{\Psi^*(0)}} \neq 0$!

just before t, X^* is at posn X^*_{t-}

jumps arrive at rate $\Pi(dx)\,dt$

a jump arrives @ time t, sending X^* to 0 with rate

$$\Pi(-\infty, -X^*_{t-})\,dt = \left(\int_{-\infty}^{-X^*_{t-}} \frac{c_2}{|x|^{1+\alpha}}\,dx\right) dt$$

$$= \boxed{\frac{c_2}{\alpha} (X^*_{t-})^{-\alpha}\, dt}$$

On the other hand: (set $x = 1$)

$\boxed{\text{killing}}$ of X^* occurs at rate $q^*\, d\varphi(t)$

$$\int_0^{\varphi(t)} e^{\alpha \xi^*_u}\, du = t$$

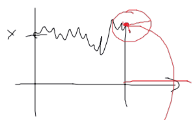
$$X^*_t \to \boxed{e^{\alpha \xi^*_{\varphi(t)}}}\, d\varphi(t) = dt$$

i.e. $d\varphi(t) = dt \big/ (X^*_t)^\alpha$

killing of X^* occurs at rate $\boxed{\frac{q^*}{(X^*_t)^\alpha}\, dt}$

Comparing the two: $q^* = \frac{c_2}{\alpha} = \frac{\Gamma(\alpha) \sin \alpha\rho\pi}{\pi}$

For the rest of Ψ^*:



$$X^*_t = x\, e^{\xi^*_{\varphi(x^{-\alpha}t)}}$$

$(x=1)$ $\quad e^{\underbrace{\xi^*_{\mathbf{e}_{q^*}-}}_{\text{indep. exp. clock of } \xi^*}}$

Note: if ξ is a L.p. (no killing) with exponent Ψ

$$\mathbb{E}_1\left[e^{i\theta\xi_{\mathbf{e}_q}}\right] = \frac{q}{q+\Psi(\theta)} = \int_0^\infty q e^{-qt}\,\mathbb{E}\left[e^{i\theta\xi_t}\right]dt$$

$\exp(q) \perp \xi$

$$\mathbb{E}_1\left[e^{i\theta\xi^*_{\mathbf{e}_{q^*}-}}\right] = \frac{q^*}{\Psi^*(\theta)}$$

$$\| $$

$$\mathbb{E}_1\left[\left(X_{\tau_0^- -}\right)^{i\theta}\right]$$

law of dual.

$$\mathbb{P}_1\left(X_{\tau_0^- -} \in dv\right) = \hat{\mathbb{P}}\left(1 - X_{\tau_1^+ -} \in dv\right)$$

$$= K\,dv\int_0^\infty\int_0^\infty \mathbb{1}_{(y\le 1\wedge v)}\frac{(1-y)^{\alpha\hat\rho}(v-y)^{\alpha\rho-1}}{(v+u)^{1+\alpha}}\,du\,dy$$

$$\boxed{K = \frac{\sin\alpha\hat\rho\pi}{\pi}\,\frac{\Gamma(\alpha n)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}}$$

$$\mathbb{E}_1\left[\left(X_{\tau_0^-}\right)^{i\theta}\right] = \frac{K}{\alpha}\,\frac{\Gamma(1-\alpha\hat\rho+i\theta)\Gamma(\alpha\hat\rho)}{\Gamma(1+i\theta)}\,\frac{\Gamma(\alpha\hat\rho-i\theta)\Gamma(\alpha\rho)}{\Gamma(\alpha-i\theta)}$$

$$= \frac{q^*}{\Psi^*(\theta)}$$

Hence

$$\boxed{\Psi^*(\theta) = \underbrace{\frac{\Gamma(\alpha-i\theta)}{\Gamma(\alpha\hat\rho-i\theta)}}_{\mathcal{R}(-i\theta)} \times \underbrace{\frac{\Gamma(1+i\theta)}{\Gamma(1-\alpha\hat\rho+i\theta)}}_{\hat{\mathcal{R}}(i\theta)}}$$

$$\Psi^*(0) = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat\rho)}\,\frac{\Gamma(1)}{\Gamma(1-\alpha\hat\rho)} = \Gamma(\alpha)\,\frac{\sin\alpha\hat\rho\pi}{\pi}$$

Hypergeometric Lévy processes

β-subordinators:

$$\kappa(\lambda) = q + \delta\lambda + \int_{(0,\infty)} (1-e^{-\lambda x})\,\nu(dx)$$

Proposition: Assume $0 \le \alpha \le \beta+\gamma$, $\gamma \in (0,1)$, $\beta \ge 0$, then there exists a subordinator, H, with Laplace exponent

$$\boxed{\kappa(\lambda) = (\lambda+\alpha)\,\frac{\Gamma(\lambda+\beta+\gamma)}{\Gamma(\lambda+\beta+1)}}$$

In particular, H has zero drift $\left[\frac{\kappa(\lambda)}{\lambda} \xrightarrow{\lambda\to\infty} \delta\right]$

the killing rate is $\kappa(0) = \alpha\,\Gamma(\beta+\gamma)/\Gamma(\beta+1)$

$$\nu(dx) = v(x)\,dx\,; \quad v(x) = \frac{1}{\Gamma(1-\gamma)}(1-e^{-x})^{-\gamma} e^{-(\beta+\gamma)x} \times \left[\frac{\gamma}{1-e^{-x}} + \beta - \alpha\right].$$

β-subordinators (α,β,γ)

Potential measure of β-subordinator

Prop: $U(dx) = u(x)\,dx$ where

$$u(x) = \frac{e^{-\alpha x}}{\Gamma(\gamma)}(1-e^{-x})^{\gamma-1} e^{-(1+\beta-\alpha)x} + \frac{e^{-\alpha x}}{\Gamma(\gamma)}(\beta+\gamma-\alpha)\int_0^x (1-e^{-u})^{\gamma-1} e^{-(1+\beta-\alpha)u}\,du$$

Note: if $\alpha=\beta\ge 0$, $\gamma\in(0,1)$ then

$$\kappa(\lambda) = (\lambda+\beta)\,\frac{\Gamma(\lambda+\beta+\gamma)}{\Gamma(\lambda+\beta+1)} = \boxed{\frac{\Gamma(\lambda+\beta+\gamma)}{\Gamma(\lambda+\beta)}}$$

Moreover:

$$u(x) = \frac{1}{\Gamma(\gamma)} e^{-\beta x}(1-e^{-x})^{\gamma-1}$$

$$v(x) = \frac{\gamma}{\Gamma(1-\gamma)}(1-e^{-x})^{-(\gamma+1)} e^{-(\beta+\gamma)x}$$