Positive self-similar Markov processes

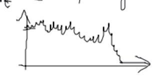
Defn: A pssMp $\{X_t : t \geq 0\}$ is a $[0,\infty)$-valued strong Markov process, with probabilities $\{P_x : x > 0\}$, such that 0 is an absorbing state, and for such $\exists\, \alpha > 0$ s.t. — index of self similarity

$$\forall c > 0 : \quad \{ c X_{c^{-\alpha} t} : t \geq 0 \} \text{ under } P_x$$

is equal in law to $\{X_t : t \geq 0\}$ under P_{cx}

Example: (I) take BM $\{B_t : t \geq 0\}$

$\underline{B}_t = \inf_{s \leq t} B_s$, define $X_t = B_t \mathbb{1}(\underline{B}_t \geq 0)$



stuff to play with:

Markov property — first show that $(B, \underline{B})$ is Markovian
then convince yourself that $\Rightarrow X_t$ —"—

— $X_t = B_t \mathbb{1}(\tau_0^- > t)$
$\tau_0^- = \inf\{s > 0 : B_s < 0\}$
this makes X a killed BM (at a stopping time)

Selfsimilarity: $\forall c > 0 \quad \{ c B_{c^{-2} t} : t \geq 0 \} \overset{d}{=} \{ B_t : t \geq 0 \}$

check that similar scaling holds for $(B, \underline{B})$ ⊛

check that this scaling passes through to X:

$(\alpha = 2)$
$$\boxed{\begin{aligned} c X_{c^{-2}t} &= c B_{c^{-2}t} \mathbb{1}(c\, \underline{B}_{c^{-2}t} \geq 0) \\ &\overset{⊛\, d}{=} B_t \mathbb{1}(\underline{B}_t \geq 0) \end{aligned}}$$

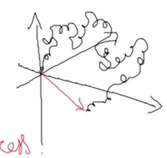
simultaneously $\forall t$

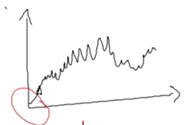
(II) Consider a n-dimensional BM, $\{\vec{B}_t : t \geq 0\}$

$X_t := \|\vec{B}_t\|$ is a pssMp.

radial distance from origin

$n = 3$ → Bessel-3 process!





apparently can start from origin.

Lévy Processes

(killed Lévy process).

In 1-D: A Lévy process $\{\xi_t : t \geq 0\}$ is a stochastic process with the following properties

(1) $\xi_0 = 0$ a.s.

(2) $\forall s < t$, $\xi_t - \xi_s \perp\!\!\!\perp \sigma(\xi_u : u \leq s)$

(3) $\forall s < t$, $\xi_t - \xi_s \stackrel{d}{=} \xi_{t-s}$

(4) ξ has paths that are a.s. right-cts with left limits.

(eg. BM & Poisson processes, compound Poisson process)

killed Lévy process: think of $\{-\infty\}$ as a cemetery state. Introduce an independent exponential clock e_q where $q \geq 0$ $(e_0 := \infty)$ [note $e_q \stackrel{d}{=} \frac{1}{q} e_1$]

ξ is a killed Lévy process if it is a Lévy process sent to $\{-\infty\}$ when the exp. clock rings. i.e. if $\exists$ a Lévy process $\tilde{\xi}$ s.t.

$$\xi_t = \begin{cases} \tilde{\xi}_t & \text{if } t < e_q \\ -\infty & \text{if } t \geq e_q \end{cases}$$

Henceforth "Lévy process" means killed L.P.

Some notation

- ξ is always a Lévy process.
- for a given α, $I_t = \int_0^t e^{\alpha \xi_u}\, du$ $\quad \left[I_\infty := \lim_{t \uparrow \infty} \int_0^t e^{\alpha \xi_u}\, du \right]$
- $\varphi(t) = \inf\{ s > 0 : I_s > t \}$.
- sometimes we'll talk about pssMp (X, P_x)

  sometimes, will indicate the initial position of X by writing $X^{(x)}$

  similarly for e.g. $\zeta = \inf\{ t > 0 : X_t = 0 \}$

  $\zeta^{(x)} = \inf\{ t > 0 : X_t^{(x)} = 0 \}$

Theorem (Lamperti's transform). Fix $\alpha > 0$

(I) If $X^{(x)}$ is a pssMp with stability index α then

$$\boxed{X_t^{(x)} \mathbb{1}_{(t < \zeta^{(x)})} = x \exp\{ \xi_{\varphi(x^{-\alpha} t)} \}}$$

where either

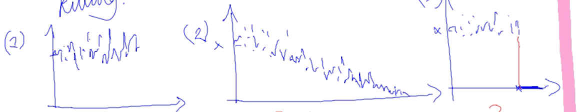
(1) $\zeta^{(x)} = \infty$ a.s. $\forall x > 0$, in which case ξ is a Lévy process (no killing!) satisfying $\limsup_t \xi_t = \infty$,

(2) $\zeta^{(x)} < \infty$ a.s. $\forall x > 0$ and $X^{(x)}_{\zeta^{(x)}-} = 0$

in which case ξ is a Lévy process satisfying $\lim_{t \to \infty} \xi_t = -\infty$ (no killing!)

(3) $\zeta^{(x)} < \infty$ a.s. $\forall x > 0$ and $X^{(x)}_{\zeta^{(x)}-} > 0$

in which case ξ is a Lévy process with killing.



e.g. Bessel-3 $\qquad$ e.g. $B_t \mathbb{1}(B_t \geq 0)$ $\qquad$?

(II) Conversely, given any (killed) Lévy process, ξ, for each $x > 0$, define

$$X_t^{(x)} := x \exp\{ \xi_{\varphi(x^{-\alpha} t)} \} \mathbb{1}_{(t < x^{\alpha} I_\infty)}$$

is a pssMp (index of self-similarity is hidden in I.)

Lamperti's theorem says: there is a bijection between L.p.s and pssMps.

<u>Note</u>: for all Lévy processes (unkilled) either

(1) $\lim_{t \to \infty} \xi_t = +\infty$

(2) $\limsup_{t \to \infty} \xi_t = -\liminf_{t \to \infty} \xi_t = \infty$

(3) $\lim_{t \to \infty} \xi_t = -\infty$

Lemma: Simultaneously for all $x>0$, either $\boxed{P_x(\zeta=\infty)=1}$

or $P_x(\zeta<\infty, X_{\zeta-}=0)=1$

or $P_x(\zeta<\infty, X_{\zeta-}>0)=1$

Pf/ First note: $\forall c>0$

$$\zeta^{(cx)}=\inf\{t>0: X_t^{(cx)}=0\}$$

$$\stackrel{d}{=} c^{\alpha}\inf\{\underbrace{c^{-\alpha}t}_{s}>0 : \cancel{c}\, X^{(x)}_{\underbrace{c^{-\alpha}t}_{s}}=0\}$$

$$=c^{\alpha}\zeta^{(x)}$$

(1) $\Rightarrow$ $x^{-\alpha}\zeta^{(x)}$ is indep. of x!

(2) $\Rightarrow$ $P_x(\zeta<\infty)=P_{cx}(\zeta<\infty)\ \forall c>0.$

define $p:=P_x(\zeta<\infty)\in[0,1]$

Thanks to M.P.:

$(\forall t\geq 0)$ $P_x(t<\zeta<\infty)=E_x\left[\mathbb{1}_{(t<\zeta)}\underbrace{\boxed{P_{X_t}(\zeta<\infty)}}_{p}\right]$

and $p=P_x(\zeta\leq t)+P_x(t<\zeta<\infty)$

$=P_x(\zeta\leq t)+p(1-P_x(\zeta\leq t))$

$\Rightarrow \cancel{p}=\cancel{p}+(1-p)P_x(\zeta\leq t)$

$\Rightarrow$ either $p=1$ or $\boxed{P_x(\zeta\leq t)=0\ \forall t}$

$\hookrightarrow P_x(\zeta<\infty)=0.$

i.e. $p=1$ or $p=0$

i.e. $P_x(\zeta<\infty)=0$ or 1 $\forall x>0.$

Now assume that $P_x(\zeta<\infty)=1\ \forall x>0.$

consider $P_x(X_{\zeta-}=0, \zeta<\infty)$

exercise $X^{(x)}_{\zeta^{(x)}-}$ is indep. of x

$\Rightarrow \eta=P_x(X_{\zeta-}=0, \zeta<\infty)$ is indep. of x

Now fix $y\in(0,x)$

$\tau_y^-=\inf\{t\geq 0: X_t<y\}.$

note $\{\tau_y^-=\zeta\}\cap\{X_{\zeta-}=0\}=\phi$

$\eta=P_x(X_{\zeta-}=0, \zeta<\infty)$

$=E_x\left[\mathbb{1}_{(\tau_y^-<\zeta)}\underbrace{\boxed{P_z(X_{\zeta-}=0, \zeta<\infty)\Big|_{z=X_{\tau_y^-}}}}_{\eta}\right]$

$\Rightarrow \eta=\eta P_x(\tau_y^-<\zeta)$

$\Rightarrow$ either $\eta=0$ or $\boxed{P_x(\tau_y^-<\zeta)=1\ \forall y\in(0,x)}$

$\hookrightarrow P_x(X_{\zeta-}=0, \zeta<\infty)=1$

i.e. $\eta=0$ or $\eta=1$

i.e. $P_x(X_{\zeta-}=0, \zeta<\infty)=0$ for $\forall x>0$

or $=1$ for $\forall x>0$

$$\zeta^{(cx)} \stackrel{d}{=} c^{\alpha} \zeta^{(x)}$$

$$\left\{ c X^{(x)}_{c^{-\alpha} t} : t \geq 0 \right\}$$

$$\stackrel{d}{=} \left\{ X^{(cx)}_{t} : t \geq 0 \right\}$$

$$c X^{(x)}_{c^{-\alpha} c^{\alpha} \zeta^{(x)} -} \stackrel{d}{=} X^{(cx)}_{\zeta^{(cx)}} .$$

$$X^{(x)}_{\zeta -}$$