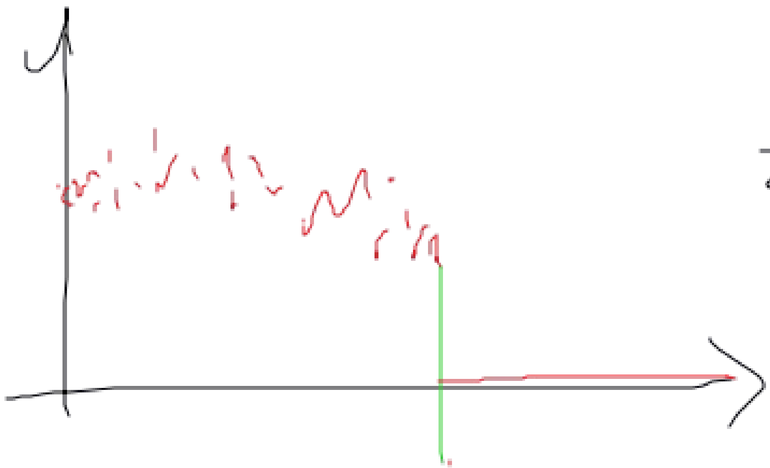
Example: Stable process killed on exiting $[0,\infty)$

$$\{X_t : t \geq 0\} \text{ — stable } \alpha \in (0,2)$$

$$Z_t = \mathbb{1}_{(\underline{X}_t \geq 0)} X_t$$



how to characterise explicitly ξ ?

$$Z_t = e^{\xi_{e(t)}}$$

Drifting & oscillating of L.P.

- $\mathbb{E}\left[e^{i\theta\xi_t}\right] = \exp(-\Psi(\theta)t)$

generic L.p.

characteristic exponent

When it exists $\Psi'(0) = \lim_{\theta\to 0} \Psi'(\theta)$

we get $\mathbb{E}[\xi_t] = t\,\Psi'(0) = t\,\mathbb{E}[\xi_1]$.

If SLLN can be stated then

$$\frac{\xi_t}{t} \to \mathbb{E}[\xi_1]$$

not surprising: either

(i) $\exists \lim_{t\to\infty} \xi_t = \infty$ a.s.

or (ii) $\exists \lim_{t\to\infty} \xi_t = -\infty$ a.s

or (iii) $\limsup_{t\to\infty} \xi_t = -\liminf_{t\to\infty} \xi_t = \infty$

<u>Theorem</u>: Suppose ξ is a Lévy process (Lp) with jump measure Π.

(i) If $\mathbb{E}[\xi_1]$ is <u>defined</u> and valued in $[-\infty, 0)$ or if $\mathbb{E}[\xi_1]$ is undefined and

$$\int_{(1,\infty)} \frac{x\,\Pi(dx)}{\int_0^x \Pi(-\infty,-y)\,dy} < \infty,$$

then $\exists \lim_{t\to\infty} \frac{\xi_t}{t} = \gamma_-$ a.s. where $\gamma_- = \mathbb{E}[\xi_1]$ if defined or $\gamma_- = -\infty$ if mean undefined

(ii) Similar statement for the case $\mathbb{E}[\xi_1]$ exists and valued in $(0,\infty]$ or undefined and

$$\int_{(-\infty,-1)} \frac{|x|\,\Pi(dx)}{\int_0^{|x|} \Pi(y,\infty)\,dy} < \infty$$

i.e. $\exists \lim_{t\to\infty} \frac{\xi_t}{t} = \gamma_+$ a.s. where $\gamma_+ = \mathbb{E}[\xi_1]$ if defined or $= \infty$ if undefined

(iii) If $\mathbb{E}[\xi_1]$ is defined and equal to 0, or if $\mathbb{E}[\xi_1]$ is undefined and both integral tests in parts (i) & (ii) fail then $\exists \lim_{t\to\infty} \frac{\xi_t}{t} = 0$ if $\exists\, \mathbb{E}[\xi_1] = 0$,

and o/w $\limsup_{t\to\infty} \frac{\xi_t}{t} = -\liminf_{t\to\infty} \frac{\xi_t}{t} = \infty$

(hence in both cases $\limsup_{t\to\infty} \xi_t = -\liminf_{t\to\infty} \xi_t = \infty$.)

Note: for α-stable process:

$$\Psi(\theta) = |\theta|^\alpha \left(e^{\pi i \alpha (\frac{1}{2}-\rho)} \mathbb{1}_{(\theta>0)} + e^{-\pi i \alpha (\frac{1}{2}-\rho)} \mathbb{1}_{(\theta<0)} \right)$$

$\theta \in \mathbb{R}$

$\alpha \in (0,2)$

$\exists\ \Psi'(0)$ when $\alpha \in (1,2)$
$\quad \overset{\shortparallel}{0}$

problem when $\alpha \in (0,1]$.

Recall: $\alpha \in (0,1)$ then $\rho \in [0,1]$

$\alpha = 1$ then $\rho = 1/2$

Fact: When $\alpha \in (0,1)$ then $\mathbb{E}[\xi_1]$ is defined iff $\rho = 1$ or 0 and is $\pm\infty$

when $\alpha = 1$ then $\mathbb{E}[\xi_1]$ is undefined

$A = \{\alpha \in (0,1),\ \rho \in [0,1]\}$

$\cup \{\alpha = 1,\ \rho = 1/2\}$

$\cup \{\alpha \in (1,2),\ \boxed{\rho \in [1-\frac{1}{\alpha}, \frac{1}{\alpha}]}\}$

$\alpha \in (0,1),\ \rho = 0$ or 1 then $\lim \xi_t = -\infty$ or $+\infty$

$\alpha \in (0,1),\ \rho \in (0,1)$ then $\limsup \xi_t = -\liminf \xi_t = +\infty$

$\alpha = 1,\ \rho = 1/2$ then ———— , ————

$\alpha \in (1,2)\ \boxed{\rho \in [1-\frac{1}{\alpha}, \frac{1}{\alpha}]}$ $\limsup_t \xi_t = -\liminf \xi_t = \infty$

$\mathbb{E}\xi_1 = 0$

Transience & Recurrence

$\forall$ L.p. either

$$\mathbb{P}\left(\int_0^\infty \mathbb{1}_{(|\xi_t| < a)} \, dt < \infty \right) = 0 \text{ or } 1, \quad \forall a > 0$$

recurrence — transience

Theorem: Suppose ξ is a L.p. with exponent Ψ. Then there is transience $(=1)$ iff for some ε suff small

$$\int_{|\theta| < \varepsilon} \mathrm{Re}\left(\frac{1}{\Psi(\theta)} \right) d\theta < \infty$$

and o/w there is recurrence $(=0)$

In the stable case:

$$\int_{(-\varepsilon, \varepsilon)} \mathrm{Re}\left(\frac{1}{\Psi(\theta)} \right) d\theta \propto \int_{(-\varepsilon, \varepsilon)} \frac{1}{|z|^\alpha} \, dz$$

$\Rightarrow$ ξ is transient whenever $\alpha \in (0,1]$ and recurrent whenever $\alpha \in (1,2)$

Hitting points: ξ is a General L.p.

ξ can "hit a point x" if

$p_x := \mathbb{P}(\xi_t = x$ for at least one $t) > 0$

define $\mathcal{C} := \{ x \in \mathbb{R} : p_x > 0 \}$

Theorem Suppose ξ is not a CPP, then $\mathcal{C} \neq \emptyset$ (ξ can hit pts) iff

$$\int_{\mathbb{R}} \mathrm{Re}\left(\frac{1}{1 + \Psi(\theta)} \right) d\theta < \infty$$

Moreover, If σ is the Gaussian coefficient in the Lévy–Itô decomposition, then:

(i) $\sigma^2 > 0$ then $\mathcal{C} = \mathbb{R}$

(ii) $\sigma = 0$ and ξ has paths of unbounded variation and ξ can hit points, then $\mathcal{C} = \mathbb{R}$

(iii) If ξ has paths of bounded variation $(\Rightarrow \sigma^2 = 0)$ then $\mathcal{C} \neq \emptyset$ iff

$$\lim_{|\theta| \to \infty} \frac{\Psi(\theta)}{\theta} \neq 0$$

(the limit always exists for b.v. processes)

In that case $\mathcal{C} = \mathbb{R}$ or $\mathcal{C} = \mathbb{R}_+$ if ξ is a subordinator or $\mathcal{C} = \mathbb{R}_-$ if $-\xi$ is a subordinator

For stable processes:

$\alpha \in (0,1)$ the process has paths of bounded variation and $\frac{\Psi(\theta)}{\theta} \xrightarrow{|\theta| \to \infty} 0$

hence $\mathcal{C} = \emptyset$

$\alpha = 1$, ξ has paths of unbounded variation, then integral test fails and $\mathcal{C} = \emptyset$.

$\alpha \in (1,2)$ ——"—— and integral test ok $\Rightarrow \mathcal{C} = \mathbb{R}$.

The Wiener-Hopf factorization

ξ is a general L.p. (not a CPP)

$$\overline{\xi}_t := \sup_{s \le t} \xi_s \quad \text{and} \quad \underline{\xi}_t := \inf_{s \le t} \xi_s$$

Lemma: The range of $\overline{\xi}$ agrees with the range of a (killed) subordinator, say H.

[Similarly range of $-\underline{\xi}$ agrees with the range of a killed subordinator, say $\hat{H}$.]

$$H_t = \begin{cases} \tilde{H}_t & t < e_q \\ +\infty & t \geq e_q \end{cases}$$

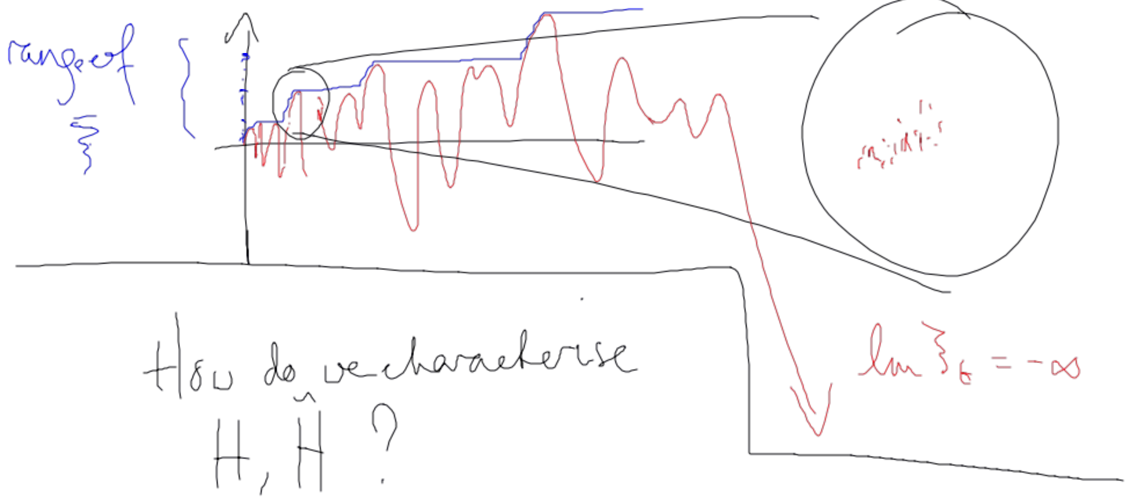
ascending ladder height process

where $\tilde{H}$ is a subordinator (no killing)

$e_q \perp\!\!\!\perp \tilde{H}$

$e_q \sim \exp(q)$

[as usual $q = 0$ means no killing i.e. $e_0 := \infty$]



How do we characterise $H, \hat{H}$?

Introduce $\kappa(\lambda) := -\log \mathbb{E}[e^{-\lambda H_t}]$, $\lambda \geq 0$

$$\hat{\kappa}(\lambda) := -\log \mathbb{E}[e^{-\lambda \hat{H}_t}], \ \lambda \geq 0.$$

<u>Remark</u> For <u>all</u> subordinators (with killing) it necessarily the case that

$$(*) \quad \kappa(\lambda) = q + \delta\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \Upsilon(dx)$$

where $\int_{(0,\infty)} (1 \wedge x) \Upsilon(dx) < \infty$

<u>Wiener-Hopf factorisation:</u>

$$\forall \theta \in \mathbb{R}, \quad \Psi(\theta) = \kappa(-i\theta)\hat{\kappa}(i\theta)$$

(up to a multiplicative constant)

<u>Note</u>: extremely rare to observe a WHF!

<u>Example</u>: Stable processes!

$$\Psi(\theta) = |\theta|^\alpha \left(e^{\pi i \alpha (\frac{1}{2} - \rho)} \mathbb{1}_{(\theta > 0)} + e^{-\pi i \alpha (\frac{1}{2} - \rho)} \mathbb{1}_{(\theta < 0)} \right)$$

<u>where there are non-monotone paths!</u>

<u>Aside</u>: for monotone paths: $\alpha \in (0,1)$, $\rho = 0$ or 1

$$-\frac{\log}{t} \mathbb{E}[e^{i\theta \xi_t}] = \Psi(\theta) = |\theta|^\alpha \left(e^{-\frac{\pi i \alpha}{2}} \mathbb{1}_{(\theta > 0)} + e^{\pi i \alpha / 2} \mathbb{1}_{(\theta < 0)} \right)$$

$$= (-i\theta)^\alpha$$

<u>Laplace</u> exponent $\eta(\lambda) := -\frac{1}{t} \log \mathbb{E}[e^{-\lambda \xi_t}]$

$$\text{"="} \ \Psi(i\lambda)$$

$$= \lambda^\alpha$$

---

look for stable factors!

When $\theta > 0$: $\Psi(\theta) = \theta^\alpha e^{i\pi\alpha(\frac{1}{2} - \rho)}$

$$= \kappa(-i\theta)\hat{\kappa}(i\theta)$$

where $\kappa(\lambda) = \lambda^{\alpha_1}$, $\hat{\kappa}(\lambda) = \lambda^{\alpha_2}$

$$\theta^\alpha e^{i\pi\alpha(\frac{1}{2} - \rho)} = (-i\theta)^{\alpha_1} (i\theta)^{\alpha_2}$$

$$= \theta^{\alpha_1 + \alpha_2} e^{-\frac{\pi i \alpha_1}{2}} e^{\frac{\pi i}{2} \alpha_2}$$

we need (radial and angular parts!)

$$\alpha = \alpha_1 + \alpha_2$$

$$\cancel{\pi}\alpha\left(\frac{1}{2} - \rho\right) = \frac{\cancel{\pi}\alpha_2}{2} - \frac{\cancel{\pi}\alpha_1}{2}$$

$$\alpha - 2\alpha\rho = \alpha_2 - \alpha_1$$

$$\boxed{\alpha_1 = \alpha\rho} \quad \boxed{\alpha_2 = \alpha(1-\rho) =: \alpha\hat{\rho}}$$

$$0 < \alpha\rho < 1 \qquad 0 < \alpha(1-\rho) < 1$$

Creeping:

A L.p. creeps (at x) upwards if $\xi_{\tau_x^+} = x$ (on $\tau_x^+ < \infty$)

$$\tau_x^+ = \inf\{t > 0 : \xi_t > x\}$$

L.p. creep upwards over x

$\iff$ x is in the range of ξ

$\iff$ x is in the range of H

$\iff$ H can hit points!

$\iff$ H has a "drift component"

i.e. $\delta > 0$ in $\circledast$