Remains to prove that

in case (1) $\limsup_{t\to\infty} \xi_t = \infty$

Case (2) $\lim_{t\to+\infty} \xi_t = -\infty$.

Recall that from previous calculations, we can represent

$$\zeta^{(x)} = \inf\{t>0 : X_t^{(x)} = 0\}$$
$$= x^\alpha \int_0^\infty e^{\alpha \xi_s}\, ds.$$

$$\left[\quad \xi_t = \log\left(X^{(x)}_{\boxed{x^\alpha I_t}}/x\right) \quad \right]$$

We need a Lemma

(No killing on ξ)

$$\boxed{\int_0^\infty e^{\alpha \xi_t}\, dt < \infty} \text{ a.s.}$$
$$\iff \lim_{t\to\infty} \xi_t = -\infty$$

Note: $\forall$ L.P. we have either

$\lim_{t\to\infty} \xi_t = \infty$ (a.s) or $\lim_{t\to\infty} \xi_t = -\infty$ (a.s).

or $\limsup_{t\to\infty} \xi_t = -\liminf_{t\to\infty} \xi_t = \infty$ (a.s)

Infact: either $\lim_{t\to\infty} \frac{\xi_t}{t} = C_+ \in (0,\infty]$ a.s. $\Rightarrow \int_0^\infty e^{\alpha\xi_t}\, dt = \infty$

or $\lim_{t\to\infty} \frac{\xi_t}{t} = C_- \in [-\infty, 0)$ a.s $\Rightarrow \int_0^\infty e^{\alpha \xi_t}\, dt < \infty$

or $\underbrace{\limsup_{t\to\infty} \xi_t = -\liminf_{t\to\infty} \xi_t = \infty.}$

look at $T_1 = \inf\{t>0 : \xi_t > 2\}$

$S_1 = \inf\{t > T_1 : \xi_t < 1\}$

$T_2 = \inf\{t > S_1 : \xi_t > 2\}$

$S_2 = \inf\{t > T_2 : \xi_t < 1\}$

$\vdots$

$$I_\infty := \int_0^\infty e^{\alpha \xi_t}\, dt \geq e^\alpha \sum_{n=1}^\infty \underbrace{(S_n - T_n)}$$

$\hookrightarrow$ stochastically dominating indep. copies of $\tau_{-1}^- = \inf\{s>0 : \xi_s < -1\}$ under $\xi_0 = 0$.

i.e. $I_\infty \geq I'_\infty =^d e^\alpha \sum_{n\geq 1} \tau^-_{-1}(n) = \infty$ a.s.

where $\tau^-_{-1}(n) \sim$ iid τ^-_{-1}

Stable Process :

A stable process is a Lévy process which respects the scaling property

$$\{ c Y^{(x)}_{c^{-\alpha} t} : t \geq 0 \} \overset{d}{=} \{ Y^{(cx)}_t : t \geq 0 \}$$

Note : A stable process is NOT a pssMp !!! *
because in general Y is $\mathbb{R}$-valued !

Note : necessarily $\alpha \in (0,2]$ with $\alpha = 2$ being (scaled) BM.

First parenthesis on Lévy processes :

A L.P. (1) $\xi_0 = 0$

(2) $0 \leq s < t < \infty$, $\xi_t - \xi_s \perp\!\!\!\perp \{ \xi_u : u \leq s \}$

(3) —— | —— $\xi_t - \xi_s \overset{d}{=} \xi_{t-s}$

(4) ξ has a.s right-cts & left limits in its paths.

* unless Y is a stable subordinator

Characteristic exponent:

$$(*)\quad \xi_t = \xi_{t/n} + \underbrace{\left(\xi_{\frac{2t}{n}} - \xi_{\frac{t}{n}}\right)}_{=^d \xi_{t/n}} + \underbrace{\left(\xi_{\frac{3t}{n}} - \xi_{\frac{2t}{n}}\right)}_{=^d \xi_{t/n}} + \cdots + \underbrace{\left(\xi_t - \xi_{\frac{t(n-1)}{n}}\right)}_{=^d \xi_{\frac{t}{n}}}$$

$n \in \mathbb{N}$. (distⁿ of ξ_t is infinitely divisible).

$$\Psi_t(\theta) := -\log \mathbb{E}\left[e^{i\theta\xi_t}\right]$$

$(*) \Rightarrow$ on the one hand $\quad \Psi_m(\theta) = m\Psi_1(\theta)$

on the other hand $\quad \Psi_m(\theta) = n\Psi_{m/n}(\theta) \qquad m, n \in \mathbb{N}$

$$\Rightarrow \Psi_{\frac{m}{n}}(\theta) = \frac{m}{n}\Psi_1(\theta)$$

$$\forall t \in \mathbb{Q} \cap [0,\infty) \quad \Psi_t(\theta) = t\,\boxed{\Psi_1(\theta)} =: \Psi(\theta)$$

$$(**)\quad \mathbb{E}\left[e^{i\theta\xi_t}\right] = e^{-t\Psi(\theta)} \qquad \text{for } t \in \mathbb{Q} \cap [0,\infty)$$

use DCT + right-continuity to deduce ($t_n \in \mathbb{Q}$: $t_n \downarrow t \in [0,\infty)$)

$\Rightarrow (**)$ holds $\forall t$.

$\Psi(\theta)$ is the char. exp. of $\xi_1 \longleftarrow$ inf.-div. distⁿ.

all inf.-div. distⁿs necessarily have char. exponent of the form (Lévy–Khintchine formula)

$$\Psi(\theta) = \boxed{ia\theta + \frac{1}{2}\sigma^2\theta^2} + \boxed{\int_{\mathbb{R}} \left(1 - e^{i\theta x} + i\theta x \mathbb{1}_{(|x|<1)}\right)\Pi(dx)}$$

$a \in \mathbb{R}$, $\sigma^2 \geq 0$ $\quad \Pi$ is a measure on $\mathbb{R}\setminus\{0\}$

s.t. $\displaystyle\int_{\mathbb{R}} (1 \wedge x^2)\,\Pi(dx) < \infty$.

In fact: <u>Theorem</u> Class of L.P.s corresponds to the class of inf. div. distⁿ through the L-K-formula.

Exercise: Consider a compound Poisson process with drift: λ arrival rate, iid jumps have dist^n F and c is the drift.

$$\xi_t = ct + \sum_{i=1}^{N_t} J_i \quad : \quad J_i \overset{iid}{\sim} F.$$

N_t — PP(λ)

is a L.P.

$$\mathbb{E}\left[e^{i\theta \xi_t}\right] = e^{i\theta ct}\,\mathbb{E}\left[e^{i\theta \sum_{k=1}^{N_t} J_k}\right]$$

$$= e^{i\theta ct}\,\mathbb{E}\left[\mathbb{E}\left[e^{i\theta J_1}\right]^{N_t}\right]$$

$$= e^{i\theta ct}\,\mathbb{E}\left[\left(\int_{\mathbb{R}} e^{i\theta x} F(dx)\right)^{N_t}\right]$$

$$\mathbb{E}\left[s^{N_t}\right] = e^{\lambda t(s-1)}$$

$$= e^{i\theta ct} \exp\left(\lambda t \left(\int_{\mathbb{R}} \left(e^{i\theta x} - 1\right) F(dx)\right)\right)$$

$$= e^{-t\Psi(\theta)}$$

$$\Psi(\theta) = -ci\theta + \boxed{\int_{\mathbb{R}} \left(1 - e^{i\theta x}\right)\lambda F(dx).}$$

Suppose $\int_{\mathbb{R}} |x| F(dx) < \infty$ we take $c = -\int_{\mathbb{R}} x \lambda F(dx)$

(Exercise: this makes ξ_t a m.g.)

also $$\boxed{\Psi(\theta) = \int_{\mathbb{R}} \left(1 - e^{i\theta x} + i\theta x\right) \lambda F(dx)}$$

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(1 - e^{i\theta x} + i\theta x \mathbb{1}_{(|x|<1)}\right) \Pi(dx)$$

$$\int_{|x|\geq 1} \left(1 - e^{i\theta x}\right) \frac{\Pi(dx)}{\underbrace{\Pi(x : |x| \geq 1)}_{F_0(dx)}} \cdot \underbrace{\Pi(x : |x| \geq 1)}_{\lambda_0}$$

$$+ \sum_{n=1}^{\infty} \int_{|x| \in [2^{-n}, 2^{-(n-1)})} \left(1 - e^{i\theta x} + i\theta x\right) \frac{\Pi(dx)}{\underbrace{\Pi(|x| \in [2^{-n}, 2^{-(n-1)}))}_{F_n(dx)}} \cdot \underbrace{\Pi(|x| \in [2^{-n}, 2^{-(n-1)}))}_{\lambda_n}$$

Note: $\int_{\mathbb{R}} (1 \wedge x^2) \Pi(dx) < \infty$

$$\Rightarrow \Pi(A) < \infty \quad \forall A \text{ s.t. } \{0\} \notin \bar{A}$$

Note: If ξ^1 and ξ^2 are two indep. L.P. with exponents $\Psi^1(\theta)$ & $\Psi^2(\theta)$

$$\mathbb{E}\left[e^{i\theta(\xi_t^1 + \xi_t^2)}\right] = e^{-t(\Psi^1(\theta) + \Psi^2(\theta))}$$

and $\xi_t^1 + \xi_t^2$ is a L.P.

Can we add an infinite # L.P. together?

In the exponent

$$\lim_{k \uparrow \infty} \sum_{n=1}^{k} \int_{\boxed{|x| \in [2^{-n}, 2^{-(n-1)})}} \left(1 - e^{i\theta x} + i\theta x\right) \Pi(dx)$$

corresponds to the "infinite" sum of centred CPPs which are also mgs.

an the sum of processes converges thanks to L^2-mg theory which will use $\int_{(-1,1)} x^2 \Pi(dx) < \infty$

How do stable processes fit into this picture?

Usual def^n: $\sigma = 0$ $\quad (\alpha \in (0,2))$

$$\Pi(dx) = \frac{c_1}{x^{1+\alpha}} \mathbb{1}_{(x>0)}\, dx + \frac{c_2}{|x|^{1+\alpha}} \mathbb{1}_{(x<0)}\, dx$$

[A jump of size x arrives in $(t, t+dt)$ w.p. $\Pi(dx)\,dt + o(dt)$]

There exists an appropriate choice of $a \in \mathbb{R}$ s.t.

$$\Psi(\theta) = c|\theta|^\alpha \left(1 - i\beta \tan(\pi\alpha/2)\,\mathrm{sgn}(\theta)\right)$$

$$c = -(c_1+c_2)\cos(\pi\alpha/2)\,\Gamma(-\alpha)$$

$$\beta = \frac{c_1 - c_2}{c_1 + c_2}$$

Note: $$\mathbb{E}\left[e^{i\theta(cY_{c^{-\alpha}t})}\right] = \mathbb{E}\left[e^{i(c\theta)Y_{c^{-\alpha}t}}\right] = e^{-c^{-\alpha}t\,\Psi(c\theta)} = e^{-t\,\Psi(\theta)}$$

Henceforth: we will use a particular normalisation of Ψ.

Theorem Because of scaling $\mathbb{P}(Y_t > 0) = \rho$ where ρ is a constant in $[0,1]$ and

$$\boxed{\rho = \frac{1}{2} + \frac{1}{\pi\alpha}\tan^{-1}\left(\beta \tan(\pi\alpha/2)\right)}$$

By choosing $c_1 = \Gamma(1+\alpha)\dfrac{\sin(\pi\alpha\rho)}{\pi}$

$c_2 = \Gamma(1+\alpha)\dfrac{\sin(\pi\alpha\hat{\rho})}{\pi} \qquad \hat{\rho} = 1-\rho$

then

$$\Psi(\theta) = |\theta|^\alpha \left(e^{\pi i\alpha(\frac{1}{2}-\rho)}\mathbb{1}_{(\theta>0)} + e^{-\pi i\alpha(\frac{1}{2}-\rho)}\mathbb{1}_{(\theta<0)}\right)$$

In fact the parameter range of (α, ρ) turns out to be a bit more restrictive than $\alpha \in (0,2)$ and $\rho \in [0,1]$

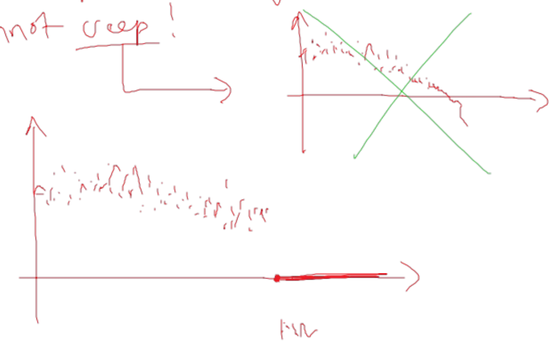
$$A = \{\alpha \in (0,1),\ \rho \in [0,1]\} \cup \underbrace{\{\alpha = 1,\ \rho = \tfrac{1}{2}\}}_{\text{Cauchy Process}}$$

$$\cup \{\alpha \in (1,2),\ \rho \in [1-\tfrac{1}{\alpha}, \tfrac{1}{\alpha}]\}$$

Two examples of pssMp using stable Lévy processes

(1) $$X_t := Y_t \, \mathbb{1}_{(Y_t \geq 0)} = Y_t \, \mathbb{1}_{(t < T_0^-)}$$

[Note: a stable process always jumps across the origin i.e. cannot creep!



(2) $$A_t = \int_0^t \mathbb{1}_{(Y_s > 0)} \, ds$$

$$\gamma(t) = A_t^{-1} \quad \text{i.e.} \quad \int_0^{\gamma(t)} \mathbb{1}_{(Y_s > 0)} \, ds = t$$

$$\check{X}_t = Y_{\gamma(t)}$$

$$T_0 = \inf\{t > 0 : \check{X}_t = 0\}.$$

$$X_t = \check{X}_t \, \mathbb{1}_{(t < T_0)}$$