Hypergeometric Lévy process

Philanthropy:

choose $\kappa, \hat{\kappa} \to$ build $\Psi(\theta) = \kappa(-i\theta)\hat{\kappa}(i\theta)$

$\kappa, \hat{\kappa}$ need have Lévy measures abs. cts and with non-increasing densities

Candidate class to choose $\kappa, \hat{\kappa}$ from

β-subordinators (α, β, γ)

$$\kappa(\lambda) = q + \delta\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \, \nu(dx)$$

$$= (\lambda + \alpha) \frac{\Gamma(\lambda + \beta - \gamma)}{\Gamma(\lambda + \beta + 1)} \qquad 0 \le \alpha \le \beta + \gamma, \quad \gamma \in [0, 1)$$

$\Gamma(1+z) = z\Gamma(z)$

$$\frac{\nu(dx)}{dx} = \frac{1}{\Gamma(1-\gamma)} (1-e^{-x})^{-\gamma} e^{-(\beta+\gamma)x} \left[\frac{\gamma}{1-e^{-x}} + \beta - \alpha \right]$$

$$\frac{U(dx)}{dx} = \frac{e^{-\alpha x}}{\Gamma(\gamma)} (1-e^{-x})^{\gamma-1} e^{-(1+\beta-\alpha)x}$$

$$+ \frac{e^{-\alpha x}}{\Gamma(\gamma)} (\beta+\gamma-\alpha) \int_0^x (1-e^{-u})^{\gamma-1} e^{-(1+\beta-\alpha)u} \, du$$

$$\alpha = \beta > 0 \,, \quad \gamma \in (0,1)$$

$$k(\lambda) = \frac{\Gamma(\lambda+\beta+\gamma)}{\Gamma(\lambda+\beta+1)}$$

$\underline{\alpha = \beta}$

$$\frac{U(dx)}{dx} = u(x) = \frac{1}{\Gamma(\gamma)} e^{-\beta x}\left(1-e^{-x}\right)^{\gamma-1}$$

$$\frac{V(dx)}{dx} = v(x) = \frac{\gamma}{\Gamma(1-\gamma)}\left(1-e^{-x}\right)^{-(\gamma+1)} e^{-(\beta+\gamma)x}$$

Defn of Hypergeometric Lévy process

Def^n Two parameter sets $HG(\beta, \gamma, \hat\beta, \hat\gamma)$

$$A_1 = \{\beta \le 1,\ \gamma \in (0,1),\ \hat\beta \ge 0,\ \hat\gamma \in (0,1)\}$$

$$A_2 = \{\beta \in (1,2],\ \gamma (0,1),\ \hat\beta \in [-1,0),\ \hat\gamma \in (0,1),\ 1-\beta+\hat\beta+(\gamma \wedge \hat\gamma) \ge 0\}$$

Theorem Let $A = A_1 \cup A_2$

(1) For $(\beta, \gamma, \hat\beta, \hat\gamma) \in A$, $\exists$ a (killed) L.p. henceforth called a hypergeometric L.p. $HG(\beta, \gamma, \hat\beta, \hat\gamma)$ s.t its char. exp.

$$\Psi(z) = \frac{\Gamma(1-\beta+\gamma-iz)}{\Gamma(1-\beta-iz)} \times \frac{\Gamma(\hat\beta+\hat\gamma+iz)}{\Gamma(\hat\beta+iz)}$$

(iv) The Lévy measure of Y has density w.r.t Leb given by

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\hat{\gamma})\Gamma(-\gamma)} e^{-(1-\beta+\gamma)x} {}_2F_1(1+\gamma, \eta; \eta-\hat{\gamma}; e^{-x}) & x>0 \\ -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\gamma)\Gamma(-\hat{\gamma})} e^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1(1+\hat{\gamma}, \eta, \eta-\gamma, e^{x}) & \end{cases}$$

$$\boxed{\eta = 1-\beta+\gamma+\hat{\beta}+\hat{\gamma}}$$

$z \in \mathbb{C}$ $\quad {}_2F_1(a, b, c, z) = \sum_{k \geq 0} \dfrac{(a)_k (b)_k}{(c)_k k!} z^k \quad |z|<1$

note $(b)_k = \cancel{\dfrac{\Gamma(b+k)}{\Gamma(b)}} = b(b-1) \cdots (b-k+1)$

(iii) $(\beta, \gamma, \hat{\beta}, \hat{\gamma}) \in \mathcal{A}_1$ then ascending ladder process is a $(1-\beta, 1-\beta, \gamma)$ β-subordinator and descending ladder process is a $(\hat{\beta}, \hat{\beta}, \hat{\gamma})$ β-subordinator

(iv) $(\beta, \gamma, \hat{\beta}, \hat{\gamma}) \in \mathcal{A}_2$ then ascending ladder process (resp. descending) is a $(-\hat{\beta}, 1-\beta, \gamma)$ β subordinator (resp. $(\beta-1, \hat{\beta}, \hat{\gamma})$ β-subordinator

(iii) $$\Psi(z) = \frac{\Gamma(1-\beta+\gamma-iz)}{\Gamma(1-\beta-iz)} \times \frac{\Gamma(\hat{\beta}+\hat{\gamma}+iz)}{\Gamma(\hat{\beta}+iz)}$$

(iv) $$\Psi(z) = (-\hat{\beta}-iz)\frac{\Gamma(1-\beta+\gamma-iz)}{\Gamma(2-\beta-iz)} \times (\beta-1+iz)\frac{\Gamma(\hat{\beta}+\hat{\gamma}+iz)}{\Gamma(1+\hat{\beta}+iz)}$$

A subclass of hypergeometric Lévy processes

Defn Lamperti-Stable Lévy process

If $\beta, \hat{\beta}$ are in A_1 regime <u>and</u> equal

Then our $HG(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ is called a $LS(\beta, \gamma, \hat{\gamma})$

(Lamperti stable)

$$(\beta, \gamma, \hat{\gamma}) \in A_3 := \{\beta \in [0,1], \gamma \in (0,1), \hat{\gamma} \in (0,1)\}$$

In the LS class: $\eta = 1 + \gamma + \hat{\gamma}$

$$ {}_2F_1(1+\gamma, \eta, \eta - \hat{\gamma}; e^{-x})$$

$$= \sum_{k \geq 0} \frac{\cancel{(1+\gamma)_k}\,(1+\gamma+\hat{\gamma})_k}{\cancel{(1+\gamma)_k}\;k!} e^{-kx}$$

Use Binomial expansion $(1-z)^{-(1+a)} = \sum_{k \geq 0} \frac{(1+a)_k}{k!} z$

$z \in \mathbb{C}$ $|z| < 1$

Using also reflection formula for Γ-function

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z} \qquad z \in \mathbb{C},\ z \neq \{-1,-2,\dots\}$$

$$-\frac{\Gamma(\eta)}{\Gamma(\eta-\hat{\gamma})\,\Gamma(-\gamma)} = -\frac{\Gamma(1+\gamma+\hat{\gamma})}{\Gamma(1+\gamma)\,\Gamma(-\gamma)}$$

$$= \frac{-\Gamma(1+\gamma+\hat{\gamma})}{\gamma\,\Gamma(\gamma)\,\Gamma(-\gamma)}$$

$$= \frac{\Gamma(1+\gamma+\hat{\gamma})}{\Gamma(\gamma)\,\Gamma(1-\gamma)}$$

__Lemma__ For $LS(\beta,\gamma,\hat{\gamma})$ $\quad (\beta,\gamma,\hat{\gamma}) \in A_3$

$$\pi(x) = \frac{\Gamma(1+\gamma+\hat{\gamma})}{\Gamma(\hat{\gamma})\,\Gamma(1-\hat{\gamma})}\,\frac{e^{(\beta+\hat{\gamma})x}}{(e^{x}-1)^{1+\gamma+\hat{\gamma}}}\,\mathbb{1}_{(x>0)} + \frac{\Gamma(1+\gamma+\hat{\gamma})}{\Gamma(\hat{\gamma})\,\Gamma(1-\hat{\gamma})}\,\frac{e^{-(1-\beta+\gamma)x}}{(e^{-x}-1)^{1+\gamma+\hat{\gamma}}}\,\mathbb{1}_{(x<0)}$$

for $x>0$ and large

$$\frac{e^{(\beta+\hat{\gamma})x}}{(e^{x}-1)^{1+\gamma+\hat{\gamma}}} = O\left(e^{(\beta-1-\gamma)x}\right)$$

for $x>0$ and small

$$\text{———}\mid\text{———} = O\left(\frac{1}{x^{1+\gamma+\hat{\gamma}}}\right)$$

(nice for mathematical finance!!)

$$\Psi^*(\theta) = \frac{\Gamma(\alpha + i\theta)}{\Gamma(\alpha\hat{\rho} - i\theta)} \times \frac{\Gamma(1 + i\theta)}{\Gamma(1 - \alpha\hat{\rho} + i\theta)}$$

note that Ψ^* is $LS(1-\alpha\hat{\rho}, \alpha\rho, \alpha\hat{\rho})$

$$\Psi^*(0) = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \quad \text{killing !}$$

$$X^*_t = e^{\xi^*_{\varphi(t)}}$$

<u>Consider</u>

$$\Psi^\uparrow(\theta) = \frac{\Gamma(\alpha\rho - i\theta)}{\Gamma(-i\theta)} \frac{\Gamma(1 + \alpha\hat{\rho} + i\theta)}{\Gamma(1 + i\theta)}$$

$$\Psi^\uparrow(\theta) = \Psi^*(\theta - i\alpha\hat{\rho}) - \Psi^*(-i\alpha\rho)$$

Esscher transform

Suppose X is a L.p. with char. exp. Ψ

define Laplace exponent:

$\lambda \in \mathbb{R}$ $\quad \mathbb{E}\left[e^{\lambda X_t}\right] = e^{\psi(\lambda) t}$ (*) whenever it exists

$\boxed{\text{if it exists}}$ $\psi(\lambda) = -\Psi(-i\lambda)$

$\mathcal{E}_t(\lambda) := e^{\lambda X_t - \psi(\lambda) t}$ is a mg.

$$\mathbb{E}\left[\mathcal{E}_{t+s}(\lambda) \mid \mathcal{F}_t\right] = \mathcal{E}_t(\lambda) \underbrace{\mathbb{E}\left[e^{\lambda \overbrace{(X_{t+s} - X_t)}^{\overset{X_s \perp \mathcal{F}_t}{\overset{\|d}{}}} - \psi(\lambda)s} \,\middle|\, \mathcal{F}_t\right]}_{= 1 \text{ by (*)}}$$

Introduce change of measure $\left.\dfrac{d\mathbb{P}^\lambda}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}_t(\lambda)$

Under $\mathbb{P}^\lambda$, X is a L.P. with char. exp $\Psi_\lambda(\theta) = \Psi(\theta - i\lambda) - \Psi(-i\lambda)$

$$e^{-\Psi_\lambda(\theta)} := \mathbb{E}^\lambda\left[e^{i\theta X_t}\right] = \mathbb{E}\left[e^{i\theta X_t} e^{\lambda X_t - \psi(\lambda)t}\right]$$

$$= \mathbb{E}\left(e^{i(\theta - i\lambda)X_t + \Psi(-i\lambda)t}\right)$$

$$= e^{-\Psi(\theta - i\lambda)t + \Psi(-i\lambda)t}$$

$$\boxed{\Psi_\lambda(\theta) = \Psi(\theta - i\lambda) - \Psi(-i\lambda)}$$

Recalling that $\Psi^\uparrow(\theta) = \Psi^*(\theta - i\alpha\hat{\rho}) - \Psi^*(-i\alpha\hat{\rho})$

we see that $\Psi^\uparrow(\theta)$ is the result of an Esscher transform to Ψ^*

(obvious notation) $\left.\frac{d\mathbb{P}^\uparrow}{d\mathbb{P}^*}\right|_{\mathcal{F}_t} = e^{\alpha\hat{\rho}\xi^*_t - \cancel{\psi^*(\alpha\hat{\rho})}t}$ $\qquad \underline{\psi^*(\lambda) = -\Psi^*(-i\lambda)}$

lamperti-stable $X^*_t = e^{\xi^*_{\varphi(t)}}$; $\varphi(t) = \inf\{s : \int_0^s e^{\alpha\xi^*_u}du > t\}$

for any stopping time τ for ξ^*

$$\frac{d\mathbb{P}^\uparrow}{d\mathbb{P}^*}\Big|_{\mathcal{F}_\tau} = e^{\alpha\hat{\rho}\,\xi^*_\tau}\,\mathbb{1}_{(\tau<\infty)}$$

specifically: $\mathbb{E}^\uparrow\left[f(\xi^*_\tau)\mathbb{1}_{(\tau<\infty)}\right]$

$$= \mathbb{E}^*\left[e^{\alpha\hat{\rho}\,\xi^*_\tau} f(\xi^*_\tau)\,\mathbb{1}_{(\tau<\infty)}\right]$$

choose $\varphi(t) = \tau$ $\quad \{\varphi(t) < \infty\} = \{t < \zeta\}$

absorption at zero of X^*

$$\mathbb{E}_1^\uparrow\left[g(X_t^*)\,\mathbb{1}_{(t<\zeta)}\right]$$

$$= \mathbb{E}^*\left[(X_t^*)^{\alpha\hat{\rho}}\, g(X_t^*)\,\mathbb{1}_{(t<\zeta)}\right]$$

<u>In fact</u>: (exercise)

$$\left(X^*_t\right)^{\alpha\hat\rho}\mathbb{1}_{(t<\zeta)} \quad \text{is a mg}$$

$x>0$ $$\boxed{\left.\frac{d\mathbb{P}^\uparrow_x}{d\mathbb{P}_x}\right|_{\mathcal{G}_t} = \frac{\left(X^*_t\right)^{\alpha\hat\rho}}{x^{\alpha\hat\rho}}\mathbb{1}_{(t<\overset{\tau_0^-}{\zeta})}} \qquad \mathcal{G}_t = \sigma(X^*_u : u\le t)$$

$\hookrightarrow$ induces a new pssMp with underlying L.p. having char. exponent

$\Psi^\uparrow(\theta)$ as before.

Why notation $\mathbb{P}^\uparrow$?

The change of measure is a special example of (for any L.p. X)

$$\left.\frac{d\mathbb{P}^\uparrow_x}{d\mathbb{P}_x}\right|_{\mathcal{G}_t} = \frac{\hat{u}(\cancel{X}_t)}{\hat{u}(x)}\mathbb{1}_{(t<\tau_0^-)}$$

In the general case $(\mathbb{P}^\uparrow, X)$ is called the Lévy process X conditioned to stay positive.

<u>Hence</u> $\Psi^\uparrow(\theta) = \Psi^*(\theta - i\alpha\hat\rho)$ is nothing more than stable process X conditioned to stay positive! $\boxed{\text{Which is also a pssMp.}}$

because Lamperti transform is still in place under $\mathbb{P}^\uparrow$

$$\Psi^\uparrow(0) = \Psi^*(-i\alpha\hat\rho) = 0 \Rightarrow$$

(underlying Lévy process, say $\xi^\uparrow$ has no killing!)

<u>Exercise</u> $\Psi^{\uparrow\prime}(0) = \mathbb{E}^\uparrow(\xi_1^\uparrow) > 0.$

$$\xi^\uparrow_t \xrightarrow{t\to\infty} \infty$$

Can do this story all over again using pole ~~[illegible]~~ $\Gamma(1-\alpha\hat{\rho}+i\theta)$ at

$$\theta = i(1-\alpha\hat{\rho})$$

$$\Psi^{\downarrow}(\theta) := \Psi^{*}(\theta + i(1-\alpha\hat{\rho})) - \underbrace{\Psi^{*}(i(1-\alpha\hat{\rho}))}_{=0}$$

$\Psi^{\downarrow}$ is the Esscher transform of Ψ^{*} using the martingale

$$\left.\frac{d\mathbb{P}^{\downarrow}}{d\mathbb{P}^{*}}\right|_{\mathcal{F}_t} = e^{-(1-\alpha\hat{\rho})\xi^{*}_t}$$

$\longrightarrow$ induces change of measure on X^{*}_t

$$\left.\frac{d\mathbb{P}^{\downarrow}_x}{d\mathbb{P}^{*}_x}\right|_{\mathcal{G}_t} = \frac{(X^{*}_t)^{\alpha\hat{\rho}-1}}{x^{\alpha\hat{\rho}-1}} \mathbb{1}(t < \zeta)$$

τ_0^-

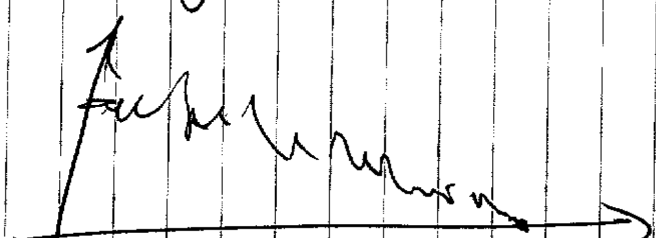
<u>Note</u>: for a suitable class of Lévy processes, X,

$\exists$ a density of $\hat{U}(dx) = \hat{u}(x)\,dx$

[stable case $\hat{U}(x) \propto x^{\alpha\hat{\rho}}$ $\rightarrow$ $\hat{u}(x) \propto x^{\alpha\hat{\rho}-1}$]

$$\frac{d\mathbb{P}_x^{\downarrow}}{d\mathbb{P}_x}\bigg|_{\mathcal{F}_t} = \frac{\hat{u}(X_t)}{\hat{u}(x)}\,\mathbb{1}_{(t<\tau_0^-)}$$

corresponds to conditioning the L.p. to <u>die</u> continuously at 0



$\xi^{\uparrow}$ has char. exponent $\Psi^{\uparrow}(\theta) = \Psi^{*}(\theta + i(1-\alpha\hat{\rho}))$

<u>Hence</u> $\mathbb{P}^{\uparrow}$ for stable processes gives a new pssMp = stable process conditioned to die at 0 continuously

Finally: check

$\overline{\Psi}^*$, $\overline{\Psi}^\uparrow$, $\overline{\Psi}^\downarrow$ are all

LS Lévy processes.