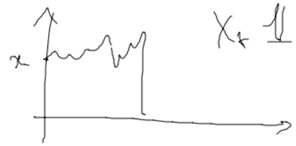
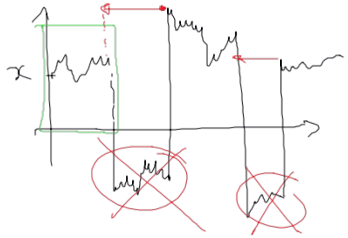
Censored Stable





Where $X_t \mathbb{1}(t<\tau_0^-)$ kills a L.p. (in the Lamperti transform) Z_t "introduces" an additional jump. By Markov property, after this jump, repeat "killed" stable with extra jump instead of killing

→ Suppose $Z_t^{(x)} = x e^{\tilde{\xi}_{\varphi(x^{-\alpha}t)}}$

$$\tilde{\xi}_t = \xi^L + \underline{\xi^c}$$

where ξ^L is the same Lévy process as ξ^* but without killing i.e. $\mathbb{E}\left[e^{i\theta\xi^L_t}\right] e^{-(\Psi^*(\theta)-q^*)t}$

ξ^c is a process which jumps at the times of a Poisson process with rate q^*

(hopefully ξ^c is a CPP process!)

has to be a CPP otherwise $\tilde{\xi}$ is not a L.p. and $\tilde{\xi}$ has to be a L.p. because the Lamperti transform says so!

Intuitively speaking $\xi^c \perp\!\!\!\perp \xi^L$

From previous lecture:

$$\Psi^*(\theta) = \frac{\Gamma(\alpha - i\theta)}{\Gamma(\alpha\hat{\rho} - i\theta)} \times \frac{\Gamma(1+i\theta)}{\Gamma(1-\alpha\hat{\rho}+i\theta)}$$

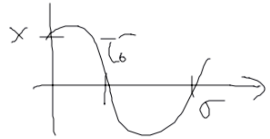
$$q^* = \Psi^*(0) = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}$$

$$\underline{LS}(\beta, \gamma, \hat{\gamma}) = (1-\alpha\hat{\rho}, \alpha\rho, \alpha\hat{\rho})$$

For full description of $\tilde{\Psi}(\theta) := -\log \mathbb{E}[e^{i\theta\tilde{\xi}_1}]$

we need to work out jump dist^n of ξ^L.

Lemma: For each $x>0$,
the joint law of $(X_{\tau_0^-}, X_{\tau_0^- -}, X_\sigma)$ under $\mathbb{P}_x$
is equal to $(x X_{\tau_0^-}, x X_{\tau_0^- -}, x X_\sigma)$ under $\mathbb{P}_1$
where $\sigma = \inf\{t > \tau_0^-, X_t > 0\}$.



Pf $\forall c>0$, $X^c_t = c X_{c^{-\alpha} t}$, $t \geq 0$.

$$(X^c, \mathbb{P}_x) \overset{d}{=} (X, \mathbb{P}_{cx})$$

$$\tau_0^{c,-} = \inf\{t>0 : X^c_t < 0\}$$
$$\sigma^c = \inf\{t > \tau_0^{c,-} : X^c_t > 0\}$$

$$c^\alpha \tau_0^- = c^\alpha \inf\{t>0 : X_{tc^\alpha c^{-\alpha}} < 0\}$$
$$= \inf\{s>0 : cX_{sc^{-\alpha}} < 0\}$$
$$= \tau_0^{c,-}$$

Similarly $c^\alpha \sigma = \sigma^c$

$$(cX_{\tau_0^-}, cX_{\tau_0^- -}, cX_\sigma)$$ under $\mathbb{P}_x$

$$= (cX_{c^{-\alpha}\underbrace{c^\alpha \tau_0^-}_{\tau_0^{c,-}}}, cX_{c^{-\alpha}\underbrace{c^\alpha \tau_0^-}_{\tau_0^{c,-}} -}, cX_{c^{-\alpha}\underbrace{c^\alpha \sigma}_{\sigma^c}})$$

$$\overset{d}{=} (X_{\tau_0^-}, X_{\tau_0^- -}, X_\sigma)$$ under $\mathbb{P}_{cx}$

Set $x = \frac{1}{c}$

What are the jumps of ξ^c?

$$\boxed{\tilde{\xi}_t = \xi^L_t \oplus \xi^c_t}$$

$$\Delta Z_{\tau_0^-} = Z_{\tau_0^-} - Z_{\tau_0^- -}$$

$$\tau_0^- = \inf\{t>0 : X_t < 0\}$$

$$= T_1 = \inf\{t>0 : \Delta\xi^c_t \neq 0\}$$

$$\Delta Z_{\tau_0^-} = e^{\tilde{\xi}_{T_1}} - e^{\tilde{\xi}_{T_1-}}$$

$$= e^{\tilde{\xi}_{T_1-}}\left(e^{\Delta\xi^c_{T_1}} - 1\right)$$

$$= X_{\tau_0^- -}\left(e^{\boxed{\Delta\xi^c_{T_1}}} - 1\right)$$

under $\mathbb{P}_1$

$$e^{\Delta\xi^c_{T_1}} = 1 + \frac{\Delta Z_{\tau_0^-}}{X_{\tau_0^- -}} = 1 + \frac{(X_\sigma - X_{\tau_0^-})}{X_{\tau_0^-}}$$

$$= \frac{X_\sigma}{X_{\tau_0^-}}$$

Note: take $0 < t_1 < t_2 < \dots < t_n \leq t$

wlog

$$\mathbb{E}_1\left[f(X_{t_1},\dots,X_{t_n})\mathbb{1}_{(t<\tau_0^-)}\, g\left(\frac{X_\sigma}{X_{\tau_0^-}}\right)\right]$$

condition on $\mathcal{G}_t$

$$= \mathbb{E}_1\left[f(X_{t_1},\dots,X_{t_n})\mathbb{1}_{(t<\tau_0^-)}\,\mathbb{E}_{X_t}\left[g\left(\frac{X_\sigma}{X_{\tau_0^-}}\right)\right]\right]$$

$$\stackrel{\text{Lemma}}{=} \mathbb{E}_1\left[f(X_{t_1},\dots,X_{t_n})\mathbb{1}_{(t<\tau_0^-)}\right]\mathbb{E}_1\left[g\left(\frac{X_\sigma}{X_{\tau_0^-}}\right)\right]$$

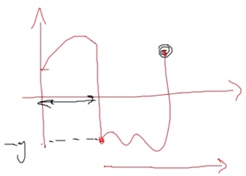
This shows independence of jumps of ξ^c with

Lemma The r.v. $e^{\Delta \xi^c_{T_1}}$

is equal in dist^n to $\left(-\frac{X_{\tau_0^-}}{X_{\tau_0^- -}}\right) \times \left(-\hat{X}_{\hat{\tau}_0^-}\right)$

where $\hat{X} \perp\!\!\!\perp X$, $\hat{X} =^d -X$ and

$$\hat{\tau}_0^- = \inf\{t > 0 : \hat{X}_t < 0\}.$$



Pf/ From previously $e^{\Delta \xi^c_{T_1}} = \frac{X_\sigma}{X_{\tau_0^- -}}$

$$\mathbb{E}_1\left[f(X_\sigma) \middle| \mathcal{F}_{\tau_0^-}\right]$$

$$= \mathbb{E}_{-y}\left[f\left(-\hat{X}_{\hat{\tau}_0^-}\right)\right]\Big|_{y = X_{\tau_0^-}}$$

$$\overset{\text{Lemma}}{=} \mathbb{E}_1\left[f\left(y \hat{X}_{\hat{\tau}_0^-}\right)\right]\Big|_{y = X_{\tau_0^-}}$$

$$\Rightarrow \boxed{\mathbb{P}_1\left(X_\sigma \in \cdot \middle| \mathcal{F}_{\tau_0^-}\right) = \mathbb{P}_1\left(y \hat{X}_{\hat{\tau}_0^-} \in \cdot\right)\Big|_{y = X_{\tau_0^-}}}$$

Now compute

$$\mathbb{E}_1\left[f\left(\frac{X_\sigma}{X_{\tau_0^- -}}\right)\right] = \mathbb{E}_1\left[\int_{(0,\infty)} f\left(\frac{x}{X_{\tau_0^- -}}\right) \mathbb{P}_1\left(X_\sigma \in dx \middle| \mathcal{F}_{\tau_0^-}\right)\right]$$

$$= \mathbb{E}_1\left[\int_{(0,\infty)} f\left(\frac{x}{X_{\tau_0^- -}}\right) \mathbb{P}_1\left(y \hat{X}_{\hat{\tau}_0^-} \in dx\right)\Big|_{y = X_{\tau_0^-}}\right]$$

$$= \mathbb{E}_1\left[\mathbb{E}_1\left[f\left(\frac{y \hat{X}_{\hat{\tau}_0^-}}{z}\right)\right]\Big|_{y = X_{\tau_0^-},\, z = X_{\tau_0^- -}}\right]$$

$$\Rightarrow \frac{X_\sigma}{X_{\tau_0^- -}} =^d \left(\frac{X_{\tau_0^-}}{X_{\tau_0^- -}}\right) \otimes \hat{X}_{\hat{\tau}_0^-}$$

Note $\Psi_c(\theta) = q^* \int (1 - e^{i\theta}) F_c(dx)$
$= q^* \left(1 - \mathbb{E}\left[e^{i\theta \Delta \xi^c_{T_1}}\right]\right)$

$$\mathbb{E}\left[e^{i\theta \Delta \xi^c_{T_1}}\right] = \mathbb{E}_1\left[\left(\frac{X_{\tau_0^-}}{X_{\tau_0^-}}\right)^{i\theta}\right] \times \mathbb{E}_1\left[(-\hat{X}_{\hat{\tau}_0})^{i\theta}\right]$$

Recall:

$$\mathbb{P}(X_{\tau_a^+} - a \in du,\ a - X_{\tau_a^+-} \in dv,\ a - \bar{X}_{\tau_a^+-} \in dy)$$
$$= \frac{\sin \alpha\rho\pi}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \frac{(a-y)^{\alpha\hat{\rho}-1} (v-y)^{\alpha\rho-1}}{(v+u)^{1+\alpha}} \, dy\,dv\,du$$

$$\mathbb{E}_1\left[(-\hat{X}_{\hat{\tau}_0})^{i\theta}\right] = \mathbb{E}\left[(X_{\tau_1^+} - 1)^{i\theta}\right]$$
$$\cdots = \frac{\sin(\pi\alpha\rho)}{\pi} \int_0^\infty t^{i\theta - \alpha\rho} (1+t)^{-1}\, dt$$
$$= \frac{\sin \pi\alpha\rho}{\pi} \Gamma(1 - \alpha\rho + i\theta)\Gamma(\alpha\rho - i\theta)$$

$$\mathbb{E}_1\left[\left(\frac{X_{\tau_0^-}}{X_{\tau_0^- -}}\right)^{i\theta}\right] = \mathbb{E}\left[\frac{(\hat{X}_{\hat{\tau}_1^+} - 1)^{i\theta}}{(1 - \hat{X}_{\hat{\tau}_1^+ -})^{i\theta}}\right]$$
$$= \frac{\sin \pi\alpha\hat{\rho}}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}$$
$$\times \int_0^1 \int_y^\infty \int_0^\infty \frac{u^{i\theta} (1-y)^{\alpha\hat{\rho}-1} (v-y)^{\alpha\rho-1}}{v^{i\theta} (v+u)^{1+\alpha}} \, du\,dv\,dy$$
$$\overset{\vdots}{=} \frac{\Gamma(i\theta+1)\Gamma(\alpha - i\theta)}{\Gamma(\alpha)}$$

In conclusion: Lemma, $\theta \in \mathbb{R}$,

$$\mathbb{E}\left[e^{i\theta \Delta \xi^c_{T_1}}\right] = \frac{\sin(\pi\alpha\rho)}{\pi\Gamma(\alpha)} \Gamma(1 - \alpha\rho + i\theta)\Gamma(\alpha\rho - i\theta) \times \Gamma(1 + i\theta)\Gamma(\alpha - i\theta).$$

With stamina!

$$\mathbb{P}(\Delta \xi^c_{T_1} \in dx) = \frac{\alpha\Gamma(\alpha\rho+1)\Gamma(\alpha\hat{\rho}+1)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)\Gamma(\alpha+2)} \times e^{-\alpha\rho x} {}_2\mathcal{F}_1(1, \alpha\rho+1; \alpha+2; 1-e^{x})$$

[Need to understand ${}_2\mathcal{F}_1(a, b; c, x)$ in the sense of analytic continuation]

Putting the pieces together

$$\vec{\Psi}(\theta) = \Psi_L(\theta) + \Psi_C(\theta)$$

$$= \left(\Psi^*(\theta) - \cancel{q_L^*}\right) + \cancel{q_L^*}\left(1 - \mathbb{E}\left[e^{i\theta\Delta\xi_{T_1}}\right]\right)$$

$$= \Psi^*(\theta) - q_L^* \mathbb{E}\left[e^{i\theta\Delta\xi^c_{T_1}}\right]$$

$$= \frac{\boxed{\Gamma(\alpha - i\theta)}\ \boxed{\Gamma(1+i\theta)}}{\Gamma(\alpha\hat{\rho} - i\theta)\ \Gamma(1-\alpha\hat{\rho}+i\theta)}$$

$$\frac{\cancel{\Gamma(\alpha)}}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}\left[\frac{1}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \quad \frac{\Gamma(1-\alpha\rho+i\theta)\Gamma(\alpha\rho-i\theta)\boxed{\Gamma(1+i\theta)}\boxed{\Gamma(\alpha-i\theta)}}{\cancel{\Gamma(\alpha)}}\right]$$

$$= \Gamma(\alpha - i\theta)\Gamma(1+i\theta)$$

$$\times\left[\frac{1}{\Gamma(\alpha\hat{\rho}-i\theta)\Gamma(1-\alpha\hat{\rho}+i\theta)} - \frac{\Gamma(1-\alpha\rho+i\theta)\Gamma(\alpha\rho-i\theta)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)}\right]$$

$$= \Gamma(\alpha - i\theta)\Gamma(1+i\theta)\Gamma(1-\alpha\rho+i\theta)\Gamma(\alpha\rho-i\theta)$$

$$\times\left[\frac{\sin(\alpha\hat{\rho}-i\theta)\pi \quad \sin(\alpha\rho-i\theta)\pi}{\pi^2} - \frac{\sin(\alpha\hat{\rho}\pi)\sin(\alpha\rho\pi)}{\pi^2}\right]$$

$$\sin(\pi(\alpha\hat{\rho}-i\theta))\sin(\pi(\alpha\rho-i\theta)) + \boxed{\sin(\pi i\theta)\sin(\pi(\alpha-i\theta))}$$

$$= \sin(\pi\alpha\hat{\rho})\sin(\pi\alpha\rho)$$

$$= \cancel{\Gamma(\alpha-i\theta)}\cancel{\Gamma(1+i\theta)}\boxed{\Gamma(1-\alpha\rho+i\theta)}\ \boxed{\Gamma(\alpha\rho-i\theta)}$$

$$\times\left[\cancel{-}\frac{1}{\cancel{\Gamma(i\theta)}\underbrace{\Gamma(1-i\theta)}_{-i\theta\Gamma(-i\theta)}} \cdot \frac{1}{\cancel{\Gamma(\alpha-i\theta)}\Gamma(1-\alpha+i\theta)}\right] \qquad \Gamma(1+z) = z\Gamma(z)$$

$$= \frac{\boxed{\Gamma(\alpha\rho-i\theta)}\ \boxed{\Gamma(1-\alpha\rho+i\theta)}}{\Gamma(-i\theta)\ \Gamma(1-\alpha+i\theta)}$$

$$\in HG(\underbrace{\beta, \gamma, \hat{\beta}, \hat{\gamma}})$$

$$= (1, \alpha\rho, 1-\alpha, \alpha\hat{\rho}) \in \underset{\text{f } \alpha\in(1,2)}{A_3}$$

$$\in A_1 \text{ f } \alpha\in(0,1)$$