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Semi-Stable Markov Processes. I

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1. Introduction

A real valued random function $\{x_t\}$, continuous in probability and with $x_0=0$, is called *semi-stable* if there is a constant $\alpha>0$ (called the *order* of the process) such that for every $a>0$ the random functions $\{x_{at}\}$ and $\{a^\alpha x_t\}$ have the same joint distributions. If $\{x_t\}$ is Markovian with the stationary transition function $P_t(x,E)$, it is obvious that this condition holds provided that $x_0=0$ and that

$$P_{at}(x,E)=P_t(a^{-\alpha}x, a^{-\alpha}E) \tag{1.1}$$

for all $a>0$, $t>0$, $x\in R^1$, and all measurable sets E. Markov processes whose transition probabilities satisfy (1.1), and which fulfill certain regularity conditions stated in § 2 below, are the object of study in this paper.

The term "semi-stable" was introduced in [9], and although the present paper is very largely independent of that one, motivation, examples, additional facts and some alternative methods may be found there. The main point of [9] was to show the connection between semi-stable processes and the following common and important kind of stochastic limit: Suppose that $\{y_t\}$ is any real random process, and that for some norming function $f(\eta)\nearrow\infty$ we have

$$\operatorname*{wk\,lim}_{\eta\to\infty} P\left(\frac{y_{\eta t_1}}{f(\eta)}\leqq x_1,\ldots,\frac{y_{\eta t_n}}{f(\eta)}\leqq x_n\right)=P(x_{t_1}\leqq x_1,\ldots,x_{t_n}\leqq x_n), \tag{1.2}$$

where $\{x_t\}$ is a non-constant random function, continuous in probability. Then $f(\eta)=\eta^\alpha L(\eta)$ for some $\alpha>0$, where L is a slowly varying function, and $\{x_t\}$ is semi-stable of order α. Conversely, it is obvious that every semi-stable process arises as a limit of this sort. (The process $\{y_t\}$ may be chosen as $\{x_t\}$ itself.) In other words, the semi-stable processes form exactly the class of possible "asymptotes" which can be obtained by taking some fixed random process and expanding indefinitely the units in which space and time are measured. This is the basic reason, in my opinion, why such processes play a very large role in many aspects of probability theory and its applications.

The point of view of this paper is a little different from that of [9]. Of course, in the (stationary) Markov case the above theorem applies and (with some regularity assumptions) yields the condition (1.1). However, in the Markov case it is not always desirable to fix the starting state y_0. In the theory of branching processes, in particular, interesting limits satisfying (1.1) arise when y_0 is allowed to tend to

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∞ as a function of η; in these cases, the limit process $\{x_t\}$ cannot be started in state 0 without remaining there forever, and when y_0 is taken as fixed only a trivial limit can be obtained (see [12]). Therefore in the present paper the condition (1.1) itself seems a more natural starting point than the assumption that $\{x_{at}\}$ and $\{a^\alpha x_t\}$ have the same distribution, and the condition that $\{x_t\}$ can be nontrivially started in state 0 may be imposed later if desired. We accordingly will not dwell further on the relation between semi-stable processes and limit theorems, although it continues to provide the major motivation for this study.

It should be mentioned that random functions obeying what I have called the "semi-stable" condition have attracted attention in other contexts, where different names, such as "self-similar", are sometimes used. (See, for example, [7] and [16].) The term "semi-stable" was chosen in my former ignorance of some of this other work, and in order to reflect the way in which the limit theorems described above generalize the role played by stable laws and processes within the limiting theory of sums of independent random variables. Both names seem reasonable to me, and, with this much apology, I will continue to use the one I have adopted earlier. The other works just refered to are not concerned especially with Markov processes, and have very little direct connection with the content of the present paper.

The goal, then, is to study the transition functions and Markov processes satisfying (1.1); we consider here only the simplest case in which the state space of $\{x_t\}$ is $R^+=[0,\infty)$, together with the field $\mathscr{B}^+$ of its Borel subsets. The main idea is as follows[1]: Let $P_t(x,E)$ satisfy (1.1), and define a corresponding semigroup of operators as usual by setting

$$Q_t f(x)=\int_0^\infty P_t(x,dy)\,f(y) \tag{1.3}$$

for f belonging to some space of bounded Borel functions. It is easy to see that (1.1) implies

$$Q_{at}=H_{a^{-\alpha}}\,Q_t\,H_{a^\alpha}, \qquad \text{all } a>0, \tag{1.4}$$

where $H_u f(x)=f(ux)$ for any $u>0$. If A is the generator of the semigroup $\{Q_t\}$, then (1.4) yields

$$aA=H_{a^{-\alpha}}\,A\,H_{a^\alpha}. \tag{1.5}$$

Now suppose we take an initial state $x_0>0$ and introduce the additive functional

$$\varphi_\tau(\omega)=\int_0^\tau (x_t)^{-1/\alpha}\,dt, \tag{1.6}$$

which is continuous and strictly increasing for $\tau<\xi=\inf\{t: x_t=0\}$. (We are now assuming $\{x_t\}$ is strongly Markov with "nice" paths.) Let $T(t)=T(t,\omega)$ be the inverse function to φ_τ, and set

$$y_t(\omega)=x_{T(t,\omega)}(\omega) \qquad \text{for } \omega\in\{T(t)<\xi\}; \tag{1.7}$$

[1] This idea has been stated in abstract form in Notices A.M.S. **13**, 121 (1966) and in Časopis pro Pěstováni Matematiky **93**, 231 (1968).

the variable y_t is not defined otherwise, or it may be assigned to a fictitious state. Then $\{y_t\}$ is again a strong Markov process, obtained from $\{x_t\}$ by a random time change plus killing at the moment when x_t first enters the state 0. Its transition probabilities have a generator B which takes the form $B = x^{1/\alpha} \mathscr{A}$ for functions in its domain, where $\mathscr{A}$ is the Dynkin "characteristic operator" of $\{x_t\}$. But substituting in (1.5) (which also holds for $\mathscr{A}$) we see that B satisfies the corresponding equation without the factor "a" on the left side. Formally, therefore, the semi-group $\{S_t\}$ which B generates should be expected to have the property

$$S_t = H_{u^{-1}} S_t H_u, \qquad \text{all } u > 0, \tag{1.8}$$

so that the transition probabilities of $\{y_t\}$ ought to be homogeneous with respect to the action of the multiplicative group of positive reals.

Finally, the transformation of state space defined by setting

$$z_t(\omega) = \log y_t(\omega) \qquad \text{for } \omega \in \{T(t) < \xi\} \tag{1.9}$$

ought to result in a process $\{z_t\}$ on R^1 with translation independent transition probabilities, and hence with independent increments. The nature of $\{z_t\}$ will be known, therefore, and this correspondence yields, in principle, a complete description of the behavior of $\{x_t\}$ up until its first entry into state 0. Moreover, since all possible candidates for $\{z_t\}$ can be represented through the Lévy-Khintchine formula for their characteristic functions, we obtain a promising beginning on the problem of *constructing* all semi-stable Markov processes on R^+.

The plan of this paper is as follows: In § 2, we give precise definitions and establish certain elementary properties of a semi-stable Markov process, in preparation for carrying out the idea sketched above. In § 3, the functional φ_τ defined in (1.6) is studied. The main theorem – asserting that $\{z_t\}$ is indeed a process with independent increments – is proved in § 4. This theorem is applied in § 5 to give a new determination of all the semi-stable diffusion processes on $[0, \infty)$. In § 6 we present a general formula for the characteristic operator of $\{x_t\}$, while the final § 7 contains a simple application to the study of the local properties of $\{x_t\}$.

The second part of this paper will deal with the "construction problem". Here the idea will be to start with any additive process $\{z_t\}$ on R^1 and reverse the procedure outlined above, in order to arrive at one or more semi-stable processes on R^+. At the time of writing this program is incomplete in certain respects, although the overall picture is clear; in particular, the construction of all semi-stable Markov processes with non-decreasing paths can be given quite explicitly. Certain extensions, including especially that to processes on R^1 (rather than on R^+) and in higher dimensions, will also be discussed in the future.

The properties of semi-stable Markov processes have already received some attention. Papers by Stone [17], by Wendel [19] and by Taylor and Wendel [20] treat various aspects of the set of zeros of such a process; there is also a study of this subject in [9]. In another work of Stone [18], convergence of birth and death processes to semi-stable diffusions was studied. Related results can be found in [8]. (See § 5 below.) A few other fragments of information about classes of semi-stable Markov processes can be found in [9]. There are, of course, many papers showing how particular classes of semi-stable processes arise as limits in various contexts, and no attempt will be make to provide a bibliography of these. Finally, it may be remarked that the limiting theory of one-dimensional branching processes as presented in [12, §4], [13], and [14] has some overlap with the present subject (as already mentioned) and offers as well an interesting overall analogy, especially with regard to the use of random time changes, with the method of this paper.

2. General Preliminaries

Definition 2.1. A function $P_t(x, E)$ is a *semi-stable transition function* provided that (i) it is a conservative Markov transition function in the sense of [2, p. 47] with respect to the state space $(R^+, \mathscr{B}^+)$ consisting of $[0, \infty)$ and its Borel subsets; that (ii) the *semi-stable* condition

$$P_{at}(x, E) = P_t(a^{-\alpha} x, a^{-\alpha} E) \tag{2.1}$$

holds for some $\alpha > 0$, all $a > 0$, $t > 0$, $x \in R^+$, $E \in \mathscr{B}^+$; and that (iii) P_t is uniformly stochastically continuous in some neighborhood of $x = 0$. The constant α is called the *order* of the function (or associated process).

Associated with such a transition function is the semi-group of operators $\{Q_t\}$ defined by

$$Q_t f(x) = \int_0^\infty P_t(x, dy) f(y) \tag{2.2}$$

for any bounded, measurable function f. We will first derive some simple consequences of Definition 2.1. Let C_0 denote the space of continuous real functions on $[0, \infty)$ which tend to 0 at ∞.

Lemma 2.1. *Suppose that P_t is a semi-stable transition function, and that $f \in C_0$. Then $Q_t f(x)$ is continuous in t for each x, while for each t it is continuous in x when $x > 0$ and tends to 0 as $x \to \infty$.*

Proof. By assumptions (i) and (iii), there is an interval $[0, A]$, $A > 0$, with the property that for any ε, $\delta > 0$ there exists $h > 0$ such that

$$P_t(x, (x - \delta, x + \delta)) \geqq 1 - \varepsilon \quad \text{for all } x \leqq A, \ t \leqq h. \tag{2.3}$$

We note first that the existence of one such A implies that every finite interval satisfies (2.3). Indeed, from (2.1) we have

$$P_t(x, (x - \delta, x + \delta)) = P_{at}(a^\alpha x, a^\alpha (x - \delta, x + \delta)), \tag{2.4}$$

and so changing δ to δa^α and h to $h a$ allows us to replace A by $A a^\alpha$; thus A can be taken as large as desired. In particular, $P_t(x, \cdot)$ is stochastically continuous for all x.

We now assert that $Q_t f(x) \to f(x)$ uniformly in x as $t \to 0$, provided $f \in C_0$. Take any $\eta > 0$, choose A so that $|f(x)| \leqq \eta$ for $x \geqq A/2$ and choose δ so that $|f(x) - f(y)| \leqq \eta$ whenever $|x - y| \leqq \delta$. We then write

$$|Q_t f(x) - f(x)| \leqq \int_0^\infty P_t(x, dy) \, |f(y) - f(x)| \tag{2.5}$$

and estimate the right side for small t. If $x \leqq A$, we have

$$\int_0^{x-\delta} + \int_{x+\delta}^{\infty} + \int_{x-\delta}^{x+\delta} P_t(x, dy)\,|f(x)-f(y)| \leqq 2\varepsilon \max|f(\bullet)| + \eta \tag{2.6}$$

for all $t \leqq h$, where h is chosen in accordance with (2.3). To deal with the case $x > A$, note that (2.1) implies

$$P_t(x, (x/2, \infty)) = P_{t x^{-1/\alpha}}(1, (\tfrac{1}{2}, \infty)). \tag{2.7}$$

By stochastic continuity, we can find h' such that the right side is greater than $1-\varepsilon$ provided $t x^{-1/\alpha} < h'$. For $x > A$, therefore, the estimate

$$\int_0^{x/2} + \int_{x/2}^{\infty} P_t(x, dy)\,|f(x)-f(y)| \leqq 2\varepsilon \max|f(\bullet)| + 2\eta \tag{2.8}$$

certainly holds for all $t \leqq A^{1/\alpha} h'$. Combining (2.5) with (2.6) and (2.8), the assertion above is established. The continuity in t of $Q_t f(x)$ (for fixed x) is an immediate corollary.

There remains to prove only the continuity of $Q_t f(x)$ in x, for $x > 0$. Suppose $f \in C_0$, $x, y > 0$, and write by (2.1)

$$\begin{aligned} Q_t f(y) &= \int_0^{\infty} P_t(y, du)\, f(u) = \int_0^{\infty} P_{t(x/y)^{1/\alpha}}(x, x/y\, du)\, f(u) \\ &= \int_0^{\infty} P_{t(x/y)^{1/\alpha}}(x, dv)\, f(v\, y/x). \end{aligned} \tag{2.9}$$

But since $f \in C_0$ it is evident that $f(\theta v) \to f(v)$ uniformly in v as $\theta \to 1$. Using this fact and the continuity of $Q_t f(x)$ in t (in the reverse order), we have

$$\begin{aligned} Q_t f(x) &= \lim_{y \to x} \int_0^{\infty} P_{t(x/y)^{1/\alpha}}(x, dv)\, f(v) \\ &= \lim_{y \to x} \int_0^{\infty} P_{t(x/y)^{1/\alpha}}(x, dv)\, f(v\, y/x) = \lim_{y \to x} Q_t f(y). \end{aligned} \tag{2.10}$$

This completes the proof of Lemma 2.1. As will be seen later on, under certain conditions $Q_t f(x)$ may actually be discontinuous at $x = 0$.

Lemma 2.2. *A semi-stable transition function satisfies the conditions* $L(\Gamma)$ *and* $M(\Gamma)$ *of* [2, pp. 91, 92].

Proof. $M(\Gamma)$ states that P_t is uniformly stochastically continuous on compact sets Γ; this was shown at the very beginning of the proof of Lemma 2.1. Condition $L(\Gamma)$ states that

$$\lim_{x \to \infty} \sup_{t \leqq s} P_t(x, \Gamma) = 0 \tag{2.11}$$

for each compact set Γ and each s. But, by (2.1) again,

$$P_t(x, \Gamma) = P_{t x^{-1/\alpha}}(1, x^{-1}\Gamma), \tag{2.12}$$

and the stochastic continuity of P_t at the state 1 thus implies (2.11).

Lemma 2.3. *Corresponding to each semi-stable transition function there is a (simple) Markov process* $\{x_t\}$ *whose path functions are almost surely right-continuous for all t and have no discontinuities other than jumps. If* $\{x_t\}$ *is strong Markov, its paths are also quasi-continuous from the left.*

Proof. These assertions follow from the general theory, more precisely from Theorems 3.2, 3.6 and 3.13 of [2][2], in view of the properties of P_t established in Lemma 2.2 above.

Definition 2.2. A *semi-stable Markov process* (ssmp) is a strong Markov process with right-continuous paths having no discontinuities except jumps, whose transition probabilities are given by a semi-stable transition function.

Remarks. As we have seen in Lemma 2.3, the possibility of constructing a (simple) Markov process with "nice" paths does not restrict the generality of the transition functions of Definition 2.1. This is not true, however, of the strict Markov property. For example, adapting an example of Dynkin, suppose that $P_t(x, E)$ is defined as for the reflecting barrier Brownian notion on $[0, \infty)$ when $x>0$, but $P_t(0, \{0\})=1$. It is easy to see that this defines a semi-stable transition function, but it is one which does not correspond to a strong Markov process. The example is typical, in the sense that difficulty can arise only through bad behavior at $x=0$. We will explore this point in more detail below.

From now on, $\{x_t\}$ will always denote a Markov process with a semi-stable transition function and with the nice paths guaranteed by Lemma 2.3. We denote the space of right-continuous functions with left limits everywhere by Ω, the Borel field generated by the cylinder sets by $\mathscr{F}$, and the probability measures of the process $\{x_t\}$, with $x_0=x$, by P_x. For the process $\{x_t\}$ we can now define the first passage time

$$\xi=\inf\{t>0: x_t=0 \text{ or } x_{t-}=0\}; \tag{2.13}$$

it is quite possible, of course, that $\xi=\infty$. If $\{x_t\}$ is strong Markov and so quasi-left continuous, then $x_{t-}=0$ implies $x_t=0$, but this might not hold in general. In any event, as we have defined ξ the events $\{\xi>t\}$ belong to the Borel field $\mathscr{F}$, and so the law of ξ is determined by the transition function P_t. We next prove a fact to be used several times below:

Lemma 2.4. *Suppose a set* $F\in\mathscr{F}$ *has the property that* $x_t\in F$ *implies* $a^{-\alpha}x_{at}\in F$ *for all* $a>0$. *Then* $P_x(F)$ *is independent of* x, $x>0$.

Proof. From the semi-stable condition (2.1), it is very easy to see that

$$P_x(F)=P_{a^\alpha x}(\{x_{(\cdot)}: a^{-\alpha}x_{at}\in F\}) \tag{2.14}$$

for any cylinder set F. Then by the uniqueness of the extension of measures to $\mathscr{F}$, (2.14) must also hold for all sets $F\in\mathscr{F}$. But if F satisfies the condition of the lemma, the right side of (2.14) equals $P_{a^\alpha x}(F)$. Since $a>0$ is arbitrary, we have the desired conclusion.

Lemma 2.5. *Either* $P_x(\xi<\infty)=1$ *for all* $x>0$, *or else* $P_x(\xi<\infty)=0$ *for all* $x>0$.

[2] In fact, these theorems are proved in an earlier work of Dynkin's, but [2] is the more convenient reference for their statements.

Proof. The set $F=\{x_t: \exists\, \tau\in(0,\infty)$ such that $x_\tau=0$ or $x_{\tau-}=0\}$ belongs to $\mathscr{F}$ and satisfies the condition of Lemma 2.4. Accordingly, then, $P_x(F)=p$ independent of x, $x>0$. For any $t>0$

$$P_x(t<\xi<\infty)=E_x\big(\chi_{\{t<\xi\}}P_x(x_\tau=0 \text{ or } x_{\tau-}=0 \text{ for some } \tau\in(t,\infty)\,|\,\mathscr{F}_t)\big), \quad (2.15)$$

where χ_A is the indicator function of A and $\mathscr{F}_t$ is the subfield of $\mathscr{F}$ generated by $\{x_s; s\leqq t\}$. However, by the (simple) Markov property, with probability one (P_x) we have (when $x_t>0$)

$$P_x(x_\tau=0 \text{ or } x_{\tau-}=0 \text{ for some } \tau\in(t,\infty)\,|\,\mathscr{F}_t)=P_{x_t}(\xi<\infty)=p, \quad (2.16)$$

and so for all t

$$P_x(t<\xi<\infty)=p\,P_x(t<\xi). \quad (2.17)$$

But then

$$p=P_x(\xi\leqq t)+P_x(t<\xi<\infty)=P_x(\xi\leqq t)+p\,P_x(t<\xi) \quad (2.18)$$

so that $(1-p)\,P_x(\xi\leqq t)=0$. Thus, unless $p=1$, we have $P_x(\xi\leqq t)=0$ for all $x>0$ and all t so that there is a.s. no zero; this proves the lemma.

Theorem 2.1. *Let P_t be a semi-stable transition function, and $\{x_t\}$ the right continuous Markov process corresponding to P_t. Suppose that $P_x(\xi<\infty)=0$ for all $x>0$. Then $\{x_t\}$ is a Feller process on $(0,\infty)$ and strongly Markov. If instead $P_x(\xi<\infty)=1$ for all $x>0$, then $\{x_t\}$ is a strong Markov process if and only if for each $t>0$, $f\in C_0$,*

$$\lim_{x\to 0} Q_t f(x)=Q_t f(0). \quad (2.19)$$

Proof. In case $P_x(\xi<\infty)=0$, the paths of $\{x_t\}$ remain right-continuous even when the range space is the open interval $(0,\infty)$. It was proved in Lemma 2.1 that $Q_t f(x)$ is continuous in $x>0$ for $f\in C_0$; this is sufficient to show the weak convergence of the measures $P_t(y,\cdot)$ to the limit $P_t(x,\cdot)$ as $y\to x>0$. But since $P_t(x,\cdot)$ has no mass at 0, it is also clear that $Q_t f$ is still continuous even if f is merely bounded and continuous on the open interval $(0,\infty)$. Thus, in this case, $\{x_t\}$ is a right-continuous Feller process on $(0,\infty)$ and so, by Theorem 3.10 of [2], it is also strongly Markov. Moreover, it is easily seen that either $P_t(0,\{0\})=1$ for all t, or else $\{x_t\}$, started at 0, immediately enters the open interval and remains. In either case, we may add 0 to the state space without disturbing the strong Markov property. (But the Feller property may then fail at $x=0$.)

Now suppose $P_x(\xi<\infty)=1$ for all $x>0$, and that (2.19) holds. The path functions are of course again right-continuous, on $[0,\infty)$ this time, while (2.19) plus the last part of Lemma 2.1 provide a Feller property on $[0,\infty)$. It follows by the theorem quoted just above that $\{x_t\}$ must be strongly Markov in this case also.

It remains to show that (2.19) is necessary for the strong Markov property (but only when $P_x(\xi<\infty)=1$). To do this we first establish the fact that

$$\lim_{x\to 0} P_x(\xi\geqq t)=0 \qquad \text{for each } t>0. \quad (2.20)$$

To see this, note that (2.14) can be applied with $F=\{\xi \geqq t\}$; taking $a=x^{-1/\alpha}$ the result is

$$P_x(\xi \geqq t)=P_1(\xi \geqq t\, x^{-1/\alpha}). \tag{2.21}$$

But since ξ is a.s. finite (P_1), (2.21) obviously implies (2.20).

Now fix $t>0$, and write

$$P_x(x_t \leqq y)=P_x(\xi \geqq t, x_t \leqq y)+P_x(\xi<t, x_t \leqq y). \tag{2.22}$$

The first term tends to 0 as $x \to 0$ by (2.20). The second can be transformed with the aid of the strong Markov property as follows:

$$\begin{aligned} P_x(\xi<t, x_t \leqq y) &= E_x\big(\chi_{\xi<t} P_x(x_t \leqq y \mid \mathscr{F}_\xi)\big) \\ &= E_x\big(\chi_{\xi<t} P_{t-\xi}(0,[0,y]\big). \end{aligned} \tag{2.23}$$

But by Lemma 2.1, $p_{t-h}(0,[0,y])$ has limit $p_t(0,[0,y])$ as $h \to 0$, for all continuity points y of $p_t(0,[0,y])$. Since $\xi \to 0$ in law as $x \to 0$, the integrand in the last expression in (2.23) converges in probability to $p_t(0,[0,y])$ and we can conclude by the bounded convergence theorem that

$$\lim_{x \to 0} P_x(x_t \leqq y)=P_t(0,[0,y]) \tag{2.24}$$

for all continuity points y of the right-hand side. This is the same as (2.19) and completes the proof of the theorem.

Remark. The theorems shows that a semi-stable (strong) Markov process $\{x_t\}$ always has the Feller property, although it may (when $P_x(\xi<\infty)=0$) be necessary to exclude $x=0$ from the state space. It is always possible, however, to include the point $x=+\infty$ by defining $P_t(\infty,\{\infty\})=1$ for all t. That the Feller property is still valid with this extension is an immediate consequence of Lemma 2.1.

3. The Functional φ_t

In this section $\{x_t\}$ will always denote a semi-stable Markov process (ssmp) of order α in the sense of Definition 2.2. In order to carry out the program sketched in the introduction, it is necessary to establish certain properties of the functional which determines the random time substitution. We define

$$\varphi_\tau(\omega)=\int_0^\tau x_s(\omega)^{-1/\alpha}\, ds, \tag{3.1}$$

which is obviously a continuous and strictly increasing function of τ as long as $\tau<\xi$. (We now, as remarked earlier, can use the simpler definition $\xi=\min\{t\colon x_t=0\}$ in view of the quasi-left continuity of $\{x_t\}$.)

Lemma 3.1. *In the case $P_x(\xi<\infty)=0$ for all $x>0$, we have*

$$P_x\big(\lim_{\tau \to \infty} \varphi_\tau(\omega)=+\infty\big)=1, \qquad x>0. \tag{3.2}$$

Proof. It is very easy to see that the event $F_z=\{\omega: \lim_{\tau\to\infty}\varphi_\tau(\omega)\leqq z\}$ belongs to $\mathscr{F}$ and satisfies the homogeneity condition: $x_t\in F_z$ implies $a^{-\alpha}x_{at}\in F_z$ for $a>0$. By Lemma 2.4, therefore, $P_x(F_z)$ is independent of x for $x>0$. But write

$$\varphi_\infty(\omega)=\int_0^1+\int_1^\infty x_s(\omega)^{-1/\alpha}\,ds=X+Y, \tag{3.3}$$

and note that for $x>0$

$$\begin{aligned} P_x(Y\leqq z)&=E_x\left(P_x\left[\int_1^\infty x_s(\omega)^{-1/\alpha}\,ds\leqq z\,|\,\mathscr{F}_1\right]\right)\\ &=E_x\left(P_{x_1}\left[\int_0^\infty x_s(\omega)^{-1/\alpha}\,ds\leqq z\right]\right)=P_x(F_z)\end{aligned} \tag{3.4}$$

because of the (simple) Markov property and the fact above. In other words, the variable Y in (3.3) has the same distribution as φ_∞. But since X is positive a.s., this is impossible unless $Y=\varphi_\infty=\infty$ a.s., which proves the lemma.

To discuss the behavior of φ_τ when $\{x_t\}$ can reach the state $x=0$, another "$0-1$" distinction is relevant: either the paths are a.s. continuous at $t=\xi$, or else they a.s. have a jump there.

Lemma 3.2. *Suppose that $P_x(\xi<\infty)=1$ for all $x>0$. Then either $P_x(x_{\xi-}=0)=1$ for all $x>0$ or else $P_x(x_{\xi-}>0)=1$ for all $x>0$.*

Proof. Let $F=\{x: \lim_{t\to\xi-}x_t=0\}$. It is easy to see that F again satisfies the hypothesis of Lemma 2.4, and so $P_x(F)=p$ is independent of x, $x>0$. We must show that $p=0$ or $p=1$. Let

$$\xi_\varepsilon=\min\{t>0: x_t\leqq\varepsilon\} \tag{3.5}$$

for any $\varepsilon>0$; ξ_ε is of course a Markov time for $\{x_t\}$. But for $x>0$ we have using the strong Markov property that

$$\begin{aligned} p=P_x(F)&=P_x(F\cap\{\xi_\varepsilon<\xi\})=E_x\big(P_x(F\cap\{\xi_\varepsilon<\xi\}\,|\,\mathscr{F}_{\xi_\varepsilon})\big)\\ &=E_x\big(\chi_{\xi_\varepsilon<\xi}P_{x_{\xi_\varepsilon}}(F)\big)=p\,E_x(\chi_{\xi_\varepsilon<\xi}).\end{aligned} \tag{3.6}$$

If $p\neq 0$, therefore, $P_x(\xi_\varepsilon<\xi)=1$ for all $x>0$ and all $\varepsilon>0$. It follows that a.s. $\{x_t\}$ does not jump to 0, so that $p=1$ and the proof is complete.

In the case of continuous approach to 0, the functional φ_τ again increases continuously to ∞:

Lemma 3.3. *Suppose that $P_x(\xi<\infty)=1$ and $P_x(x_{\xi-}=0)=1$ for all $x>0$. Then*

$$P_x\big(\lim_{\tau\to\xi-}\varphi_\tau(\omega)=+\infty\big)=1,\qquad x>0. \tag{3.7}$$

Proof. We define the event $F_z=\{\omega: \varphi_{\xi-}\leqq z\}$; it follows just as in the proof of Lemma 3.1 that $P_x(F_z)$ is independent of $x>0$ for each z. (This is so regardless of whether $x_{\xi-}=0$ a.s. or not.) Take any $y\in(0,x)$ and consider

$$\varphi_{\xi-}(\omega)=\int_0^{\xi_y}+\int_{\xi_y}^{\xi-}x_s(\omega)^{-1/\alpha}\,ds=X+Y, \tag{3.8}$$

where ξ_y is defined by (3.5). We now proceed just as in (3.4), except that $\mathscr{F}_{\xi_y}$ is used in place of $\mathscr{F}_1$ and hence the strong Markov property is required; the hypothesis that $P_x(x_{\xi-}=0)=1$ also comes in because we need to know that $x_{\xi_y}>0$ a.s. (P_x). The conclusion again is that Y has the same probability law as $\varphi_{\xi-}$ itself. But obviously $X>0$ a.s., and the conclusion that $\varphi_{\xi-}=\infty$ a.s. follows just as before.

There remains only the case when $\{x_t\}$ a.s. attains the state $x=0$ by jumping there. In this situation, the functional φ_τ always jumps from a finite value to ∞:

Lemma 3.4. *Suppose $P_x(\xi<\infty)=1$ and $P_x(x_{\xi-}>0)=1$ for all $x>0$. Then*

$$P_x\big(\varphi_{\xi-}(\omega)<\infty, \varphi_{\xi+}(\omega)=\infty\big)=1, \qquad x>0. \tag{3.9}$$

Moreover, the probability distribution of $\varphi_{\xi-}$ is independent of $x>0$.

Proof. It is obvious that $\varphi_{\xi-}<\infty$ a.s., since by quasi-left continuity the paths of $\{x_t\}$ (a.s.) do not approach 0 prior to $t=\xi$, at which time they jump there. Also, the non-dependence of the law of $\varphi_{\xi-}$ on the initial state x has already been noted. It the only remains to show that $\varphi_{\xi+\varepsilon}=+\infty$ a.s., for every $\varepsilon>0$. By the strong Markov property, we have

$$P_x(\varphi_{\xi+\varepsilon}-\varphi_{\xi-}\leqq z)=P_0\left(\int_0^\varepsilon x_s(\omega)^{-1/\alpha}\,ds\leqq z\right). \tag{3.10}$$

But by applying (2.14) to the right side, it is seen that the probability is independent of the integration limit ε, which is clearly impossible unless the integral is a.s. infinite.

Remark. It is not hard to prove at this point that the random variable $\varphi_{\xi-}$ must have an exponential distribution. We shall not pause to do so, however, as this will be shown automatically during the proof of the main theorem in Section 4 below.

4. The Main Theorem

Again $\{x_t\}$ will always be a semi-stable Markov process as defined in Section 2, and φ_τ will denote the function (3.1). Let $T(t)$ be the inverse function to φ; that is, define the random variables $T(t)=T(t,\omega)$ as the (unique) solutions of the equation

$$t=\int_0^T x_s(\omega)^{-1/\alpha}\,ds=\varphi_T, \tag{4.1}$$

which exist when $t<\varphi_{\xi-}$. (We assume $x_0=x>0$, so that $\xi>0$ a.s.) Next, define

$$y_t(\omega)=x_{T(t,\omega)}(\omega) \tag{4.2}$$

provided $t<\varphi_{\xi-}$; y_t is not defined otherwise. According to a theorem first published by Volkonski (see [2, Chapter 10]) the random variables $\{y_t\}$ constitute a new strong Markov process whose paths again have the pleasant properties of right continuity and absence of discontinuities other than jumps. Finally, we note that always $y_t>0$ if it is defined, so that we can set $z_t=\log y_t$; clearly $\{z_t\}$ must be a strong Markov process on $(-\infty,\infty)$ with nice paths.

Theorem 4.1. *The process* $\{z_t\}$ *defined above has stationary independent increments. The lifetime of* $\{z_t\}$ *is a.s. infinite either if* $\{x_t\}$ *does not reach* 0, *or if it does so by continuous approach. If* $\{x_t\}$ *jumps to* 0, *then* $\{z_t\}$ *may be considered to be an additive process with infinite lifetime which has been killed by an exponentially-distributed random variable independent of the process.*

Proof. The essential step to justify the formal arguments given in the introduction is to show that $\{y_t\}$ is a Feller process on the *compact* state space $[0, \infty]$; once this is known the characteristic operator of $\{y_t\}$ can be indentified with the generator and the Hille-Yosida theorem applied. The main tools will be the lemmas of Sections 2 and 3 and the results of [11].

Consider first the case when $P_x(\xi<\infty)=0$ for all $x>0$. Then as we saw in Section 2, $\{x_t\}$ is a Feller process on $(0, \infty]$; by Lemma 3.1, $T(t)$ is a.s. defined for all t and so $\{y_t\}$ has an infinite lifetime. The transition function P_t is uniformly stochastically continuous with respect to a bounded metric such as

$$\rho(x, y)=\frac{|x-y|}{1+|x-y|}, \qquad \rho(x, \infty)=1,$$

although not in the usual one. The "time change function" $v(x)=x^{1/\alpha}$ occuring in (4.1) is positive and continuous on $(0, \infty)$, and the state "∞" is never reached from $x\in(0, \infty)$. Under these conditions, Theorem 2 of [11] applies and yields the conclusion that $\{y_t\}$ is Feller on $(0, \infty)$.

We extend the process $\{y_t\}$ to the closed interval $[0, \infty]$ by making 0 and ∞ stationary points; it must be shown that this does not disturb the Feller property. First consider the behavior near 0. We apply (2.14) (extended) to the set $F=\{\omega\colon \varphi_u>t\}$ and set $x=1$; the result is

$$P_1(T(t)<u)=P_{a^\alpha}(T(t)<a\,u). \tag{4.3}$$

From (4.3) it is evident that $T(t)\to 0$ in law for each t as the intitial state $\to 0$. Combining this with the uniform stochastic continuity of P_t near 0, it is easy to see that $y_t=x_{T(t)}$ has a law which also tends to concentrate at 0 when x_0 is taken close to 0, and so the Feller property holds there for the extended process $\{y_t\}$.

The situation is a little different at ∞, for here the time change has the opposite effect of speeding up the process $\{x_t\}$. Fix t and choose u to make the left side of (4.3) at least $1-\varepsilon$; then we have

$$P_x(T(t)\leqq u\,x^{1/\alpha})\geqq 1-\varepsilon. \tag{4.4}$$

Thus it is enough to show that

$$\lim_{x\to\infty} P_x(\{x_s\in\Gamma \text{ for some } s\leqq u\,x^{1/\alpha}\})=0 \tag{4.5}$$

for any compact set Γ, since (4.4) and (4.5) together yield

$$\limsup_{x\to\infty} P_x(y_t\in\Gamma)\leqq\varepsilon \tag{4.6}$$

for any $\varepsilon>0$, which is equivalent to the Feller property at $x=\infty$. However, by (2.14) again we have

$$P_x(\{x_s\in\Gamma \text{ for some } s\leqq u\,x^{1/\alpha}\})=P_1(\{x_s\in x^{-1}\Gamma \text{ for some } s\leqq u\}). \tag{4.7}$$

Since $\{x_t\}$ never attains the state 0 and is quasi-continuous, the paths do not approach 0 during $[0, u]$ a.s., and so as $x \to \infty$ the right side of (4.7) tends to 0. This proves (4.5), and finishes the proof that $\{y_t\}$ is Feller on $[0, \infty]$ when $\xi = \infty$ a.s.

Next suppose that $P_x(\xi < \infty) = 1$ and $P_x(x_{\xi-} = 0) = 1$, $x > 0$. In this case, $\{x_t\}$ is Feller on the compact space $[0, \infty]$ (Theorem 2.1), and again for $x > 0$ the functional φ_τ grows continuously to ∞ and $T(t)$ is defined a.s. (Lemma 3.3). Once again, Theorem 2 of [11] yields the result that $\{y_t\}$ is a Feller process on $(0, \infty)$. The extension to $[0, \infty)$ preserving the Feller property is justified by the same argument as above, but the behavior at ∞ requires a different discussion. Let ξ_u denote the first passage time to $[0, u]$ as in (3.5). Applying (2.14) once more, we can obtain without difficulty

$$P_1(\varphi_{\xi_u} > t) = P_x(\varphi_{\xi_{xu}} > t). \tag{4.8}$$

But clearly

$$P_x(\varphi_{\xi_u} > t) = P_x(T(t) < \xi_u) \leqq P_x(y_t \geqq u). \tag{4.9}$$

Since $\xi < \infty$ and $\varphi_{\xi-} = \infty$, we have $\varphi_{\xi_u} \nearrow \infty$ as $u \searrow 0$; thus the left side of (4.8) will exceed $1 - \varepsilon$ for sufficiently small $u > 0$. But then combining (4.8) and (4.9) gives

$$P_x(y_t \geqq x u) \geqq P_x(\varphi_{\xi_{ux}} > t) = P_1(\varphi_{\xi_u} > t) \geqq 1 - \varepsilon \tag{4.10}$$

for all x, which yields the Feller property at $x = \infty$.

Finally we have the case $P_x(\xi < \infty, x_{\xi-} > 0) = 1$, $x > 0$, in which $\{y_t\}$ has the finite lifetime $\varphi_{\xi-}$. Consider the modified process $\{y'_t\}$, where $y'_t = y_t$ if the latter is defined (if $t < \varphi_{\xi-}$), but $y'_t = 0$ otherwise; 0 is clearly then an absorbing state. The original process $\{x_t\}$ was Feller on $[0, \infty]$, and Theorem 3 of [11] can be applied to obtain the conclusion that $\{y'_t\}$ is a Feller process on $[0, \infty)$. From this we can obtain a Feller property for $\{y_t\}$. Indeed, since 0 is an absorbing state for $\{y'_t\}$, we have $P_x(y_t \in E) = P_x(y'_t \in E)$ for every $x > 0$ and Borel set E not containing 0. The Feller property of $\{y'_t\}$ thus implies that the distribution of y_t in $(0, \infty)$ is also weakly continuous in x, so $\{y_t\}$ is Feller in the open interval.

As before, the endpoints can be included. To do so, we recall that the distribution $P_x(y'_t > 0) = P_x(t < \varphi_{\xi-})$ of the killing time for $\{y_t\}$ is independent of $x > 0$ (Lemma 3.4). We therefore define both 0 and ∞ to be states from which $\{y_t\}$ can not move to other states, and which can not be reached from others, but from which killing still occurs with that same distribution. It is rather obvious, as before, that the extension to 0 does not disturb the Feller property, and the details can be omitted. To handle the extension to ∞, first note that

$$P_x(y_t \in (0, \gamma]) \leqq P_x(x_s \in (0, \gamma] \text{ for some } s < \xi). \tag{4.11}$$

Using (2.14) much as before we transform the right side:

$$P_x(x_s \in (0, \gamma] \text{ for some } s < \xi) = P_1(x_s \in (0, \gamma x^{-1}] \text{ for some } s < \xi). \tag{4.12}$$

But because the paths of $\{x_t\}$ a.s. jump to 0, and do not approach 0 before ξ by quasi-continuity, it is clear that the right side of (4.12) tends to 0 as $x \to \infty$. For large x, therefore, the law of y_t will have mass $P_x(T(t)$ is defined$)$ distributed over large states, and since the total mass is independent of x we obtain as $x \to \infty$ the limit we have specified; i.e., $\{y_t\}$ is Feller at ∞.

It is easy to see that the lifetime distribution is exponential. Indeed, since it is independent of x we have by the Markov property

$$\begin{aligned} P_x(y_{t+s}\text{ defined}) &= \int_0^\infty P_u(y_t\text{ defined})\, dP_x(y_s\text{ defined}, y_s \leqq u) \\ &= P_x(y_t\text{ defined})\, P_x(y_s\text{ defined}), \end{aligned} \tag{4.13}$$

from which the result follows at once.

In each of the cases considered above, which include all semi-stable Markov processes on R^+, we have seem that $\{y_t\}$ with the natural extension is a Feller process on the compact space $[0, \infty]$. Using this result, it is easy to justify the formal argument sketched in Section 1. We begin with the fact that the characteristic operator $\mathscr{A}$ of $\{x_t\}$ satisfies

$$a\mathscr{A} f(x) = H_{a^{-\alpha}} \mathscr{A} H_{a^\alpha} f(x), \qquad a, x > 0, \tag{4.14}$$

for any f such that $\mathscr{A} f(x)$ is defined for each $x>0$. To see this we consider the two Markov processes

$$x'_t = x_{at}; \qquad x''_t = a^\alpha x_t, \tag{4.15}$$

where $\{x_t\}$ is semi-stable of order α. By (2.1), these processes have the same transition function; both also have nice paths and are strongly Markov. They therefore have the same characteristic operator. But it is clear that these operators are related to that of $\{x_t\}$ by $\mathscr{A}' = a\mathscr{A}$, $\mathscr{A}'' = H_{a^{-\alpha}} \mathscr{A} H_{a^\alpha}$, and the identity of the two sides of (4.14) – domains as well as values – follows.

Let $\mathscr{B}$ denote the characteristic operator of $\{y_t\}$. By the theory of random time substitutions, we know that $\mathscr{B} f(x)$ exists if and only if $\mathscr{A} f(x)$ does, $0<x<\infty$, and that in this case

$$\mathscr{B} f(x) = x^{1/\alpha} \mathscr{A} f(x). \tag{4.16}$$

For f such that $\mathscr{B} f(x)$ exists throughout $(0, \infty)$, combining (4.16) and (4.14) yields at once

$$\mathscr{B} f(x) = H_{u^{-1}} \mathscr{B} H_u f(x), \qquad u>0. \tag{4.17}$$

We shall see that (4.17) implies a multiplicative homogeneity property for $\{y_t\}$.

Let q_t, S_t denote respectively the transition function and associated semigroup of $\{y_t\}$, and define

$$q'_t(x, E) = q_t(u^{-1} x, u^{-1} E), \qquad u>0. \tag{4.18}$$

Then q'_t is a transition function, and the corresponding semigroup is $H_{u^{-1}} S_t H_u$. Both q_t and q'_t are Feller functions on the compact space $[0, \infty]$, so that their (weak or strong) generators coincide with their characteristic operators restricted to those $f \in C_0$ for which the operator is defined and leads to a function again in C_0 [2, Theorem 5.5]. But it is clear that at 0 and at ∞ the characteristic operator $\mathscr{B}'$ of the function q'_t is the same as that of q_t, while for $x \in (0, \infty)$ we have $\mathscr{B}' f(x) = H_{u^{-1}} \mathscr{B} H_u f(x)$. From (4.17), therefore, q'_t and q_t have the same characteristic operators and hence the same generators (including the identity of their domains). By the Hille-Yosida theorem we conclude that $q_t = q'_t$; in other words that

$$q_t(x, E) = q_t(u^{-1} x, u^{-1} E) \qquad \text{for any } u>0. \tag{4.19}$$

Finally we consider the effect of the transformation $z_t = \log y_t$. (We can let $\log 0 = -\infty$, $\log \infty = \infty$, although it is also satisfactory to now consider $\{y_t\}$ only on the open interval.) Defining $Q_t(z, W)$ to be the transition function of $\{z_t\}$, we have by (4.19) that

$$Q_t(z, W) = P_{e^z}(y_t \in e^W) = P_{e^{z+a}}(y_t \in e^{a+W}) = Q_t(z+a, W+a);$$

in other words, Q_t is translation invariant and hence $\{z_t\}$ has independent increments when the lifetime is infinite. In the contrary case, we know that $Q_t(z, R^1) = e^{-\beta t}$ for some $\beta > 0$. It is then easy to see that $Q_t'(z, W) = e^{\beta t} Q_t(z, W)$ is the transition function of a translation invariant process with infinite lifetime, from which $\{z_t\}$ can be obtained by killing at a time η independent of the process with $P(\eta > t) = e^{-\beta t}$. This completes the proof of the theorem.

Example. A simple example of a semi-stable process on R^+ arises in studying the statistics of extremes [10]. This process has a transition function defined by

$$P_t(x, [0, y]) = \begin{cases} 0 & \text{if } y < x; \\ \exp(-c\,t\,y^{-1/\alpha}) & \text{if } y \geqq x. \end{cases} \tag{4.20}$$

For $x > 0$, the generator (and the characteristic operator) is given by the formula

$$A f(x) = \frac{c}{\alpha} \int_x^\infty [f(y) - f(x)] \frac{dy}{y^{1+1/\alpha}}. \tag{4.21}$$

It is trivial to introduce the time-change factor $x^{1/\alpha}$ and thus obtain a formula for the characteristic operator $\mathscr{B}$ of $\{y_t\}$. But then the characteristic operator $\mathscr{C}$ of $\{z_t\}$ is obtained by the change of variable

$$\mathscr{C} g(z) = \mathscr{B} f(x),$$

where $z = \log x$ and $g(z) = f(x)$. We find, therefore, that

$$\mathscr{C} g(z) = \frac{c}{\alpha} \int_0^\infty [g(v+z) - g(z)]\, e^{-v/\alpha}\, dv. \tag{4.22}$$

Formula (4.22) shows that in this case $\{z_t\}$ is a compound Poisson process, with jumps having an exponential distribution. That $\{z_t\}$ should be compound Poisson is to be expected, of course, since $\{x_t\}$ was a process of the "pure jump" type.

5. The Diffusion Case

In this section we apply and illustrate Theorem 4.1 by specializing to the case of diffusion processes on R^+; i.e., we assume that the semi-stable process $\{x_t\}$ has (a.s.) trajectories which are continuous for all t. It is now very easy to obtain a new derivation of the form of the general semi-stable diffusion on R^+. It may be remarked in this connection that the proof of Theorem 4.1 could have been much shorter if attention had been restricted to the diffusion case throughout.

According to Theorem 4.1, the process $z_t = \log x_{T(t)}$ has independent increments; it cannot have finite lifetime since $\{x_t\}$ does not jump to 0. Since $\{x_t\}$ has continuous paths, so does $\{z_t\}$; it follows that $\{z_t\}$ can be nothing other than Brownian motion, possibly degenerate and possibly with a constant drift superimposed. In the degenerate case, $\{z_t\}$ is the deterministic motion $z_t = bt + z_0$. It is trivial to reverse the transformations embodied in Theorem 4.1 in this case, and the result is

$$x_t = \left(x_0^{1/\alpha} + \frac{bt}{\alpha}\right)^{\alpha} \tag{5.1}$$

as long as $x_t > 0$, where $\alpha > 0$ is the order of the process and $x_0 = e^{z_0} > 0$. In case $b < 0$, after reaching 0 the process can only be continued by "sticking" at 0 if path continuity is to be maintained. When $b > 0$ and $x_0 = 0$, either $x_t \equiv 0$ or $x_t = (bt/\alpha)^{\alpha}$ are possible choices for completing the definition of $\{x_t\}$.

The description of $\{x_t\}$ in all other cases is as follows:

Theorem 5.1. *Any non-degenerate semi-stable Markov process on R^+ with (a.s.) continuous paths has a generator of the form*

$$Af(x) = b\,x^{1-1/\alpha} f'(x) + d x^{2-1/\alpha} f''(x), \qquad x > 0, \tag{5.2}$$

where $d > 0$ and b are constants. When $b \geqq d$, there is a unique process on $(0, \infty)$ generated by (5.2). This process may be extended to $[0, \infty)$ either by making 0 a stationary state, or by defining

$$P_t(0, \cdot) = \operatorname{wk} \lim_{x \to 0+} P_t(x, \cdot), \tag{5.3}$$

and in either case the resulting process is semi-stable with $\xi = \infty$ a.s. If $b \leqq d(1 - 1/\alpha)$, (5.2) generates a unique diffusion; it is semi-stable, has $\xi < \infty$ a.s. and 0 is an absorbing state. Finally, when $d(1 - 1/\alpha) < b < d$, there are many continuous processes corresponding to (5.2), but among them only those in which the boundary state 0 is completely absorbing or completely reflecting are semi-stable.

Remark. Formula (5.2) was obtained in [9] by a very simple argument, which is, however, only valid with unnecessary restrictive assumptions. This class of diffusions has been studied by Stone in [18], where there are detailed results concerning the convergence of random walks or birth-and-death processes to limits satisfying (5.2). Limit theorems leading to such processes (with $\alpha = \frac{1}{2}$) were also given in [8].

Proof. As already noted, for the process $\{x_t\}$ considered in the theorem the corresponding additive process $\{z_t\}$ must be simply a Wiener process with a constant drift. Thus the generator C of $\{z_t\}$ is a differential operator with constant coefficients

$$Cg(z) = c g'(z) + d g''(z), \tag{5.4}$$

where $d > 0$ since the deterministic case is now excluded.

Let us see what this means for the original process $\{x_t\}$. First, we return to $(0, \infty)$ via the transformation $y_t = \exp(z_t)$; let B again denote the generator of $\{y_t\}$. It is easy to see that a function f on $(0, \infty)$ is in the domain of B if and only if $g(z) = f(e^z)$ is in the domain of C, and that then $Bf(e^z) = Cg(z)$. In this way we obtain

$$Bf(x) = (c + d)\, x f'(x) + d x^2 f''(x). \tag{5.5}$$

But since the relation $B = x^{1/\alpha} A$ holds between the generator A of $\{x_t\}$ and that of the process $\{y_t\}$ obtained by time change, (5.2) follows at once with $b = c + d$.

The interpretation of (5.2) naturally requires more care than is necessary for (5.4) and (5.5). The latter correspond to processes which always have natural boundaries at the ends of their state-intervals (respectively $\pm\infty$ and $0, \infty$), and so there is in each case only one process whose generator takes the form indicated. This is no longer true in the case of (5.2), for the boundary at 0 is now attainable under certain conditions, and so the domain of A must be restricted by a boundary condition, or some other specification imposed, before the process is uniquely determined.

We apply the classification of boundary points given by Feller in [3]. It turns out that $x=\infty$ is always a *natural* boundary, while $x=0$ is *entrance*, *regular* or *exit* in case $b \geqq d$, $d>b>(1-1/\alpha)\,d$, or $b \leqq d(1-1/\alpha)$ respectively. In the entrance case there is only one process on $(0, \infty)$ generated by (5.2). It is easy to see that this process can be started at 0, preserving both path continuity and the strong Markov property, only in the two ways described in the statement of the theorem. Next, in the exit case the process $\{x_t\}$, started with $x_0>0$, must reach 0 in finite time. From 0 it can not return by continuous movement into the interior; since in our case jumps are excluded, an absorbing state at 0 is the only possibility and a unique process results.

In the regular case, many diffusions correspond to (5.2), even when the requirements of infinite lifetime and continuity at $x=0$ are taken into account. But from the semi-stable condition (2.1) it is obvious that $P_t(0, \{0\})$ must be independent of t. In fact, it must be 0 or 1, for we have

$$p=P_{t+s}(0, \{0\})=\int_{0+}^{\infty} P_t(0, dy)\, P_s(y, \{0\})+p^2. \tag{5.6}$$

Letting $s \to 0$, $P_s(y, \{0\}) \to 0$ for each $y>0$ by stochastic continuity and the integral term in (5.6) tends to 0 by bounded convergence. Thus $p=p^2$. Hence $x=0$ must be a completely absorbing state ($p=1$) or completely reflecting when $p=0$; the "sticky" boundary diffusions are not semi-stable. It is also not difficult to obtain the same conclusion by a purely analytical argument.

In each case, the fact that the processes we have selected are actually semi-stable is easily seen. If $P_t(x, E)$ is the transition function of $\{x_t\}$, it is clear that for any $a>0$

$$q_t(x, E)=P_{at}(a^\alpha x, a^\alpha E) \tag{5.7}$$

is the transition function of a diffusion whose generator also is given by (5.2), and whose behavior at 0 (fully reflecting or absorbing) agrees with that of $\{x_t\}$. Since in each case there is only one such process, $q_t=P_t$ and the semi-stable property holds. This completes the proof of the theorem.

Remark. It is easy to see that for $\{x_t\}$ we will have $\xi<\infty$ a.s. if and only if $c<0$, so that $\{z_t\}$ drifts toward $-\infty$. Thus we could conclude that 0 is an unattainable boundary when $c \geqq 0$, or when $b \geqq d$, without appealing to Feller's theory. This observation applies even if the paths of $\{z_t\}$ and $\{x_t\}$ are not continuous, when the classification is not available. Unfortunately, the distinction between the regular and the exit boundary cases does not seem to have any such simple interpretation.

The above theorem completely describes ssmp's on R^+ whose paths are *everywhere* continuous. If discontinuities are allowed at $x=0$, however, many new possibilities arise. This question will be studied in detail in the second part of this paper; here we will discuss as a suggestive example the case when $\{x_t\}$, away from 0, is the Brownian motion process. The problem is to determine the most general way to continue $\{x_t\}$ after the time $t=\xi$ of reaching 0 so that the resulting process satisfies the semi-stable condition. This question is easily answered, since *all* extensions of $\{x_t\}$ after $t=\xi$ have been analyzed by Feller, Wentzel and Ito and McKean.

By Theorem 2.1, all of the processes we wish to construct will be Feller processes. The domain of the generator, accordingly, will be the space C_2 of functions twice continuously differentiable on $(0, \infty)$, restricted by a boundary condition whose most general form (see [21] or [5]) is

$$p_1 u(0) - p_2 u'(0) + p_3 u''(0) = \int_{0+}^{\infty} [u(x) - u(0)]\, p_4(dx). \tag{5.8}$$

Here $p_1, p_2, p_3 \geqq 0$ and p_4 is a non-negative measure satisfying

$$\int_0^1 x\, p_4(dx) < \infty \quad \text{and} \quad \int_1^{\infty} p_4(dx) < \infty. \tag{5.9}$$

Every choice of such $p_1 - p_4$ (not all 0), conversely, determines a domain and hence a process; non-proportional p_i's give different processes. The paths are continuous a.s. for all t iff $p_4 = 0$.

If $\{x_t\}$ is semi-stable as we have defined it, the lifetime is infinite. This immediately gives $p_1 = 0$. Moreover, except in the case of an absorbing boundary, we have seen that a semi-stable process must visit $x=0$ on a t-set of Lebesgue measure 0. This implies that either we have the absorbing case, for which the boundary condition is $u''(0)=0$, or else $p_3=0$. The case $p_4=0$, $p_2>0$ corresponds to the reflecting barrier – the only other possibility for a semi-stable process with everywhere continuous paths. It therefore remains to explore the cases where $p_2 \geqq 0$, $p_4 \neq 0$.

To continue, we again invoke the fact that when $\{x_t\}$ is semi-stable, the domain of its generator A is not altered by the operators H_a, $a>0$. (See the proof of Theorem 4.1.) In other words, *if a function* $u(x)$ *satisfies* (5.8), *so does the function* $u(a x)$ *for each* $a>0$. Let V denote the vector space of C_2 functions which vanish at 0, and define the linear functionals

$$f_a(u) = p_2\, a\, u'(0) + \int_0^{\infty} u(x)\, p_4\left(\frac{dx}{a}\right). \tag{5.10}$$

Clearly $f_a(u) = f_1(H_a u)$, and so if (5.8) is the boundary condition for a semi-stable process, the functionals f_a all have the same null space. It follows that they are proportional; i.e., that

$$p_2\, a\, u'(0) + \int_0^{\infty} u(x)\, p_4\left(\frac{dx}{a}\right) = c_a\, p_2\, u'(0) + c_a \int_0^{\infty} p_4(dx) \tag{5.11}$$

for all $u \in V$.

It is possible to choose a sequence of functions $u_n \in V$ such that $u_n'(0)=1$, but for which the integral terms on both sides of (5.11) tend to 0. As a result we must have $p_2\, a = c_a\, p_2$. This will hold for all $a>0$, so that either $p_2=0$ or $c_a=a$. Assume the latter; then

$$\int_0^\infty u(x)\frac{1}{a}\,p_4\left(\frac{dx}{a}\right)=\int_0^\infty u(x)\,p_4(dx) \tag{5.12}$$

for all $u\in V$. This is enough to ensure the identity of the measures $p_4(E)$ and $\frac{1}{a}p_4(E/a)$; hence $p_4([x,\infty))=m/x$, where $m=p_4([1,\infty))$. But this measure does not satisfy (5.9) unless $m=0$; thus if $\{x_t\}$ is semi-stable, $p_4\neq 0$ implies $p_2=0$. Incidently, a similar argument would have shown that $p_3=0$ without appealing to the impossibility of a positive measure for $\{t\colon x_t=0\}$.

Returning to (5.11) with $p_2=0$, we see just as above that

$$p_4\left(\left[\frac{x}{a},\infty\right)\right)=c_a\,p_4([x,\infty)),\qquad x>0. \tag{5.13}$$

It is then obvious that $c_a c_b=c_{ab}$, so that $c_a=a^\beta$ for some real β, and then that $p_4([x,\infty))=m\,x^{-\beta}$. Condition (5.9) will be satisfied iff $0<\beta<1$, so that these cases, finally, do yield a one-parameter family of new semi-stable processes which agree with classical Brownian motion in the open interval. We summarize these results:

Theorem 5.2. *The totality of semi-stable Markov processes whose generator has the form $A\,u(x)=u''(x)/2$ for $x>0$ are the classical absorbing and reflecting barrier Brownian motions, plus those determined by one of the boundary conditions*

$$\int_{0+}^\infty [u(x)-u(0)]\,\frac{dx}{x^{\beta+1}}=0,\qquad 0<\beta<1. \tag{5.14}$$

The probabilistic meaning of the processes corresponding to (5.14) is explained in [5]; roughly, upon reaching $x=0$ the process instantly jumps into the interior according to the measure $dx/x^{\beta+1}$. Since this measure puts infinite mass near the origin "almost all" of the jumps are very small, and infinitely many occur in a finite time once zero is reached. Ito has recently shown how such "jumping in processes" can be constructed under very general conditions [4]. His results will be applied to the construction of semi-stable processes in the sequel to this paper.

6. The Generator of a ssmp

In this section we generalize the first part of Theorem 5.1 to obtain a formula for the generator of any ssmp on R^+ when $x>0$. It is not easy in the general case to determine the *domain* of the generator, however, and the treatment of this question will be deferred to the second part of this paper.

Let $\{x_t\}$ be a ssmp of order α, and let $\{z_t\}$ be that additive process which corresponds to $\{x_t\}$ according to Theorem 4.1. By the theorem of Lévy and Khintchine, we have

$$E(e^{i\lambda(z_t-z_0)})=\exp\left\{i\mu t\lambda+t\int_{-\infty}^{\infty}\left(e^{i\lambda y}-1-\frac{i\lambda y}{1+y^2}\right)\frac{1+y^2}{y^2}\,dG(y)\right\}, \tag{6.1}$$

where dG is a finite measure uniquely determined by $\{z_t\}$. More precisely, (6.1) holds when $\{z_t\}$ has an infinite lifetime; in the contrary case, when $P_z(z_t$ is defined)

$=e^{-\beta t}$ for some $\beta\in(0,\infty)$, (6.1) holds for the "unkilled" process from which $\{z_t\}$ is obtained as discussed at the end of Section 4. The process $\{z_t\}$ is, of course, a Feller process on $[-\infty,\infty]$. According to a theorem due to Ito and Neveu (see [15], pp. 628–630), the generator C of $\{z_t\}$ has the form

$$C g(z)=\mu g'(z)+\int_{-\infty}^{\infty} h(z,y)\,dG(y), \tag{6.2}$$

where

$$h(z,y)=\begin{cases}\left[g(z+y)-g(z)-\dfrac{y}{1+y^2}\,g'(z)\right]\dfrac{1+y^2}{y^2} & \text{for } y\neq 0;\\ g''(z)/2 & \text{for } y=0.\end{cases} \tag{6.3}$$

The domain of C contains at least all functions g such that g, g' and g'' are continuous on $[-\infty,\infty]$. If there is exponential killing, the term $-\beta g(z)$ must be added to the right-hand side of (6.2). Using these facts, it is easy to prove the following:

Theorem 6.1. *Let $\{x_t\}$ be a ssmp of order α, whose corresponding (unkilled) additive process $\{z_t\}$ satisfies* (6.1). *Then the characteristic operator of $\{x_t\}$ has for $x>0$ the form*

$$\mathscr{A}f(x)=\mu x^{1-1/\alpha}f'(x)+x^{-1/\alpha}\int_0^{\infty} h^*(x,u)\,dG^*(u)-\beta x^{-1/\alpha}f(x), \tag{6.4}$$

where $G^*(u)=G(\log u)$ and

$$h^*(x,u)=\begin{cases}\left[f(xu)-f(x)-\dfrac{\log u}{1+\log^2 u}\,xf'(x)\right]\dfrac{1+\log^2 u}{\log^2 u} & \text{for } u\neq 1;\\ x^2f''(x)/2 & \text{for } u=1.\end{cases} \tag{6.5}$$

Formula (6.4) *holds at least for all f such that f, xf' and x^2f'' are continuous on* $[0,\infty]$, *and uniquely determines the process $\{x_t\}$ for all $t<\xi$. Conversely, given $\{x_t\}$, μ, G^* and β are determined.*

Proof. Formulas (6.4) and (6.5) follow formally from (6.2) and (6.3) in exactly the way that (5.2) was derived from (5.4); indeed, the latter is a special case. Since $g\in\mathscr{D}_C$ when g, g', g'' are continuous on $[-\infty,\infty]$, we find that $f(x)=g(\log x)\in\mathscr{D}_B$ (B being the generator of $\{y_t\}=\{e^{z_t}\}$) when f, xf' and x^2f'' are continuous on $[0,\infty]$. For at least these functions, therefore, the characteristic operator of $\{x_t\}$ is given by $\mathscr{A}=x^{-1/\alpha}B$. But the formula for Bf is obtained from (6.2) and (6.3) by a change of variable; combining the result with $\mathscr{A}=x^{-1/\alpha}B$ leads to (6.4) and (6.5). The uniqueness assertions are probabilistically obvious, for $\{z_t\}$ completely determines the paths of $\{x_t\}$ for all $t<\xi$ and conversely, and it is known that the correspondence between unkilled additive processes and the elements (μ, G) which appear in (6.1) and (6.2) is biunique. The relationship between $\{z_t\}$ and $\{x_t\}$ will be discussed much more fully in part II.

Example. Suppose that a ssmp $\{x_t\}$ is itself an additive process, with increasing path functions so that it can be considered as a process on R^+. Then $\{x_t\}$ must be a *stable* process of index $p=\alpha^{-1}<1$, and its characteristic function will be of the form

$$E(e^{i\lambda(x_t-x_0)})=\exp\left\{ct\int_0^\infty (e^{i\lambda y}-1)\frac{dy}{y^{1+1/\alpha}}\right\} \tag{6.6}$$

(see [9]). According to the result of Ito and Neveu, the generator of $\{x_t\}$ has the form

$$Af(x)=\int_0^\infty [f(x+y)-f(x)]\frac{c\,dy}{y^{1+1/\alpha}}=x^{-1/\alpha}\int_1^\infty [f(ux)-f(x)]\frac{c\,du}{(u-1)^{1+1/\alpha}}. \tag{6.7}$$

(The last expression holds for $x>0$.) Comparing (6.7) with (6.4), we easily identify the canonical measure dG of the corresponding process $\{z_t\}$:

$$\frac{1+z^2}{z^2}dG(z)=\frac{c\,e^z\,dz}{(e^z-1)^{1+1/\alpha}},\qquad z>0. \tag{6.8}$$

Thus $\{z_t\}$ is not again a stable process. On the other hand, if $\{z_t\}$ is a stable process, $\{x_t\}$ will not be additive.

7. Local Properties of a ssmp

There are many possibilities for exploiting Theorem 4.1 to obtain information about $\{x_t\}$ from known results for processes with independent increments. A systematic study of these matters will not be made here, but we will mention as an illustration one area in which some facts are very easily obtained: local Hölder conditions. Let $\{x_t\}$ be a semi-stable process with generator (6.4); equivalently, the corresponding homogeneous processes $\{z_t\}$ has killing parameter β and its "unkilled" increments satisfy (6.1). Define the constants

$$\sigma^2=G(0+)-G(0-);\qquad \gamma=\inf\left\{\eta>0\colon \int_{-1}^{1}|y|^\eta\frac{1+y^2}{y^2}dG(y)<\infty\right\}. \tag{7.1}$$

By applying results of Khintchine and Blumenthal and Getoor on the local behavior of additive processes, together with Theorem 4.1, it is easy to prove the following:

Theorem 7.1. *For all* $x>0$, $\sigma^2\geqq 0$,

$$P_x\left[\limsup_{t\to 0}\frac{x_t-x}{(2t\log\log t^{-1})^{\frac{1}{2}}}=\sigma x^{1-1/2\alpha}\right]=1; \tag{7.2}$$

the same holds for the lim inf. *For every* $\varepsilon>0$ *and all* $x>0$,

$$P_x\left(\lim_{t\to\infty}\frac{x_t-x}{t^{1/\gamma-\varepsilon}}=0;\ \limsup_{t\to 0}\frac{|x_t-x|}{t^{1/\gamma+\varepsilon}}=\infty\right)=1. \tag{7.3}$$

Proof. For small values of t it is very easy to follow the relationship between increments of $\{z_t\}$ and those of $\{x_t\}$. First, by right continuity we have for $x_0=y_0=e^{z_0}$ that

$$y_t-y_0=e^{z_0}(e^{z_t-z_0}-1)\sim x_0(z_t-z_0) \tag{7.4}$$

as $t\to 0$. But from (4.1) we have $\varphi_t=t\,x_0^{-1/\alpha}+o(t)$ as $t\to 0$ for each $x>0$, and from (4.2) $x_t=y_{\varphi_t}$. Combining these facts gives

$$x_t-x_0\sim x_0(z_{\varphi_t}-z_0);\qquad \varphi(t)\sim x_0^{-1/\alpha}t, \tag{7.5}$$

where asymptotic equality holds a.s. as $t \to 0$ for each $x_0 > 0$. It is now trivial to apply Khintchine's local law of the iterated logarithm (Theorems 2 and 3 of [6]) to $\{z_t\}$ in order to obtain (7.2) or to apply theorems 3.1 and 3.3 of [1] and obtain (7.3).

There are many other results for additive processes which could also yield information about the paths of $\{x_t\}$, including some due to Blumenthal and Getoor and included in [1]. In particular, if $\{x_t\}$ has increasing paths (7.3) can be sharpened. More applications will be given in the future.

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