

Quasi-stationary distributions for Lévy processes

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Abstract

In recent years there has been some focus on the behaviour of one dimensional Lévy processes and random walks conditioned to stay positive; see for example Bertoin (1993, 1996), Bertoin and Doney (1994), Chaumont (1996) and Chaumont and Doney (2004). The resulting conditioned process is transient. In older literature however, one encounters for special classes of random walks and Lévy processes a similar, but nonetheless different, type of asymptotic conditioning to stay positive which results in a limiting quasi-stationary distribution. We continue this theme into the general setting of a Lévy process fulfilling certain types of conditions which are analogues of known classes in the random walk literature. Our results generalize those of Kyprianou (1971) for special types of one sided compound Poisson processes with drift and Martinez and San Martin (1994) for Brownian motion with drift and complement the results of Iglehart (1974) and Doney (1989) and Bertoin and Doney (1996) for random walks.

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1 Introduction

Denote by $X = \{X_t : t \geq 0\}$ a Lévy process defined on the filtered space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ is assumed to satisfy the usual assumptions of right continuity and completion. We work with the probabilities

$\{P_x : x \in \mathbb{R}\}$ such that $P_x(X_0 = x) = 1$ and $P_0 = P$. The probabilities $\{\widehat{P}_x : x \in \mathbb{R}\}$ are defined in a similar sense for the dual process, $-X$.

Define the first passage time into the lower half line $(-\infty, 0)$ by

$$\tau = \inf\{t > 0 : X_t < 0\}.$$

In this short paper, the principal object of interest is the existence and characterization of the so-called limiting quasi-stationary distribution (or Yaglom's limit)

$$\lim_{t \uparrow \infty} P_x(X_t \in B | \tau > t) = \mu(B) \quad (1)$$

for $B \in \mathcal{B}([0, \infty))$ where, in particular, μ does not depend on the initial state $x \geq 0$. The sense in which this limit is quasi-stationary follows the classical interpretations of works such as Seneta and Vere-Jones (1966), Tweedie (1974), Jacka and Roberts (1995) and other references therein. We consider the limit (1) for $x \geq 0$ in the case of Lévy process for which $(-\infty, 0)$ is irregular for 0 and for $x > 0$ in the case of Lévy process for which $(-\infty, 0)$ is regular for 0.

The existence and characterization result we obtain improves on the same result for spectrally positive compound Poisson processes with negative drift and jump distribution whose characteristic function is a rational function proved by Kyprianou (1971) (within the context of the $M/G/1$ queue), also on the same result for Brownian motion with drift proved by Martinez and San Martin (1994) and complements an analogous result for random walks obtained by Iglehart (1974).

In the next section we state the main result and give some special examples. Following that in Section 3 we give two preparatory lemmas and then the proof of the main result.

2 Main result

In order to state the main result it is necessary to introduce some notation. The method we appeal to requires the use of the classical exponential change of measure. Hence it is necessary to introduce the Laplace exponent of X given by the relation

$$E(e^{\theta X_t}) = e^{\psi(\theta)t}$$

whenever it is well-defined. Note, in that case $\psi(\theta) = -\Psi(-i\theta)$, where $\Psi(\lambda) = -\log E(e^{i\lambda X_1})$ is the Lévy-Khintchine exponent. Whenever $|\psi(\theta)|$ is well defined and finite the exponential change of measure referred to above takes the form

$$\left. \frac{dP_x^\theta}{dP_x} \right|_{\mathcal{F}_t} = e^{\theta(X_t - x) - \psi(\theta)t}.$$

It is well known that under this change of measure X is still a Lévy processes and its new Laplace exponent satisfies

$$\psi_\theta(\beta) = \psi(\theta + \beta) - \psi(\theta) \quad (2)$$

whenever $\psi(\theta+\beta)$ is well defined. One also sees from the latter equality that under the change of measure, the new Lévy measure satisfies $\Pi_\theta(dx) = e^{\theta x}\Pi(dx)$.

In order to state and prove the main result of this paper we need to recall some standard notation from the theory of fluctuations of Lévy processes; see for example Bertoin (1996). The process $(L^{-1}, H) = \{(L_t^{-1}, H_t) : t \geq 0\}$ denotes the ladder process of X and is characterised by its Laplace exponent $\kappa(\alpha, \beta)$ satisfying

$$E(e^{-\alpha L_t^{-1} - \beta H_t}) = e^{-\kappa(\alpha, \beta)t}$$

for $\alpha, \beta \geq 0$. The latter exponent together with the analogous quantity, $\widehat{\kappa}(\alpha, \beta)$ defined for the dual $-X$ are related to the renewal function

$$V(x) = E\left(\int_0^\infty \mathbf{1}_{(H_t \leq x)} dt\right)$$

via the Laplace transform

$$\lambda \int_0^\infty e^{-\lambda x} V(x) dx = \frac{1}{\kappa(0, \lambda)}. \quad (3)$$

The quantities \widehat{H} , \widehat{V} and $\widehat{\kappa}$ associated with $-X$ can also be identified in the obvious way. Further, κ_θ , $\widehat{\kappa}_\theta$, V_θ and \widehat{V}_θ all correspond to the aforementioned when working under the measure P^θ .

The main result will apply to two classes of Lévy process. We call them Class A and Class B and the definitions are given below. The choice of name for each class allows us to keep some consistency with existing literature. Analogues of Class A and B were defined for random walks in Bertoin and Doney (1996).

Before formally giving the definitions, recall that it can easily be checked from the Lévy-Khintchine formula that when $\theta_r = \sup\{\theta > 0 : |\psi(\theta)| < \infty\}$, ψ is strictly convex on $(0, \theta_r)$ and by monotonicity $\psi(\theta_r) = \psi(\theta_r-)$ and $\psi'(\theta_r) = \psi'(\theta_r-)$ are well defined. Also the right derivative at zero, $\psi'(0) = \psi'(0+) = E(X_1)$.

Definition 1 (Class A) *A Lévy process belongs to Class A if,*

- (i) *there exists a Laplace exponent ψ for $0 < \theta < \theta_r$, where $\theta_r > 0$,*
- (ii) *$\psi(\theta)$ attains its negative infimum at $0 < \theta_0 \leq \theta_r$ with $\psi'(\theta_0) = 0$,*
- (iii) *X_1 is non-lattice and*
- (iv) *(X, P^{θ_0}) is in the domain of attraction of a stable law with index $1 < \gamma \leq 2$.*

Definition 2 (Class B) *A Lévy process belongs to Class B if,*

- (i) *there exists a Laplace exponent ψ for $0 \leq \theta \leq \theta_r$, where $\theta_r \in (0, \infty)$,*
- (ii) *$\psi(\theta)$ attains its negative infimum at $\theta_0 = \theta_r$,*
- (iii) *$-E^{\theta_0}(X_1) = -\psi'(\theta_0) \in (0, \infty)$,*

- (iv) X_1 is non-lattice and
- (v) the function $x \mapsto \Pi_{\theta_0}(x, \infty)$ is regularly varying at infinity with index $-\beta$ where $\beta \in (2, \infty)$.

A number of important points that follow from these assumptions are now discussed.

1. The assumption that X_1 is non-lattice is not absolutely necessary for the results in this paper. Analogues of the required analysis of the Lévy processes in Classes A and B can in principle be reconstructed. To understand how, the reader is referred to the analogous definitions for random walks given in Bertoin and Doney (1996) as well as the respective results which are lifted from that paper in the analysis here.
2. In Class A, it is implicit in the definition that $E(X_1) = \psi'(0+) < 0$ and hence the process drifts to $-\infty$. Under the change of measure P^{θ_0} , the process X oscillates as, according to (2) and the definition of θ_0 , $\psi'_{\theta_0}(0-) = \psi'(\theta_0-) = 0$. Note also that if (X, P) belongs to Class A then so does (X, P^α) for any $\alpha \in [0, \theta_0)$. This is easy to check remembering that under P^α the negative infimum of ψ_α is obtained at $\theta_0^\alpha = \theta_0 - \alpha$ (a picture of ψ will help to make this obvious).
3. In Class B it is also implicit in the definition that $E(X_1) = \psi'(0) < 0$. Like Class A, it is again true that if (X, P) belongs to Class B then for any $\alpha \in [0, \theta_0)$ the process (X, P^α) also belongs to the same class. Note, to deal with the fifth condition, recall again that $\theta_0^\alpha = \theta_0 - \alpha$ plays the role of θ_0 under P^α . From the earlier mentioned effect of the change of measure $\Pi^\alpha(dx) = e^{\alpha x} \Pi(dx)$, it is now easily checked that (v) holds. Essentially it is a statement about (X, P^{θ_0}) and hence independent of α .
4. Finally, recall from the Wiener-Hopf factorisation that (up to a multiplicative constant) for all $\theta \in \mathbb{R}$,

$$\Psi(\theta) = \kappa(0, -i\theta) \widehat{\kappa}(0, i\theta). \quad (4)$$

In both Class A and B, since $\Psi(-i\theta)/\widehat{\kappa}(0, \theta)$ is well defined for all $\theta \in [0, \theta_r)$, it follows that $\kappa(0, \theta)$ may be analytically extended to $(-\theta_r, \infty)$.

We now state the main result of this paper.

Theorem 3 *If X is a Lévy process of Class A or B then there exists a quasi-stationary distribution (1) satisfying*

$$\mu(dy) = \theta_0 \kappa_{\theta_0}(0, \theta_0) e^{-\theta_0 y} \mathbf{V}_{\theta_0}(y) dy$$

on $[0, \infty)$. In particular $\mu(\{0\}) = 0$. The existence of (1) holds for all initial states $x > 0$ when 0 is regular for $(-\infty, 0)$ and for all $[0, \infty)$ when 0 is irregular for $(-\infty, 0)$.

Below we give three special cases of this theorem.

Example 4 (Spectrally negative processes) Suppose that X is spectrally negative which drifts to $-\infty$. (Note, we exclude negative subordinators from the definition of spectral negativity). This falls into Class A as $|\psi(\theta)|$ is finite for all $\theta \geq 0$ (cf. Chapter VII of Bertoin (1996)). In this case it is well known that when further the process oscillates the ascending ladder height process is a linear drift with unit rate. We thus have that $\kappa_{\theta_0}(0, \theta_0) = \theta_0$ and $\mathbf{V}_{\theta_0}(x) = x$. Consequently it follows that

$$\mu(dx) = \theta_0^2 x e^{-\theta_0 x} dx.$$

Example 5 (Brownian motion with drift) This example is a continuation of the previous one as Brownian motion with drift, having no jumps at all, is spectrally negative. Necessarily to be in Class A the drift is negative. Hence $\psi(\theta) = \frac{1}{2}\theta^2 - \rho\theta$ for some $\rho > 0$. A straightforward exercise shows that $\theta_0 = \rho$ and hence

$$\mu(dx) = \rho^2 x e^{-\rho x} dx.$$

This result was first established in Martinez and San Martin (1994).

Example 6 (Spectrally positive processes) In the case of spectral positivity one may consider either Class A or Class B. We see that in both cases, (X, P^{θ_0}) either drifts to $-\infty$ or oscillates. This implies that $\widehat{\kappa}_{\theta_0}(0, \theta) = \theta$ for all $\theta \geq 0$. Hence from the Wiener-Hopf factorisation (4) applied to (X, P^{θ_0}) it follows that

$$\psi_{\theta_0}(\theta) = -\theta \kappa_{\theta_0}(0, -\theta) \tag{5}$$

for all $\theta \leq \theta_r - \theta_0$. As will transpire from the proof, the result is established by proving that for $0 < \alpha < \theta_0$

$$\int_{[0, \infty)} e^{\alpha x} \mu(dx) = \frac{\theta_0 \kappa_{\theta_0}(0, \theta_0)}{(\theta_0 - \alpha) \kappa_{\theta_0}(0, \theta_0 - \alpha)} \tag{6}$$

from which the density given in Theorem 3 can be identified. Note since ψ is finite on $(-\infty, 0]$ the equality (6) can be analytically extended to negative values of α . Using (2) and (5) we see that the left hand side of (6) is also equal to

$$\frac{\psi_{\theta_0}(-\theta_0)}{\psi_{\theta_0}(\alpha - \theta_0)} = \frac{-\psi(\theta_0)}{\psi(\alpha) - \psi(\theta_0)}.$$

Kyprianou (1971) found the Laplace transform of the quasi-stationary distribution for the workload process of the stable $M/G/1$ queue whose service times were distributed with a rational moment generating function. In this case, it is not difficult to show that X belongs to Class A or B. The workload process whilst the queue is busy is just a compound Poisson process with a negative drift having Laplace exponent equal to

$$\psi(\theta) = -\theta + \lambda(\widehat{m}_B(\theta) - 1)$$

where B is the distribution of the service times and $\widehat{m}_B(\theta) = E(e^{\theta B})$.

Suppose now that X belongs to Class A. Note that $\theta_0 > 0$ solves $\lambda \widehat{m}'_B(\theta_0) = 1$. The Laplace transform for this process reads

$$\int_{[0, \infty)} e^{-\beta x} \mu(dx) = \frac{-\theta_0 + \lambda(\widehat{m}_B(\theta_0) - 1)}{-\theta_0 - \beta + \lambda(\widehat{m}_B(\theta_0) - \widehat{m}_B(\beta))}.$$

This agrees with the expression given in Theorem 2a(ii) of Kyprianou (1971) but now for a more general class of Lévy processes.

3 Proof of main result

The proof of Theorem 3 requires two preparatory Lemmas.

Lemma 7 *For all sufficiently large $\vartheta > \theta_0$ the following hold.*

(i) *If X belongs to Class A, then*

$$P(X_t \geq 0) + E[e^{\vartheta X_t}; X_t < 0] \sim \frac{\vartheta}{(\vartheta - \theta_0)} \frac{g_\gamma(0)}{\theta_0} t^{-1/\gamma} L_A(t) e^{\psi(\theta_0)t} \quad (7)$$

as $t \uparrow \infty$ where L_A is slowly varying and g_γ is a continuous version of the density of the stable law mentioned in the definition of Class A.

(ii) *If X belongs to Class B, then*

$$P(X_t \geq 0) + E[e^{\vartheta X_t}; X_t < 0] \sim \frac{\vartheta}{(\vartheta - \theta_0)} \frac{|\psi'(\theta_0)|^{-\beta}}{\theta_0} t^{-(\beta-1)} L_B(t) e^{\psi(\theta_0)t} \quad (8)$$

where L_B is slowly varying as $t \uparrow \infty$.

Proof. For each $h > 0$ and within Class A or B we consider a random walk $S^h = \{S_n^h : n \geq 0\}$ where

$$S_n^h = \sum_{i=1}^n X_i, \quad n \geq 0$$

and $X_i = X(ih) - X((i-1)h)$ are obviously independent and identically distributed random variables having moment generating function

$$f_h(\theta) = E e^{\theta S_1^h} = E(e^{\theta X_h}) = e^{\psi(\theta)h}.$$

for $\theta \in [0, \theta_r)$.

(i) When X belongs to Class A, it is easy to check that for each $h > 0$, S^h belongs to Class A for random walks defined in Bertoin and Doney (1996). Following the arguments in Lemma 2.6 of Gettoor and Sharpe (1994) we have that always $P(X_t \geq 0) + E(e^{\vartheta X_t}; X_t < 0)$ is a right or left continuous function (in fact it is continuous if X is not a compound Poisson process with drift). Hence we may apply Croft's Lemma (see Croft (1957) and Corollary 2 of Kingman

(1963)); that is to say, if there exists a continuous slowly varying function L_A which does not depend on h such that

$$P(S_n^h \geq 0) + E(e^{\vartheta S_n^h}; S_n^h < 0) \sim \frac{\vartheta}{(\vartheta - \theta_0)} \frac{g_\gamma(0)}{\theta_0} (nh)^{-1/\gamma} L_A(nh) e^{\psi(\theta_0)nh} \quad (9)$$

as $n \uparrow \infty$ for each $h > 0$, then (7) holds.

Note that by virtue of the fact that X is in Class A, there exists a slowly varying function, L_A , such that $X_t/(t^{1/\gamma} L_A(t))$ converges in distribution under P^{θ_0} as $t \uparrow \infty$ to a stable random variable with density g_γ . Hence the same limit holds on any subsequence. This in turn implies that the random walk S^h belongs to Class A of random walks as defined in Bertoin and Doney (1996). The limit in (9) follows directly from Lemma 4(i) of Bertoin and Doney (1996). According to Theorem 1.3.3. of Bingham et al. (1986) we may choose L_A to be continuous.

(ii) To see that S^h also belongs to Class B for random walks as defined in Bertoin and Doney (1996) for each $h > 0$ when X belongs to Class B, we argue as follows. From a classic result of Embrechts, Goldie and Veraverbeke (1979) it now follows from the assumed regular variation of $\Pi_{\theta_0}(\cdot, \infty)$ that $P^{\theta_0}(X_h > x) \sim h \Pi_{\theta_0}(x, \infty)$ as $x \uparrow \infty$. Now from Lemma 4(ii) of Bertoin and Doney (1996) we read off that

$$P(S_n^h \geq 0) + E(e^{\vartheta S_n^h}; S_n^h < 0) \sim \frac{\vartheta}{(\vartheta - \theta_0)} \frac{|\psi'(\theta_0)|^{-\beta}}{\theta_0} (nh)^{-(\beta-1)} L_B(nh) e^{\psi(\theta_0)nh}$$

as $n \uparrow \infty$. The remaining part of the proof essentially mimics the proof of (i) using Croft's Lemma. ■

Lemma 8 For $\alpha \in [0, \theta_0)$ the following two cases hold.

(i) If X belongs to Class A then we have that $|\kappa(\psi(\theta_0), -\alpha)|$ is finite and for all $x \geq 0$

$$E_x(e^{\alpha X_t}; \tau > t) \sim e^{\theta_0 x} \widehat{V}_{\theta_0}(x) \frac{1}{\kappa(\psi(\theta_0), -\alpha)} \frac{g_\gamma(0)}{(\theta_0 - \alpha)} t^{-(1/\gamma+1)} L_A(t) e^{\psi(\theta_0)t}$$

as $t \uparrow \infty$ where L_A is the same as in Lemma 7 (i).

(ii) If X belongs to Class B then we have that $|\kappa(\psi(\theta_0), -\alpha)|$ is finite and for all $x \geq 0$

$$E_x(e^{\alpha X_t}; \tau > t) \sim e^{\theta_0 x} \widehat{V}_{\theta_0}(x) \frac{1}{\kappa(\psi(\theta_0), -\alpha)} \frac{|\psi'(\theta_0)|^{-\beta}}{(\theta_0 - \alpha)} t^{-\beta} L_B(t) e^{\psi(\theta_0)t}$$

as $t \uparrow \infty$ where L_B is the same as in Lemma 7 (ii).

Note that if 0 is regular for $(-\infty, 0)$ then the descending ladder height process is not a compound poisson process and hence $\widehat{V}_{\theta_0}(0) = 0$ consistently with the expectations mentioned in the above lemma for $x = 0$. As indicated earlier, in

the regular case, we only consider quasi-stationary laws for $x > 0$. If, on the other hand, 0 is irregular for $(-\infty, 0)$ then the descending ladder height process is a compound Poisson process and hence $\widehat{V}_{\theta_0}(0) > 0$ so that the statement of the above lemma for $x = 0$ is non-trivial.

Proof of Lemma 8. Let e_η be an exponentially distributed random variable with parameter $\eta = q + \psi(\theta_0) > 0$ which is independent of X . Denote $\overline{X}_t = \sup_{s \leq t} X_s$. From the Wiener-Hopf factorisation (which gives the independence of \overline{X}_{e_η} and $X_{e_\eta} - \overline{X}_{e_\eta}$) and duality (which gives the equality in distribution of \overline{X}_{e_η} under \widehat{P} and $\overline{X}_{e_\eta} - X_{e_\eta}$ under P) we obtain

$$\begin{aligned} & \int_0^\infty e^{-\eta t} E_x[e^{\alpha X_t}; \tau > t] dt \\ &= \frac{1}{\eta} \int_{[0, \infty)} e^{\alpha y} P(\overline{X}_{e_\eta} \in dy) \cdot e^{\alpha x} \int_{[0, x]} e^{-\alpha z} \widehat{P}(\overline{X}_{e_\eta} \in dz). \end{aligned}$$

(The last calculation is essentially taken from the proof of Theorem VI.20 of Bertoin (1996) p. 176). From Theorem VI.5 of Bertoin (1996), p. 160, we have then that

$$\int_0^\infty e^{-\eta t} E_x[e^{\alpha X_t}; \tau > t] dt = \frac{1}{\eta} \frac{\kappa(\eta, 0)}{\kappa(\eta, -\alpha)} e^{\alpha x} \int_{[0, x]} e^{-\alpha z} \widehat{P}(\overline{X}_{e_\eta} \in dz). \quad (10)$$

Taking the Laplace transforms of both sides of (10) with respect to x , we obtain with the help of Fubini's theorem

$$\begin{aligned} & \int_0^\infty \int_0^\infty e^{-\eta t} e^{-\theta x} E_x[e^{\alpha X_t}; \tau > t] dx dt \\ &= \frac{1}{\eta} \frac{1}{\theta - \alpha} \frac{\kappa(\eta, 0)}{\kappa(\eta, -\alpha)} \frac{\widehat{\kappa}(\eta, 0)}{\widehat{\kappa}(\eta, \theta)} \end{aligned} \quad (11)$$

for $\theta > \alpha$.

It is known that one can put $\exp\{\int_0^\infty (e^{-t} - e^{-\eta t})t^{-1}P(X_t = 0) dt\}$ into the normalization constant of the product $\kappa(\eta, -\alpha)\widehat{\kappa}(\eta, \theta)$ (which itself is present as a consequence of an arbitrary normalization of local time at the maximum) and hence with the help of the expressions for κ and $\widehat{\kappa}$ given on p.166 of Bertoin (1996) we may write $s = \kappa(s, 0)\widehat{\kappa}(s, 0)$ and and

$$\begin{aligned} & \int_0^\infty e^{-\eta t} \left(\int_0^\infty e^{-\theta x} E_x[e^{\alpha X_t}; \tau > t] dx \right) dt \\ &= \frac{1}{\theta - \alpha} \frac{1}{\kappa(\eta, -\alpha)\widehat{\kappa}(\eta, \theta)} \\ &= \frac{1}{\theta - \alpha} \exp \left\{ - \int_0^\infty \frac{e^{-t} - a_t(\theta)e^{-(q+\psi(\theta_0))t}}{t} dt \right\}, \end{aligned} \quad (12)$$

where

$$a_t(\theta) = E[e^{\alpha X_t}; X_t \geq 0] + E[e^{\theta X_t}; X_t < 0].$$

For the right hand side of (12) we have used the integral representations of κ and $\widehat{\kappa}$ in terms of the distribution of X ; see Corollary VI.10 on p. 165 of Bertoin (1996). By exponential change of measure we have

$$a_t(\theta) = e^{\psi(\alpha)t} \left(P^\alpha(X_t \geq 0) + E^\alpha[e^{(\theta-\alpha)X_t}; X_t < 0] \right).$$

(i) Now assume that (X, P) belongs to Class A and recall that for $\alpha \in [0, \theta_0)$ the Lévy process (X, P^α) remains in the same class as assumed and hence Lemma 7(i) may be applied to the latter process. In that case one should replace ψ by ψ_α and θ_0 by $\theta_0^\alpha := \theta_0 - \alpha$. Importantly, one should also note that the limiting density g_γ appearing in the definition of Class A applied to (X, P^α) does not depend on $\alpha \in [0, \theta_0)$ as it concerns the distribution of (X, P^{θ_0}) .

Taking note of (2) it follows from Lemma 7(i) that for all sufficiently large $\theta > \theta_0$

$$a_t(\theta) \sim \frac{(\theta - \alpha)}{(\theta - \theta_0)} \frac{g_\gamma(0)}{(\theta_0 - \alpha)} t^{-1/\gamma} L_A(t) e^{\psi(\theta_0)t}. \quad (13)$$

as $t \uparrow \infty$. This implies that the right hand side of equation (12) is finite for $q \geq 0$. In particular when $q = 0$ one may check that in the integral $\int_0^\infty (e^{-t} - a_t(\theta) e^{-(q+\psi(\theta_0))t}) t^{-1} dt$ the integrand behaves like $O(1)$ as $t \downarrow 0$ and $O(t^{-(1+1/\gamma)})$ as $t \uparrow \infty$. It is also the case that $|\widehat{\kappa}(\psi(\theta_0), \theta)| < \infty$. To see this, note that $\liminf_{t \uparrow \infty} X_t = -\infty$ and hence \widehat{L}_1^{-1} is an almost surely finite stopping time. We may use Doob's Optional Stopping Theorem (cf. Jacod and Shiryaev (2003) Theorem III.3.4) and the fact that $\widehat{\kappa}(0, \theta - \theta_0) < \infty$ to deduce that

$$\begin{aligned} e^{-\widehat{\kappa}_{\theta_0}(0, \theta - \theta_0)} &= E^{\theta_0}(e^{-(\theta - \theta_0)\widehat{H}_1}) \\ &= E(e^{\theta_0 X_{\widehat{L}_1^{-1}} - \psi(\theta_0)\widehat{L}_1^{-1}} e^{-(\theta - \theta_0)\widehat{H}_1}) \\ &= E(e^{-\psi(\theta_0)\widehat{L}_1^{-1} - \theta\widehat{H}_1}) \\ &= e^{-\widehat{\kappa}(\psi(\theta_0), \theta)} \end{aligned} \quad (14)$$

proving the last claim. Since the right hand side of (12) for $q = 0$ is finite and from Corollary VI.10 of Bertoin (1996) is equal to $\lim_{q \downarrow 0} \kappa(q + \psi(\theta_0), -\alpha)\widehat{\kappa}(q + \psi(\theta_0), \theta)$, and since $|\widehat{\kappa}(\psi(\theta_0), \theta)|$ is finite, it follows that $|\kappa(\psi(\theta_0), -\alpha)| < \infty$.

Now both sides of (12) are analytical functions in the variable $q > 0$ the identity can be extended for this parameter range. Differentiating equation (12) with respect to q yields

$$\begin{aligned} &\int_0^\infty e^{-qt} e^{-\psi(\theta_0)t} t \left(\int_0^\infty e^{-\theta x} E_x[e^{\alpha X_t}; \tau > t] dx \right) dt \\ &= \frac{1}{(\theta - \alpha)} \exp \left\{ - \int_0^\infty \frac{e^{-t} - a_t(\theta) e^{-(q+\psi(\theta_0))t}}{t} dt \right\} \int_0^\infty e^{-qt} e^{-\psi(\theta_0)t} a_t(\theta) dt. \end{aligned}$$

Now taking $q \downarrow 0$, we obtain from (13) and the converse to the Monotone Density Theorem (which does not require a monotone density cf. Feller (1971) XIII.5 p.

446) that

$$\begin{aligned} & \int_0^\infty e^{-qt} e^{-\psi(\theta_0)t} \left(\int_0^\infty e^{-\theta x} E_x[e^{\alpha X_t}; \tau > t] dx \right) dt \\ & \sim \frac{1}{(\theta - \alpha)} \frac{1}{\kappa(\psi(\theta_0), -\alpha) \widehat{\kappa}(\psi(\theta_0), \theta)} \\ & \quad \times \Gamma\left(1 - \frac{1}{\gamma}\right) \frac{(\theta - \alpha)}{(\theta - \theta_0)} \frac{1}{(\theta_0 - \alpha)} g_\gamma(0) q^{1/\gamma-1} L_A(1/q) \end{aligned}$$

and hence

$$\begin{aligned} & \int_0^\infty e^{-\theta x} E_x[e^{\alpha X_t}; \tau > t] dx \\ & \sim \frac{1}{(\theta - \theta_0)(\theta_0 - \alpha)} \\ & \quad \times \frac{1}{\kappa(\psi(\theta_0), -\alpha) \widehat{\kappa}(\psi(\theta_0), \theta)} g_\gamma(0) t^{-(1/\gamma+1)} L_A(t) e^{\psi(\theta_0)t}. \quad (15) \end{aligned}$$

From (3) we have

$$\begin{aligned} \int_0^\infty e^{-\theta x} e^{\theta_0 x} \widehat{\mathbf{V}}_{\theta_0}(x) dx &= \frac{1}{(\theta - \theta_0)} \frac{1}{\widehat{\kappa}_{\theta_0}(0, \theta - \theta_0)} \\ &= \frac{1}{(\theta - \theta_0)} \frac{1}{\widehat{\kappa}(\psi(\theta_0), \theta)} \end{aligned} \quad (16)$$

where the second equality follows from (14).

To conclude, let $\underline{X}_t = \inf_{s \leq t} X_s$ and observe that $\mathbb{E}_x(e^{\alpha X_t}; \tau > t) = \mathbb{E}_x(e^{\alpha X_t}; \underline{X}_t \geq 0)$ is right continuous in x on $[0, \infty)$ by temporal right continuity of the bivariate process (X, \underline{X}) . In addition, from Proposition I.15 of Bertoin (1996), $\widehat{\mathbf{V}}_{\theta_0}(x)$ is continuous on $(0, \infty)$. (Note, in the case that 0 is irregular for $(-\infty, 0)$, it is known that \widehat{H} is also a compound Poisson subordinator and the assumptions imply that its jump distribution is diffuse and hence \widehat{V} has no atoms except at zero). Comparing (16) with (15) then, the statement of the theorem follows.

(ii) The proof for the case that (X, P) belongs to Class B is essentially the same as in (i) with obvious changes using Lemma 7(i) ■

We have in particular the following corollary corresponding to the case $\alpha = 0$.

Corollary 9 *The following asymptotics holds as $t \uparrow \infty$.*

(i) *If X belongs to Class A then for all $x \geq 0$*

$$P_x(\tau > t) \sim e^{\theta_0 x} \widehat{\mathbf{V}}_{\theta_0}(x) \frac{1}{\kappa(\psi(\theta_0), 0)} \frac{g_\gamma(0)}{\theta_0} t^{-(1/\gamma+1)} L_A(t) e^{\psi(\theta_0)t}$$

as $t \uparrow \infty$.

(ii) If X belongs to Class B then for all $x \geq 0$

$$P_x(\tau > t) \sim e^{\theta_0 x} \widehat{V}_{\theta_0}(x) \frac{1}{\kappa(\psi(\theta_0), 0)} \frac{|\psi'(\theta_0)|^{-\beta}}{\theta_0} t^{-\beta} L_B(t) e^{\psi(\theta_0)t}$$

as $t \uparrow \infty$.

We are ready now to prove the main result.

Proof of Theorem 3. From Lemma 8 and Corollary 9, we get that for $0 < \alpha < \theta_0$,

$$\begin{aligned} \int_{[0, \infty)} e^{\alpha y} \mu(dy) &= \lim_{t \uparrow \infty} \frac{E_x(e^{\alpha X_t}; \tau > t)}{P_x(\tau > t)} \\ &= \frac{\theta_0 \kappa(\psi(\theta_0), 0)}{(\theta_0 - \alpha) \kappa(\psi(\theta_0), -\alpha)} \end{aligned}$$

However, with a similar calculation to the one at the end of the proof of Lemma 8 we can show that

$$\kappa(\psi(\theta_0), -\alpha) = \kappa_{\theta_0}(0, \theta_0 - \alpha).$$

Further, from (3)

$$\int_0^\infty e^{\alpha x} e^{-\theta_0 x} V_{\theta_0}(x) dx = \frac{1}{(\theta_0 - \alpha)} \frac{1}{\kappa_{\theta_0}(0, \theta_0 - \alpha)}.$$

The statement of the main theorem now follows. ■

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