Recall : $X = \{X_t : t \geq 0\}$

is a pssMp if $\exists \alpha > 0$ s.t.

$\forall c > 0$, $\{cX_{c^{-\alpha}t} : t \geq 0\}$ under P_x

is equal in law to (X, P_{cx})

and X is a strong Markov process (valued in $[0,\infty)$)

with paths that are right cts and quasi-left cts.

Quasi-left cts: $\forall$ sequences of stopping times T_n

s.t $T_n \uparrow T$ where T is also a stopping time

then on $\{T < \infty\}$ $\lim_{n\to\infty} X_{T_n} = X_T$.

Recall want to prove; Fix $\alpha > 0$.

$\forall$ pssMp with index $\alpha > 0$, necessarily can be written in the form

$$X_t^{(x)} \mathbb{1}(t < \zeta^{(x)}) = x \exp\left(\xi_{\varphi(x^{-\alpha} t)}\right)$$

where $\zeta^{(x)} = \inf\{t > 0 : X_t^{(x)} = 0\}$

$$\boxed{\int_0^{\varphi(t)} e^{\alpha \xi_u} \, du = t}$$ if exists o/w $\varphi(t) = \infty$

and either

(1) $\zeta^{(x)} = \infty$ a.s. $\forall x$ and ξ is a L.p. $\limsup_{t \to \infty} \xi_t = \infty$

(2) $\zeta^{(x)} < \infty$ and $X^{(x)}_{\zeta^{(x)}-} = 0$ a.s. $\forall x$ and ξ is L.p.

s.t. $\lim_t \xi_t = -\infty$

(3) $\zeta^{(x)} < \infty$ and $X^{(x)}_{\zeta^{(x)}-} > 0$ a.s. $\forall x$ and ξ is killed L.P.

Lemma The three catagories in pink necessarily occur for pssMp of a given index α.

Recall that in the theorem

$$\int_0^{\varphi(t)} e^{\alpha \xi_s}\, ds = t \text{ or } \infty$$

Pre-emptively to the statement about φ in the theorem, define

$$\varphi(t) = \int_0^{\boxed{x^\alpha t}} \left(X_s^{(x)}\right)^{-\alpha} ds, \quad t < x^{-\alpha}\zeta^{(x)}$$

$$= \infty \quad \text{o/w.}$$

Recall: $x^{-\alpha}\zeta^{(x)}$ doesn't depend on x!

Claim: $\varphi(t)$ doesn't depend on x!

To see why: suppose we indicate dependency and write $\varphi^{(x)}(t)$. Note $\forall c>0$, for $t < (cx)^{-\alpha}\zeta^{(cx)}$
$\stackrel{d}{=} x^{-\alpha}\zeta^{(x)}$

$$\varphi^{(cx)}(t) = \int_0^{(cx)^\alpha t} \left(X_s^{(cx)}\right)^{-\alpha} ds$$

$$\boxed{\stackrel{d}{=}_{\text{pssMp}}} \int_0^{(cx)^\alpha t} c^{-\alpha} \left(X_{c^{-\alpha}s}^{(x)}\right)^{-\alpha} ds$$

$$\stackrel{c^{-\alpha}s=u}{=} \int_0^{x^\alpha t} \left(X_u^{(x)}\right)^{-\alpha} du$$

$$= \varphi^{(x)}(t)$$

Exercise: Define $\varphi(x^{-\alpha}\zeta-) = \lim_{t \uparrow \zeta} \varphi(x^{-\alpha}t)$

claim: $\varphi(x^{-\alpha}\zeta^{(x)}-)$ is indep. of x.

Lemma

In the case that $\zeta = \infty$ (case (1)) or ($\zeta < \infty$ and $X_{\zeta-} = 0$) (case (2)) we have $P_x(\varphi(x^{-\alpha}\zeta_-) = \infty) = 1 \quad \forall x > 0$.

If $\zeta < \infty$ and $X_{\zeta-} > 0$ (case (3)) then $\varphi(x^{-\alpha}\zeta_-)$ is an exponentially distributed r.v. (which doesn't depend on x).

Pf/ $\underline{\text{Case (1)}}$ $\zeta = \infty$: use the Markov property of X

$$\lim_{t\uparrow\infty}\varphi(t) =: \varphi(\infty) = \int_0^1 (X_s^{(x)})^{-\alpha} ds + \int_1^\infty (X_s^{(x)})^{-\alpha} ds$$

$$\stackrel{d}{=} \int_0^1 (X_s^{(x)})^{-\alpha} ds + \int_0^\infty (\tilde{X}_u^{(z)})^{-\alpha} du$$

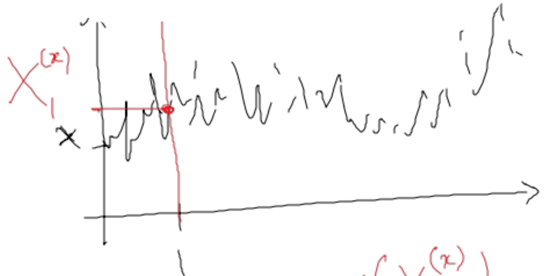
where $z = X_1^{(x)}$

$$\Longrightarrow \varphi(\infty) \stackrel{d}{=} \boxed{\int_0^1 (X_s^{(x)})^{-\alpha} ds} + \tilde{\varphi}(\infty)$$

where $\tilde{\varphi}(\infty) \perp\!\!\!\perp \{X_u : u \le t\}$

$$\stackrel{d}{=} \varphi(\infty)$$

$$\Longrightarrow \varphi(\infty) = \infty \text{ a.s.}$$



$$X_{1+s}^{(x)} = \tilde{X}_s^{(X_1^{(x)})}$$

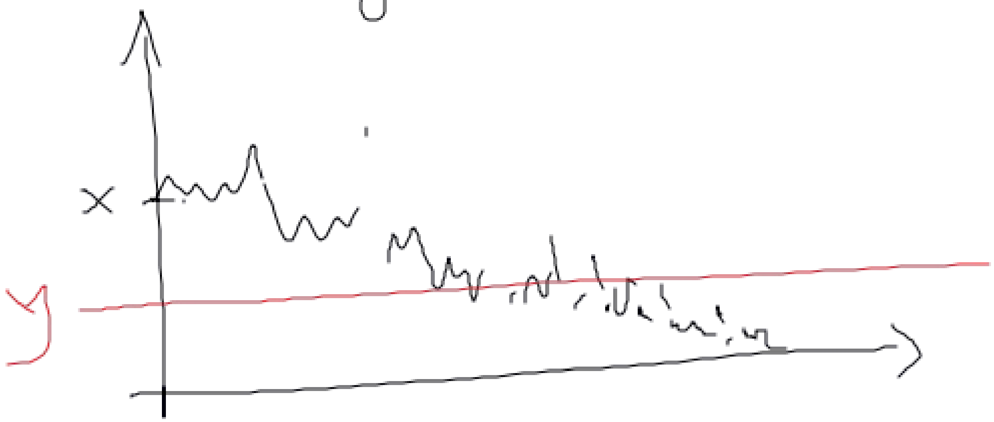
where $\tilde{X}^{(\cdot)}$ is an indep. copy of $X^{(\cdot)}$

Case (2): $\zeta < \infty$ and $X_{\zeta-} = 0$.

Cannot use previous argument. Specifically can't split integral @ $t=1$. However, $X_{\zeta-} = 0$

$\Rightarrow$ $R_y^- < \zeta$ a.s for all $y>0$ where

$$R_y^- = \inf\{t>0 : X^{(x)} < y\}$$



Now do same argument as before splitting

$$\varphi(x^{-\alpha}\zeta_-) = \int_0^{\zeta} (X_u^{(x)})^{-\alpha} du$$

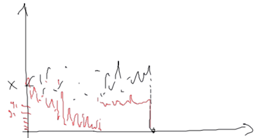
at time R_y^- for some $y>0$.

Exercise

$\Rightarrow \varphi(x^{-\alpha}\zeta-) = \infty$ a.s.

Case (3): $\zeta < \infty$ and $X_{\zeta-} > 0$

φLC $\Rightarrow$ on $[0, \zeta^{(x)})$, $X^{(x)}$ is bounded away from 0.



$$\Longrightarrow \varphi(x^{-\alpha}\zeta^{(x)}_-) = \int_0^{\zeta^{(x)}} (X^{(x)}_u)^{-\alpha}\, du < \infty \quad \text{a.s.}$$

introduce $I_u = \inf\{0 < t < x^{-\alpha}\zeta^{(x)} : \varphi(t) > u\}$

$(\inf \emptyset := \infty)$

fix $u > 0$. On $\{\varphi(x^{-\alpha}\zeta^{(x)}_-) > u\}$ $(\Rightarrow I_u < \infty)$

$$\boxed{\varphi(x^{-\alpha}\zeta_-)} = \int_0^{x^\alpha I_u} (X^{(x)}_s)^{-\alpha}\, ds + \int_{\boxed{x^\alpha I_u}}^{\zeta^{(x)}} (X^{(x)}_s)^{-\alpha}\, ds$$

$$\underset{\text{SMP}}{\overset{d}{=}} u + \int_0^{\tilde\zeta^{(z)}} (\tilde X^{(z)}_s)^{-\alpha}\, ds$$

is a stopping time for X

where $z = X^{(x)}_{x^\alpha I_u}$ and $\tilde X^{(\cdot)}$ is an indep. copy of $X^{(\cdot)}$

$x^\alpha I_u$ is a stopping time: I_u is the inverse of $\varphi(t)$

$$\int_0^{x^\alpha I_u} (X^{(x)}_s)^{-\alpha}\, ds = u$$

$\{x^\alpha I_u \le w\}$ is contained in $\sigma(X^{(x)}_s : s \le w)$

In conclusion

on $\{\varphi(x^{-\alpha}\zeta^{(x)}_-) > u\}$

$$\underbrace{\varphi(x^{-\alpha}\zeta^{(x)}_-)}_{\text{doesn't depend on } x.} \overset{d}{=} u + \underbrace{\tilde\varphi(z^{-\alpha}\tilde\zeta^{(z)}_-)}_{\text{doesn't depend on } z} \quad (*)$$

define $\mathcal{E}(u) := \underbrace{P_x(\varphi(x^{-\alpha}\zeta^{(x)}_-) > u)}_{\text{doesn't depend on } x.}$

$$\Longrightarrow \frac{\mathcal{E}(u+s)}{\mathcal{E}(u)} = \frac{P_x(\varphi(x^{-\alpha}\zeta^{(x)}_-) \ge u+s \,;\, \varphi(x^{-\alpha}\zeta^{(x)}_-) > u)}{P_x(\varphi(x^{-\alpha}\zeta^{(x)}_-) > u)}$$

$$= P_x(\varphi(x^{-\alpha}\zeta^{(x)}_-) > u+s \mid \varphi(x^{-\alpha}\zeta^{(x)}_-) > u)$$

$$= \mathcal{E}(s)$$

i.e. $\mathcal{E}(u+s) = \mathcal{E}(u)\,\mathcal{E}(s)$

note (exercise) $\mathcal{E}(\cdot)$ is right cts & strictly +ve.

$\Rightarrow \mathcal{E}(u) = e^{-qu}$ for some $q \in (0,\infty)$

$\Rightarrow \varphi(x^{-\alpha}\zeta^{(x)}_-) \sim \exp(q)$ for some $q \in (0,\infty)$

Proof of Main Theorem (i)

$$\mathbf{e} := \varphi(x^{-\alpha}\zeta^{(x)}-)$$

[case (1) & (2) $\mathbf{e} = \exp(0)$, case (3) $\mathbf{e} \sim \exp(q)$, $q>0$.]

$$\xi_t = \log\left(\frac{X^{(x)}_{\boxed{x^\alpha I_t}}}{x}\right) \qquad 0<t<\underbrace{\varphi(x^{-\alpha}\zeta^{(x)}-)}_{\mathbf{e}}$$

<u>Exercise</u>: show that ξ doesn't depend on x
[Hint: show that $I^{(x)}_{\infty}$ doesn't depend on x].

Note: ξ has right-cts paths because X does.
similarly it has left-limits ($QLC \Rightarrow \exists$ left limits)

on $\{t<\mathbf{e}\}$ we have $\varphi(I_t) = t$

$$\varphi'(I_t)\frac{dI_t}{dt} = 1 \Rightarrow \frac{dI_t}{dt} = x^{-\alpha}\left(X^{(x)}_{x^\alpha I_t}\right)^{\alpha}$$

$$\left[\varphi(t) = \int_0^{x^\alpha t}\left(X^{(x)}_u\right)^{-\alpha}du\right] \qquad \Rightarrow \frac{dI_t}{dt} = e^{\alpha\xi_t}$$

hence $I_t = \int_0^t e^{\alpha\xi_u}du$ on $t<\mathbf{e}$

accordingly $\varphi = I^{-1}$

Need to show that ξ has st. indep. increments and that $\mathbf{e}$ rings (in case (3)) indep. of increments of ξ.

To this end, fix $x>0$. and consider the event

$$\{\xi_t > -\infty\} = \{X^{(x)}_{x^\alpha I_t} > 0\} = \{x^\alpha I_t < \zeta^{(x)}\}$$
$$= \{t < \mathbf{e}\}$$

Recall that $x^\alpha I_t$ is a stopping time for X

Look @

$$\exp(\xi_{t+h} - \xi_t) \qquad \text{on } \{t<\mathbf{e}\}$$

$$= \frac{X^{(x)}_{x^\alpha I_{t+h}}}{X^{(x)}_{x^\alpha I_t}} \underset{\text{claim}}{\overset{d}{=}} z^{-1}\tilde{X}^{(z)}_{z^\alpha \tilde{I}_h}$$

where $z = X^{(x)}_{x^\alpha I_t}$ and $\tilde{X}^{(\cdot)}$ is an indep. copy of $X^{(\cdot)}$

To see why claim holds:

On $\{t<\mathbf{e}\}$

$$x^\alpha I_{t+h} = x^\alpha I_t + x^\alpha \inf\left\{s>0 : \int_0^{x^\alpha s}\left(X^{(x)}_{\boxed{x^\alpha I_t + u}}\right)^{-\alpha} > h\right\}$$

$$\left[I = \varphi^{-1} \text{ and } \varphi(t) = \int_0^{x^\alpha t}\left(X^{(x)}_s\right)^{-1}ds\right]$$

$$z^\alpha \cdot \boxed{\frac{x^\alpha}{z^\alpha}s}$$

$$= x^\alpha I_t + x^\alpha \inf\left\{s>0 : \int_0^{\boxed{x^\alpha s}}\left(\tilde{X}^{(z)}_u\right)^{-\alpha}du > h\right\}$$

$$x^\alpha I_{t+h} = x^\alpha I_t + z^\alpha \tilde{I}_h \qquad \boxed{\tilde{I} = \tilde{\varphi}^{-1} : \tilde{\varphi}(t) = \int_0^{z^\alpha t}(\tilde{X}^{(z)}_u)^{-\alpha}du}$$

where $\tilde{I}$ is obviously defined. $\perp\!\!\!\perp \{X^{(x)}_s : s \le x^\alpha I_t\}$

Hence

$$\exp(\xi_{t+h}-\xi_t) = \frac{X^{(x)}_{x^\alpha I_{t+h}}}{X^{(x)}_{x^\alpha I_t}} \overset{d}{=} \frac{\tilde{X}^{\left(\boxed{X^{(x)}_{x^\alpha I_t}}\right)}_{z^\alpha \tilde{I}_h}}{\boxed{X^{(x)}_{x^\alpha I_t}}}$$

z

by scaling $\dfrac{\tilde{X}^{(z)}_{z^\alpha \tilde{I}_h}}{z}$ is indep. of z.

$$\overset{d}{=} \exp(\xi_h)$$

note also by the defn of $\tilde{X}^{(\cdot)} \perp\!\!\!\perp \boxed{\{X_s : s \le x^\alpha I_t\}}$

$$\Rightarrow \xi_{t+h} - \xi_t \perp\!\!\!\perp \boxed{\{\xi_u : u \le t\}}$$

hence on $\{t<\mathbf{e}\}$, $\xi_{t+h} - \xi_t \perp\!\!\!\perp \{\xi_u : u \le t\}$
$\overset{d}{=} \xi_h$.

in the case $\mathbf{e} = \infty$ done!

in the case $\mathbf{e} \sim \exp(q)$: $q>0$

look back to eq (*) $\Rightarrow$ increments of ξ on $\{\xi > -\infty\}$ indep. of $\mathbf{e}$.