

Refracted Lévy processes

Andreas E. Kyprianou¹

Department of Mathematical Sciences, University of Bath

¹Joint work with Ronnie Loeffen

Basic data

- $X = \{X_t : t \geq 0\}$ with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ will always denote a spectrally negative Lévy process (i.e. $\Pi(0, \infty) = 0$ and $-X$ is not a subordinator).

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- For each $q \geq 0$, the, so-called, q -scale function $W^{(q)} : \mathbb{R} \mapsto [0, \infty)$ is defined by $W^{(q)}(x) = 0$ for $x < 0$ and otherwise is continuous satisfying

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- For convenience we shall write W for $W^{(0)}$.

Sample fluctuation identities

For example, if $\tau_0^- = \inf\{t > 0 : X_t < 0\}$ and $\tau_a^+ = \inf\{t > 0 : X_t > a\}$ then

- The oldest one in the book (Takács 1966, Zolotarev 1964) (the 'ruin probability' - in fact the Pollaczek-Khintchine formula in disguise)

$$\mathbb{P}_x(\tau_0^- < \infty) = 1 - (\mathbb{E}_0(X_1) \vee 0) W(x)$$

for $x \geq 0$.

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- Resolvent in a strip: For any $a > 0$, $x, y \in [0, a]$, $q \geq 0$

$$\int_0^\infty e^{-qt} \mathbb{P}_x(X_t \in dy, t < \tau_a^+ \wedge \tau_0^-) dt \\ = \left\{ \frac{W^{(q)}(x) W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y) \right\} dy.$$

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- Define the aggregate process $U_t^\xi = X_t - L_t^\xi$ when paying dividends with strategy ξ and let

$$\sigma^\xi = \inf\{t > 0 : U_t^\xi < 0\}$$

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- A strategy ξ is called admissible if $L_{t+}^\xi - L_t^\xi \leq U_t^\xi$ for $t < \sigma^\xi$ (i.e. ruin of the aggregate process does not result as a consequence of a dividend payment).

De Finetti's control problem

An 'old' actuarial problem of the 'modern' probabilistic age proposed by de Finetti 1957: find the value function and matching dividend strategy ξ^* such that

$$v(x) = \sup_{\xi} \mathbb{E}_x \left(\int_0^{\sigma^{\xi}} e^{-qt} dL_t^{\xi} \right) = \mathbb{E}_x \left(\int_0^{\sigma^{\xi^*}} e^{-qt} dL_t^{\xi^*} \right)$$

where $q > 0$ and the supremum is taken over all admissible dividend strategies.

Reflection strategies

It has been shown that the optimal strategy is of a 'barrier type with reflection':

$$L_t^a = (a \vee \sup_{s \leq t} X_s) - a$$

for some optimal level a . These cases are:

- 1 (Gerber 1969) Cramér-Lundberg process with exponentially distributed jumps $X_t = ct - \sum_{i=1}^{N_t} \mathbf{e}_i$,

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- 4 (K. Rivero and Song 2008) Any spectrally negative Lévy process whose jump measure has a log-convex density.

Restricted class of control strategies

- Many variations on this theme have been examined for the case of diffusions (Jeanblanc & Shiryaev 1995, Elena Boguslavskaya's Ph.D. thesis 2005) as well as the Cramér-Lundberg case with exponential jumps (Gerber & Shiu 2006) including the following:

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- The class of admissible strategies is further restricted to the case that

$$U_t^\phi = X_t - L_t^\phi = X_t - \int_0^t \phi(U_s^\phi) ds \quad (1)$$

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- Immediate problem: (1) can be a stochastic differential equation of the *degenerate* type. Does it even have a unique weak solution? (possible bad cases: X has no Gaussian component).
- Could one at least investigate (1) for the optimal strategies that have appeared in the aforementioned articles?

Refraction strategies (Loeffen and K. 2008)

- A refraction strategy refers to the control $\phi(x) = \delta \mathbf{1}_{(x>b)}$ for some threshold level $b \geq 0$. Thus the controlled process would need to solve the stochastic differential equation

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- When X has a Gaussian part then classical theory gives us a unique strong solution.
- When X has paths of bounded variation, then solution can be constructed pathwise.
- When X has unbounded variation, no Gaussian part, solution can be strongly approximated by solutions from the bounded variation case:

$$\sup_{s \in [0,1]} |X_s - X_s^{(n)}| \rightarrow 0 \Rightarrow \sup_{s \in [0,1]} |U_s^* - U_s^{(n)}| \rightarrow 0$$

as $n \uparrow \infty$ for some stochastic process U^* (which is a limit point in the $(D[0,1], \|\cdot\|_\infty)$ Banach space.

When is U^* a refracted process?

- Since

$$U_t^* = X_t - \delta \lim_{n \uparrow \infty} \int_0^t \mathbf{1}_{(U_s^{(n)} > b)} ds$$

we have that U^* is a refracted process as soon as one can prove that $\mathbb{P}_x(U_s^* = b) = 0$ for Lebesgue almost every $s > 0$.

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- Amazingly this can be done because a expression for the resolvent can be found semi-explicitly in terms of scale functions.

Resolvent

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- For $x, b \in \mathbb{R}$, Borel B and $q > 0$,

$$\begin{aligned}
 & \mathbb{E}_x \left(\int_0^\infty e^{-qt} \mathbf{1}_{\{U_t \in B\}} ds \right) \\
 = & \int_{B \cap [b, \infty)} \left\{ \left(e^{\Phi(q)(x-b)} + \delta \Phi(q) e^{-\Phi(q)b} \mathbf{1}_{\{x \geq b\}} \int_b^x e^{\Phi(q)z} \mathbb{W}^{(q)}(x-z) dz \right) \right. \\
 & \left. \cdot \frac{\varphi(q) - \Phi(q)}{\delta \Phi(q)} e^{-\varphi(q)(y-b)} - \mathbb{W}^{(q)}(x-y) \right\} dy \\
 + & \int_{B \cap (-\infty, b)} \left\{ \left(e^{\Phi(q)(x-b)} + \delta \Phi(q) e^{-\Phi(q)b} \mathbf{1}_{\{x \geq b\}} \int_b^x e^{\Phi(q)z} \mathbb{W}^{(q)}(x-z) dz \right) \right. \\
 & \left. \cdot \frac{\varphi(q) - \Phi(q)}{\Phi(q)} e^{\varphi(q)b} \int_b^\infty e^{-\varphi(q)z} W^{(q)'}(z-y) dz \right. \\
 & \left. - \left(W^{(q)}(x-y) + \delta \mathbf{1}_{\{x \geq b\}} \int_b^x \mathbb{W}^{(q)}(x-z) W^{(q)'}(z-y) dz \right) \right\} dy.
 \end{aligned}$$

Uniqueness

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- We have established existence of a strong solution for all driving spectrally negative Lévy processes X .
- Uniqueness: Suppose that $U^{(1)}$ and $U^{(2)}$ are two strong solutions. Then writing

$$\Delta_t = U_t^{(1)} - U_t^{(2)} = -\delta \int_0^t (\mathbf{1}_{\{U_s^{(1)} > b\}} - \mathbf{1}_{\{U_s^{(2)} > b\}}) ds,$$

it follows from classical calculus that

$$\Delta_t^2 = -2\delta \int_0^t \Delta_s (\mathbf{1}_{\{U_s^{(1)} > b\}} - \mathbf{1}_{\{U_s^{(2)} > b\}}) ds.$$

Now note that thanks to the fact that $\mathbf{1}_{\{x > b\}}$ is an increasing function, it follows from the above representation that, for all $t \geq 0$, $\Delta_t^2 \leq 0$ and hence $\Delta_t = 0$ almost surely.

Sample identities for U

Some nice identities fall out of this analysis. Suppose that

$$\kappa_0^- := \inf\{t > 0 : U_t < 0\}.$$

For $q \geq 0$ and $x \geq 0$

$$\begin{aligned} & \mathbb{E}_x \left(\int_0^{\kappa_0^-} e^{-qt} \delta \mathbf{1}_{\{U_t > b\}} ds \right) \\ &= -\delta \int_0^{(x-b) \vee 0} \mathbb{W}^{(q)}(z) dz \\ & \quad + \frac{W^{(q)}(x) + \delta \mathbf{1}_{\{x \geq b\}} \int_b^x \mathbb{W}^{(q)}(x-y) W^{(q)'}(y) dy}{\varphi(q) \int_0^\infty e^{-\varphi(q)y} W^{(q)'}(y+b) dy}. \end{aligned}$$