

STRONG LAW OF LARGE NUMBERS FOR BRANCHING DIFFUSIONS

JÁNOS ENGLÄNDER, SIMON C. HARRIS AND ANDREAS E. KYPRIANOU

ABSTRACT. Let X be the branching particle diffusion corresponding to the operator $Lu + \beta(u^2 - u)$ on $D \subseteq \mathbb{R}^d$ (where $\beta \geq 0$ and $\beta \neq 0$). Let λ_c denote the generalized principal eigenvalue for the operator $L + \beta$ on D and assume that it is finite. When $\lambda_c > 0$ and $L + \beta - \lambda_c$ satisfies certain spectral theoretical conditions, we prove that the random measure $\exp\{-\lambda_c t\}X_t$ converges almost surely in the vague topology as t tends to infinity. This result is motivated by a cluster of articles due to Asmussen and Hering dating from the mid-seventies as well as the more recent work concerning analogous results for superdiffusions of [ET, EW]. We extend significantly the results in [AH76, AH77] and include some key examples of the branching process literature. As far as the proofs are concerned, we appeal to modern techniques concerning martingales and ‘spine’ decompositions or ‘immortal particle pictures’.

CONTENTS

1. Introduction and statement of results	1
1.1. Model	1
1.2. Motivation	3
1.3. Main results	5
1.4. Outline	6
2. Detailed comparison with some older results	6
3. Examples	7
4. Proofs	12
4.1. A spine approach	12
4.2. Proof of Theorem 6 along lattice times	14
4.3. Replacing lattice times with continuous time	17
4.4. Proof of Theorem 7	18
References	20

1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. **Model.** Write $C^{i,\eta}(D)$ to denote the space of i times ($i = 1, 2$) continuously differentiable functions with all their i th order derivatives belonging to $C^\eta(D)$. [Here $C^\eta(D)$ denotes the usual Hölder space.] Let $D \subseteq \mathbb{R}^d$ be a domain and

Date: August 20, 2007.

1991 *Mathematics Subject Classification.* Primary: 60J60; Secondary: 60J80.

Key words and phrases. Law of Large Numbers, spine decomposition, spatial branching processes, branching diffusions, measure-valued processes, h -transform, criticality, product-criticality.

consider $Y = \{Y_t; t \geq 0\}$, the diffusion process with probabilities $\{\mathbb{P}_x, x \in D\}$ corresponding to the operator

$$(1) \quad L = \frac{1}{2} \nabla \cdot a \nabla + b \cdot \nabla \quad \text{on } \mathbb{R}^d,$$

where the coefficients $a_{i,j}$ and b_i belong to $C^{1,\eta}$, $i, j = 1, \dots, d$, for some η in $(0, 1]$, and the symmetric matrix $a = \{a_{i,j}\}$ is positive definite for all $x \in D$. At this point, we do not assume that Y is conservative, i.e. Y may get killed at the Euclidean boundary of D or run out to infinity in finite time.

Furthermore let us first assume that $0 \leq \beta \in C^\eta(D)$ is bounded from above on D and $\beta \not\equiv 0$. The (strictly dyadic) $(L, \beta; D)$ -branching diffusion is the Markov process with motion component Y and with spatially dependent rate β , replacing particles by precisely two offspring when branching and starting from a finite configuration of individuals. At each time $t > 0$, the state of the process is denoted by $X_t \in \mathcal{M}(D)$ where

$$\mathcal{M}(D) = \left\{ \sum_{i=1}^n \delta_{x_i} : n \in \mathbb{N} \text{ and } x_i \in D \text{ for } i = 1, \dots, n \right\}.$$

We will also use the following notation: $X = \{X_t : t \geq 0\}$ has probabilities $\{P_\mu : \mu \in \mathcal{M}(D)\}$, and E_μ is expectation with respect to P_μ . As usual, $\langle f, \mu \rangle := \int_D f(x) \mu(dx)$ and $\langle f, g \rangle := \int_D f(x)g(x) dx$, where dx is Lebesgue measure, and so $\langle f, g dx \rangle = \langle fg, dx \rangle = \langle f, g \rangle$.

When β is not bounded from above, one may wonder if the $(L, \beta; D)$ -branching diffusion is still well defined, in particular, whether the global (or even local) mass may blow up in finite time. To treat the case with β 's which are not upper bounded we will need to consider more general branching diffusions. For a ‘weighted branching diffusion’, the particles do not necessarily carry unit mass. At each time $t > 0$, the state of the process is $\widehat{X}_t \in \widehat{\mathcal{M}}(D)$ where

$$\widehat{\mathcal{M}}(D) = \left\{ \sum_{i=1}^n \gamma_i \delta_{x_i} : n \in \mathbb{N}, \gamma_i > 0 \text{ and } x_i \in D \text{ for } i = 1, \dots, n \right\}.$$

Next we need a definition. Let

$$\lambda_c = \lambda_c(L + \beta, D) := \inf\{\lambda \in \mathbb{R} : \exists u > 0 \text{ satisfying } (L + \beta - \lambda)u = 0 \text{ in } D\}$$

denote the *generalized principal eigenvalue* for $L + \beta$ on D . By standard theory, $\lambda_c < \infty$ whenever β is upper bounded and, for general β , there exists an $h > 0$ satisfying that $(L + \beta - \lambda)h = 0$, whenever $\lambda_c < \infty$.

In the latter case, let us define the $\widehat{\mathcal{M}}(D)$ -valued process W as follows. Each particle performs a Y motion and carries weight $h(x)$ at $x \in D$, and furthermore, when the particle’s clock rings, according to the spatially varying rate β , the particle splits into two offspring, which perform independent Y motions and carry weights according to the function h , etc. Since β is not upper bounded, we have to rule out finite time explosions. Fortunately, since h is a ‘harmonic’ function, it is a straightforward exercise to show that the total mass process $|W|$ is a supermartingale, and in particular, it is a.s. finite for all $t > 0$. Then, since \widehat{X} is well defined, so is the $\mathcal{M}(D)$ -valued process X defined by $\frac{dX_t}{d\widehat{X}_t} := e^{\lambda_c t} h^{-1}$, (i.e. $X_t(B) := e^{\lambda_c t} \langle h^{-1} \mathbf{1}_B, \widehat{X}_t \rangle$, $t \geq 0$, $B \subset D$ Borel).

Therefore, from now on, we relax the assumption that $\sup_D \beta < \infty$ and replace it with the much milder assumption $\lambda_c < \infty$.

1.2. Motivation. This paper concerns growth of mass on compact domains of branching particle diffusions. In doing so we address a gap in the literature dating back to the mid-seventies when the study of growth of typed branching processes on compact domains of the type space was popularized by Asmussen and Hering. Also we complement a recent revival in this field which has appeared amongst the superprocess community.

Before discussing main results, we shall introduce the topic in detail.

Definition 1 (Local extinction). *Fix $\mu \in \mathcal{M}(D)$. We say that X exhibits local extinction under P_μ if for every Borel set $B \subset\subset D$, there exists a random time τ_B such that*

$$P_\mu(\tau_B < \infty) = 1 \text{ and } P_\mu(X_t(B) = 0 \text{ for all } t \geq \tau_B) = 1.$$

[Here $B \subset\subset D$ means that B is bounded and its closure is a subset of D .]

Local extinction has been studied by [P96], [EP] (for superprocesses) and [EK] (for branching diffusions). To explain their results, recall that we assume that the generalized principal eigenvalue for $L + \beta$ on D is finite. In fact, $\lambda_c \leq 0$ if and only if there exists a function $h > 0$ satisfying $(L + \beta)h = 0$ on D – see Section 4.4 in [P95]. Following the papers [P96, EP] where similar issues were addressed for superprocesses, in [EK] the following was shown.

Theorem 2 (Local extinction versus local exponential growth). *Let $\mathbf{0} \neq \mu \in \mathcal{M}(D)$.*

- (i) *X under P_μ exhibits local extinction if and only if there exists a function $h > 0$ satisfying $(L + \beta)h = 0$ on D , that is, if and only if $\lambda_c \leq 0$.*
- (ii) *When $\lambda_c > 0$, for any $\lambda < \lambda_c$ and $\emptyset \neq B \subset\subset D$,*

$$P_\mu(\limsup_{t \uparrow \infty} e^{-\lambda t} X_t(B) = \infty) > 0,$$

$$P_\mu(\limsup_{t \uparrow \infty} e^{-\lambda_c t} X_t(B) < \infty) = 1.$$

In particular, local extinction/local exponential growth does not depend on the initial measure $\mathbf{0} \neq \mu \in \mathcal{M}(D)$.

(In [EK] it is assumed that β is upper bounded, whereas in [EP] only the finiteness of λ_c is assumed. The proofs of [EK] go through for this latter case too.) On closer inspection this last theorem says that when $\lambda_c \leq 0$ mass ‘escapes out of B ’ even though the entire process may survive with positive probability. (If Y is conservative in D for example then it survives with probability one). Further, when $\lambda_c > 0$ mass accumulates on all nonempty bounded open domains and in such a way that with positive probability this accumulation grows faster than any exponential rate $\lambda < \lambda_c$. On the other hand, mass will not grow faster than at the exponential rate λ_c . It is natural then to ask whether in fact λ_c gives an exact growth rate or not. That is to say, for each $\emptyset \neq B \subset\subset D$ do the random measures $\{\exp\{-\lambda_c t\} X_t : t \geq 0\}$ converge in the vague topology almost surely? Further, can one identify the limit? This is precisely the object of interest of a variety of previous studies for both branching diffusions and superprocesses which we shall now review.

We note already here that the process in expectation is given by the *linear kernel* corresponding to the operator $L + \beta$ on D . Therefore, trusting in SLLN for branching processes, one should expect that the process itself grows like the linear kernel too. On the other hand, it is easy to see that the linear kernel does not

in general scale precisely with $\exp\{-\lambda_c t\}$ but rather with $f(t)\exp\{-\lambda_c t\}$, where f grows to infinity as $t \rightarrow \infty$ and at the same time is subexponential. (Take for example $L = \Delta/2$ and $\beta > 0$ constant on \mathbb{R}^d , then $f(t) = t^{d/2}$.) In fact the growth is pure exponential if and only if $L + \beta$ is product-critical (see later in this subsection). Proving SLLN seems to be significantly harder in the general case involving the subexponential term f .

In the late seventies Asmussen and Hering wrote a series of papers concerning weak and strong laws of large numbers for a reasonably general class of branching processes which included branching diffusions. See [AH76] and [AH77]. In the context of the branching diffusions we consider here one can summarize briefly their achievements by saying that, when D is *bounded*, for a special class of operators $L + \beta$, the rescaled process $\{\exp\{-\lambda_c t\}X_t : t \geq 0\}$ converges in the vague topology, almost surely for branching diffusions. Further, for the same class of $L + \beta$ when D is unbounded they proved that there exists the limit *in probability* of $\exp\{-\lambda_c t\}X_t$ as $t \uparrow \infty$ (in the vague topology). The class of $L + \beta$ alluded to they called ‘positively regular’. The latter is a subclass of the class $\mathcal{P}_p^*(D)$ (the class that we shall work with) given below.

A more detailed comparison with [AH76, AH77] as well as the discussion on related results on superprocesses is deferred to Section 2.

Before we give the definition of the basic classes of operators that we shall use, $\mathcal{P}_p(D)$ and $\mathcal{P}_p^*(D)$, we need to recall certain concepts of the so-called *criticality theory* of second order operators. The operator $L + \beta - \lambda_c$ is called *critical* if the associated space of positive harmonic functions is nonempty but the operator does not possess a (minimal positive) Green’s function. In this case the space of positive harmonic functions is in fact one-dimensional. Moreover, the space of positive harmonic functions of the adjoint of $L + \beta - \lambda_c$ is also one dimensional.

Assumption 3. Suppose we choose representatives of these two spaces to be ϕ and $\tilde{\phi}$ respectively. Throughout the paper and without further reference, we will always assume that $L + \beta - \lambda_c$ product-critical, and in this case we pick ϕ and $\tilde{\phi}$ with the normalization $\langle \phi, \tilde{\phi} \rangle = 1$.

We now define the classes $\mathcal{P}_p(D)$ and $\mathcal{P}_p^*(D)$. Since we want to talk about spatial spread on a generic domain D , we fix, for the rest of the paper, an arbitrary family of domains $\{D_t, t \geq 0\}$ with $D_t \subset\subset D$, $D_t \uparrow D$. (For $D = \mathbb{R}^d$, D_t can be the t -ball, but we can take any other family with $D_t \subset\subset D$, $D_t \uparrow D$ too.)

Definition 4 ($\mathcal{P}_p(D)$ and $\mathcal{P}_p^*(D)$). For $p \geq 1$, we write $L + \beta \in \mathcal{P}_p(D)$ if

- (i): $\lambda_c = \lambda_c(L + \beta; D) > 0$,
- (ii): $\langle \phi^p, \tilde{\phi} \rangle < \infty$, in which case we say that $L + \beta - \lambda_c$ is *product p -critical*.

Let $q(x, y, t)$ be transition density of $L + \beta$ and $Q(x, y, t) := q(x, y, t) - e^{\lambda_c t} \tilde{\phi}(y) \phi(x)$. We write $L + \beta \in \mathcal{P}_p^*(D)$ when the following additional conditions holds.

- (iii): For all $\emptyset \neq B \subset\subset D$ there exists a positive function ζ with $\zeta(t) \uparrow \infty$ as $t \uparrow \infty$ and such that

$$|Q(x, y, \zeta(t))| \leq \alpha_t \tilde{\phi}(y) \phi(x), \quad x \in D_t, \quad y \in B, \quad t \geq 0,$$

where $\lim_{t \uparrow \infty} e^{-\lambda_c t} \alpha_t = 0$.

(iv): Furthermore, for any given $x \in D$ there exists a function $a : [0, \infty) \rightarrow [0, \infty)$ such that $\zeta(a_t) = \mathcal{O}(t)$ as $t \rightarrow \infty$ and for all $\delta > 0$,

$$P_{\delta_x}(\exists n_0, \forall n > n_0 : \text{supp}(X_{n\delta}) \subset D_{a_n\delta}) = 1.$$

Let $p(x, y, t)$ denote the transition density of the diffusion corresponding to the operator $(L + \beta - \lambda_c)^\phi$. Then $p(x, y, t) = e^{-\lambda_c t} \phi(y) \phi^{-1}(x) q(x, y, t)$, and thus, (iii) is equivalent to

(iii*): With the same ζ as in (iii),

$$\lim_{t \rightarrow \infty} \sup_{x \in D_t, y \in B} \left| \frac{p(x, y, \zeta(t))}{\phi \tilde{\phi}(y)} - 1 \right| = 0.$$

Some heuristics to help find suitable a and ζ will be discussed in Section 3.

Remark 5 (Ergodicity). Note that criticality is invariant under h -transforms. Moreover, an easy computation shows that ϕ and $\tilde{\phi}$ transforms into 1 and $\phi \tilde{\phi}$ respectively when turning from $(L + \beta - \lambda_c)$ to the h -transformed ($h = \phi$) operator $(L + \beta - \lambda_c)^\phi = L + a\phi^{-1}\nabla\phi \cdot \nabla$. Therefore product criticality is invariant under h -transforms too (this is not the case with product p -criticality when $p > 1$). Further, for operators with no zeroth order term, it is equivalent to positive recurrence (ergodicity) of the corresponding diffusion process. In particular, $(L + \beta - \lambda_c)^\phi$ corresponds to an ergodic diffusion process provided $(L + \beta - \lambda_c)$ is product critical (see [P95], Section 4.9). \diamond

1.3. Main results. With the following theorem we wish to address the issue of almost sure convergence in the vague topology of $\{\exp\{-\lambda_c t\}X_t : t \geq 0\}$ for branching diffusions with $L + \beta$ belonging to $\mathcal{P}_p^*(D)$, $p > 1$ thus generalizing the results of Asmussen and Hering.

Note that since $L + \beta - \lambda_c$ is critical, ϕ is the unique (up to constant multiples) *invariant positive function* for the linear semigroup corresponding to $L + \beta - \lambda_c$ (Theorem 4.8.6. in [P95]). Let $\{S_t\}_{t \geq 0}$ denote the semigroup corresponding to $L + \beta$. It is a standard fact (sometimes called ‘the one particle picture’) that

$$(2) \quad S_t(g)(x) = E_{\delta_x} \langle g, X_t \rangle$$

for all nonnegative bounded measurable g 's. Even though ϕ is not necessarily bounded from above, $S_t(\phi)$ makes sense and (2) remains valid when g is replaced by ϕ , because ϕ can be approximated with a monotone increasing sequence of g 's and the finiteness of the limit is guaranteed precisely by the invariance property of ϕ . By the invariance of ϕ , $E_{\delta_x} e^{-\lambda_c t} \langle \phi, X_t \rangle = e^{-\lambda_c t} S_t(\phi)(x) = \phi(x)$, which is sufficient together with the branching property to deduce that $W^\phi = \{W_t^\phi; t \geq 0\}$ is a martingale where

$$W_t^\phi = e^{-\lambda_c t} \langle \phi, X_t \rangle, \quad t \geq 0.$$

Indeed note that

$$E_{\delta_x} \left(e^{-\lambda_c(t+s)} \langle \phi, X_{t+s} \rangle | \mathcal{F}_t \right) = e^{-\lambda_c t} E_{X_t} \left(e^{-\lambda_c s} \langle \phi, X_s \rangle \right) = e^{-\lambda_c t} \langle \phi, X_t \rangle.$$

Being a positive martingale, P_{δ_x} -almost sure convergence is guaranteed, and the a.s. martingale limit $W_\infty^\phi := \lim_{t \rightarrow \infty} W_t^\phi$ appears in the following main conclusion.

Theorem 6 (SLLN). *Assume that $L + \beta \in \mathcal{P}_p^*(D)$ for some $p \in (1, 2]$ and $\langle \beta\phi^p, \tilde{\phi} \rangle < \infty$. Then,*

$$(3) \quad \lim_{t \uparrow \infty} e^{-\lambda_c t} \langle g, X_t \rangle = \langle g, \tilde{\phi} \rangle W_\infty^\phi, \quad g \in C_c^+(D)$$

holds P_{δ_x} – a.s. for all $x \in D$, and $E_{\delta_x}(W_\infty^\phi) = 1$.

Moreover, if $\sup_D \beta < \infty$ then the restriction $p \in (1, 2]$ can be replaced by $p > 1$.

We close this subsection with the Weak Law of Large Numbers. Here we change the class $\mathcal{P}_p^*(D)$ to the larger class $\mathcal{P}_p(D)$ and get convergence in probability instead of a.s. convergence. It is important to point out, however, that the class $\mathcal{P}_p^*(D)$ is already quite large — see Section 3, where we verify that key examples from the literature are in fact in $\mathcal{P}_p^*(D)$ and thus obey the SLLN.

Theorem 7 (WLLN). *Suppose that $L + \beta \in \mathcal{P}_p(D)$ for some $p \in (1, 2]$ and $\langle \beta\phi^p, \tilde{\phi} \rangle < \infty$. Then for all $x \in D$, (3) holds in P_{δ_x} –probability and $E_{\delta_x}(W_\infty^\phi) = 1$. Moreover, if $\sup_D \beta < \infty$ then the restriction $p \in (1, 2]$ can be replaced by $p > 1$.*

1.4. Outline. The rest of this paper is organized as follows. In Section 2 we embed our results into the literature, while in Section 3 we discuss some key examples for the SLLN. The proofs are given in Section 4.

2. DETAILED COMPARISON WITH SOME OLDER RESULTS

The methods of Asmussen and Hering were based for the most part on classical techniques of truncation and applications of the Borel-Cantelli Lemma. Using this method, they proved the convergence of $e^{-\lambda_c t} \langle X_t, g \rangle$ for all $0 \leq g \in L^1(\tilde{\phi}(x)dx)$. It is also worth noting that the generic strength of their method extended to many other types of branching processes; discrete time, discrete space and so on.

Interestingly, preceding all work of Asmussen and Hering is the single article [W] (later improved upon by [B]). Watanabe demonstrates that when a suitable Fourier analysis is available with respect to the operator $L + \beta$, then by spectrally expanding any $g \in C_c^+(D)$, the space of nonnegative, continuous and compactly supported functions, one can show that $\{\langle g, X_t \rangle : t \geq 0\}$ is almost surely asymptotically equivalent to its mean. From this the classic Strong Law of Large Numbers for dyadic branching Brownian motion in \mathbb{R}^d is recovered. Namely that when $L = \Delta/2$ and $\beta > 0$ is a constant,

$$\lim_{t \uparrow \infty} t^{d/2} e^{-\beta t} X_t(B) = (2\pi)^{d/2} |B| \times N_\mu$$

where B is any Borel set, $|B|$ is its Lebesgue measure and N_μ is a strictly positive random variable depending on the initial configuration $\mu \in \mathcal{M}(\mathbb{R}^d)$. The operator $1/2\Delta + \beta$ does not fall into the class $\mathcal{P}_1(D)$. For an analogous result on supercritical super-Brownian motion see [E].

Let us discuss now how our assumptions relate to the assumptions imposed in the article [AH76].

In [AH76] the domain is bounded (and even one dimensional) when the Strong Law of Large Numbers is stated for branching diffusions; on general domains, only convergence in probability was obtained. Furthermore, in [AH76] the notion of *positively regular* operators was introduced. In our context it first means that

- (A) $\lambda_c > 0$ (they call this property ‘supercriticality’).
- (B) ϕ is bounded from above,

(C) $\langle \tilde{\phi}, 1 \rangle < \infty$.

Obviously, (B-C) is stronger than the assumption $\langle \phi, \tilde{\phi} \rangle < \infty$ (product-criticality).

Secondly, $\{S_t\}_{t \geq 0}$, the semigroup corresponding to $L + \beta$ (the so called ‘expectation semigroup’) satisfies the following condition. If η is a nonnegative, bounded measurable function on \mathbb{R}^d , then

(D) $S_t(\eta)(x) = \langle \eta, \tilde{\phi} \rangle \phi(x) [e^{\lambda_c t} + o(e^{\lambda_c t})]$ as $t \uparrow \infty$, uniformly in η .

Let T_t be the semigroup defined by $T_t(f) := S_t^\phi(f) = \frac{1}{\phi} S_t(\phi f)$, for all $0 \leq f$ measurable with ϕf being bounded. Then T_t correspond to the h -transformed ($h = \phi$) operator L_0^ϕ . Recall that L_0^ϕ corresponds to a positive recurrent diffusion process. Then, assuming that ϕ is bounded, it is easy to check that the following condition would suffice for (D) to hold:

$$(4) \quad \limsup_{t \uparrow \infty} \sup_{x \in D} \sup_{\|g\| \leq 1} \langle g, \phi \tilde{\phi} \rangle^{-1} \left| T_t(g) - \langle g, \phi \tilde{\phi} \rangle \right| = 0,$$

where $\|\cdot\|$ denotes sup-norm. However this is not true in most cases on unbounded domains (or even on bounded domains with general unbounded coefficients) because of the requirement on the uniformity in x . (See our examples in Section 3 — neither of the examples on \mathbb{R}^d satisfy (4).)

Turning to superprocesses, there would seem to be considerably fewer results of this kind in the literature (see the references [D, Dy, E] for superprocesses in general). The most recent and general work in this area we are aware of are [ET, EW, E].

In [ET] it was proved that (in the vague topology) $\{\exp\{-\lambda_c t\} X_t : t \geq 0\}$ converges in law where X is the so called $(L, \beta, \alpha, \mathbb{R}^d)$ -superprocess (with α being the ‘activity parameter’) satisfying that $L + \beta \in \mathcal{P}(D)$ and that $\alpha \phi$ is bounded from above. (An additional requirement was that $\langle \phi, \mu \rangle < \infty$ where $\mu = X_0$ is the deterministic starting measure.) The long and technical proof relied heavily on the theory of dynamical systems applied to the Laplace transforms of $\{e^{-\lambda_c t} \phi X_t, t \geq 0\}$.

In [EW] the convergence in law was replaced by convergence in probability. Furthermore, instead of \mathbb{R}^d a general Euclidean domain $D \subseteq \mathbb{R}^d$ was considered. The heavy analytic method of [ET] was replaced by a different, simpler and more probabilistic one. The main tool was the introduction of a ‘weighted superprocess’ obtained by a ‘space-time H -transform’.

3. EXAMPLES

In this section we give examples which satisfy all the assumptions we have, and thus, according to Theorem 6, obey the SLLN. (Those examples do not fall into the setting in [AH76, AH77].)

Before we turn to the specific examples, we give some heuristics. Although these are not actually needed for understanding the examples, we feel that the reader ‘gets a more complete picture’ by first reading them.

Remark 8 (Expectation calculations and local vs. global growth rates). From (2), we have

$$E_x \langle \mathbf{1}_{\{\cdot \in dy\}}, X_t \rangle = e^{\lambda_c t} \frac{\phi(x)}{\phi(y)} p(t, x, y)$$

and then, by ergodicity,

$$e^{-\lambda c t} E_x \langle \mathbf{1}, X_t \rangle = \phi(x) \int_{\mathbb{R}^d} \frac{p(t, x, y)}{\phi(y)} dy \rightarrow \phi(x) \int_{\mathbb{R}^d} \tilde{\phi}(y) dy, \text{ as } t \rightarrow \infty.$$

Hence, if $\langle \tilde{\phi}, \mathbf{1} \rangle < \infty$, then the global population growth is the same as the local population growth, whereas, if $\langle \tilde{\phi}, \mathbf{1} \rangle < \infty$ the global growth rate exceeds the local growth rate.

Remark 9 (Heuristics for a_t and $\zeta(t)$). One may wonder how one can find a function a as in Definition 4(iv). In fact, this will often be straightforward to find.

Suppose that Y is conservative and fix $x \in D$. (If Y is not conservative, then there is no function a satisfying (iv). Indeed, $D_t \uparrow$ implies that $\cup_{t \leq T} D_t \subset D_T$, for $T > 0$. Now, if $Y_T = \Delta$ with positive probability for some $T > 0$, then the requirement $D_{a_T} \subset \subset D$ cannot hold.) If, for example, we can pick a deterministic increasing function a such that, for all $\delta > 0$,

$$\sum_{n=1}^{\infty} P_{\delta_x}(\text{supp}(X_{n\delta}) \not\subset D_{a_{n\delta}}) < \infty,$$

then Borel-Cantelli says that the function a is an appropriately choice. Since the probability one particle is present in a set is trivially dominated by the expected numbers in that set, it will be much easier to check that

$$\sum_{n=1}^{\infty} E_{\delta_x} \langle \mathbf{1}_{D_{a_{n\delta}}^c}, X_{n\delta} \rangle < \infty.$$

If we can choose a_t such that, for some $\epsilon > 0$,

$$\int_{|y| > a_t} \frac{p(t, x, y)}{\phi(y)} dy < e^{-(\lambda_c + \epsilon)t}$$

then we will have satisfied

$$\sum_{n=1}^{\infty} E_x \langle \mathbf{1}_{D_{a_{n\delta}}^c}, X_{n\delta} \rangle < \infty.$$

Heuristically, if the spine transition density $p(t, x, y)$ converges to its equilibrium $\phi(y)\tilde{\phi}(y)$ sufficiently quickly even for very large y , we might hope to take

$$a_t \approx \tilde{\phi}^{-1}(e^{-\lambda_c t})$$

If the spine starts at a very large position, since it is ergodic it will tend to move back toward the origin and Ventcel-Friedlin large deviation theory suggests that it will ‘closely’ follow the path of a deterministic particle with the same drift function. We can use this to *guess* for a suitable form for $\zeta(t)$. At least heuristically, to find out how far away the spine particle may start in order that it both returns to the vicinity of the origin and then ergodizes towards its invariant measure before large time t , we can solve the deterministic differential equation

$$\dot{f}(t) = \mu(f(t)) - a(f(t)) \frac{\nabla \phi(f(t))}{\phi(f(t))}$$

when $L = \frac{1}{2}a(x)\Delta - \mu(x) \cdot \nabla$, and take $\zeta(t)$ a little larger than $|f(t)|$.

Indeed, these heuristics appear to the correct form for both a_t and $\zeta(t)$ in the examples considered below.

Example 10 (OU process with quadratic breeding rate). Let $\sigma, \mu, a, b > 0$ and consider

$$L := \frac{1}{2}\sigma^2\Delta - \mu x \cdot \nabla \text{ on } \mathbb{R}^d$$

corresponding to an (inward) Ornstein-Uhlenbeck process and let $\beta(x) := bx^2 + a$. Since L corresponds to a recurrent diffusion and β is a smooth function with $\beta \geq 0$ and $\beta \not\equiv 0$, it follows that $\lambda_c > 0$ (see Chapter 4 in [P95]). The equilibrium distribution for L is given by a normal distribution,

$$\pi(x) = \left(\frac{\mu}{\pi\sigma^2}\right)^{d/2} \exp\left\{-\frac{\mu}{\sigma^2}x^2\right\}.$$

Suppose that $\mu > \sigma\sqrt{2b}$. Defining $\gamma^\pm := \frac{1}{2\sigma^2}\left(\mu \pm \sqrt{\mu^2 - 2b\sigma^2}\right)$, for the principle eigenvalue problem with $(L + \beta)\phi = \lambda_c\phi$ we can take

$$\lambda_c := \sigma^2\gamma^- + a, \quad \phi(x) := c^- \exp\{\gamma^-x^2\} \quad \text{and} \quad \tilde{\phi}(x) := c^+ \exp\{-\gamma^+x^2\}$$

where $c^- := (1 - (2b\sigma^2/\mu^2))^{\frac{d}{8}}$, $c^+ := c^- (\mu/(\pi\sigma^2))^{\frac{d}{2}}$. Note that $\tilde{\phi}(x) := \phi(x)\pi(x)$ and $L + \beta$ is a self-adjoint operator with respect to π .

Some calculations using the ‘one-particle picture’ (equation 2) reveals that, in expectation, the support of the process grows like $\sqrt{\lambda_c t/\gamma^+}$ and one can pick $a(t) = \sqrt{\lambda t/\gamma^+}$ for any $\lambda > \lambda_c$ and condition (iv) in Definition 4 will hold.

The spine is also an (inward) Ornstein-Uhlenbeck process with parameter $\alpha := \mu - 2\gamma^-\sigma^2 = \sqrt{\mu^2 - 2b\sigma^2}$ with

$$(L + \beta - \lambda_c)\phi = L + \sigma^2 \frac{\nabla\phi}{\phi} \cdot \nabla = \frac{1}{2}\sigma^2\Delta - \alpha x \cdot \nabla \text{ on } \mathbb{R}^d,$$

and transition density

$$p(t, x, y) = \left(\frac{\alpha}{\pi\sigma^2(1 - e^{-2(\alpha/\sigma^2)t})}\right)^{d/2} \exp\left[-\frac{\alpha \sum_{i=1}^d (y_i - x_i e^{-(\alpha/\sigma^2)t})^2}{\sigma^2(1 - e^{-2(\alpha/\sigma^2)t})}\right].$$

We see that the drift of the inward OU reduces the influence of any starting position exponentially in time. Indeed, one can take $\zeta(x) = (1 + \epsilon)(\sigma^2/\alpha) \log x$ for any $\epsilon > 0$ for condition (iii*) in Definition 4 to hold. Finally, we trivially note that $\zeta(a(t)) = \mathcal{O}(t)$ (in fact, only $\log t$ growth), hence all necessary conditions are satisfied for our strong law theorem to hold.

Note, a strong law for a generalization of this model can be found in [H] where the convergence is proved using a martingale expansion for continuous functions $g \in L^2(\pi)$ (rather than compactly supported g). Almost sure asymptotic growth rates (and a.s. support) for the same model are studied in [GHH].

This is certainly a non-trivial model and it highlights the strength of our general result. In particular, a quadratic breeding rate is *critical* in the sense that a BBM with breeding rate $\beta(x) = \text{const} \cdot x^p$ explodes in a finite time a.s. if and only if $p > 2$, with explosion in the expected population size when $p = 2$. When a branching inward OU process with quadratic breeding is considered here, a strong enough drift with $\mu > \sigma\sqrt{2b}$ can balance the high breeding, whereas any lower drift would lead to a dramatically different behavior. \diamond

Example 11 (Outward OU process with constant breeding rate). Let $\sigma^2, \mu, b > 0$ and consider

$$L := \frac{1}{2}\sigma^2\Delta + \mu x \cdot \nabla \text{ on } \mathbb{R}^d$$

corresponding to an ‘outward’ Ornstein-Uhlenbeck process and let $\beta(\cdot) \equiv b$. As the spatial movements have no affect on the branching, the global population grows like $e^{\beta t}$ and this is achieved ‘naturally’ with particles moving freely. This corresponds to $(L + \beta)\tilde{\phi} = b\tilde{\phi}$ with $\tilde{\phi} \equiv 1$. On the other hand, the principle eigenvalue is $\lambda_c = b - \mu < b$ with $\phi(x) = \text{const} \cdot \exp\{-(\mu/\sigma^2)x^2\}$, it being associated with the *local*, as opposed to *global*, growth rate.

After some similar expectation calculations to the inward OU in quadratic potential, an upper bound on the process’ spread is roughly the same as for an individual outward OU particle, that is, we can take $a(t) = \exp\{(1 + \delta)(\mu/\sigma^2)t\}$ for any $\delta > 0$.

Despite the transient nature of the original motion, the spine is an inward OU process

$$(L + \beta - \lambda_c)\phi = L + \sigma^2 \frac{\nabla \phi}{\phi} = \frac{1}{2}\sigma^2 \Delta - \mu x \cdot \nabla \text{ on } \mathbb{R}^d,$$

with equilibrium $\phi\tilde{\phi}(x) \propto \exp\{-(\mu/\sigma^2)x^2\}$. Intuitively, this is the motion that maximizes the *local* growth rate at λ_c (here its the original motion ‘conditioned to keep returning to the origin’). We can therefore take $\zeta(x) = (1 + \epsilon)(\sigma^2/\mu) \log x$ for any $\epsilon > 0$ and hence still find that $\zeta(a_t) = (1 + \epsilon)(1 + \delta)t = \mathcal{O}(t)$. All the conditions required for the strong law to hold are again satisfied. \diamond

Example 12 (BBM with $\beta \in C_c^+(\mathbb{R}^d)$ and $\beta \not\equiv 0$ for $d = 1, 2$). Consider the $(\frac{1}{2}\Delta + \beta)$ -branching diffusion where $\beta \in C_c^+(\mathbb{R}^d)$ and $\beta \not\equiv 0$ for $d = 1, 2$. Since Brownian motion is recurrent in dimension $d = 1, 2$, it follows that $\lambda_c > 0$ and in fact, the operator $\frac{1}{2}\Delta + \beta - \lambda_c$ is product-critical and even product- p -critical for all $p > 1$ (see Example 22 in [ET]).

We now show how to find a ζ that satisfies (iii*) in Definition 4. We do it for $d = 1$, the $d = 2$ case is similar.

Let $b > 0$ be so large that $\text{supp}(\beta) \subset [-b, b]$ and let $M := \max_{\mathbb{R}} \beta$. Recall that $p(t, x, y)$ denotes the (ergodic) kernel corresponding to $(\frac{1}{2}\Delta + \beta - \lambda_c)\phi$. In this example P will denote the corresponding probability. By comparison with the constant branching rate case, it is evident that $a_t := \sqrt{2M} \cdot t$ is an appropriate choice. Therefore we have to find a ζ which satisfies that for any fixed ball B ,

$$\limsup_{t \rightarrow \infty} \sup_{|x| \leq t} \left| \frac{p(x, B, \zeta(t))}{\int_B \phi\tilde{\phi}(y) dy} - 1 \right| = 0$$

together with the condition that $\zeta(a_t) = \zeta(\sqrt{2M} \cdot t) = \mathcal{O}(t)$ as $n \rightarrow \infty$.

An easy computation (see again Example 22 in [ET]) shows that on $\mathbb{R} \setminus [-b, b]$,

$$\left(\frac{1}{2}\Delta + \beta - \lambda_c \right) \phi = \frac{1}{2}\Delta - \text{sgn}(x) \cdot \sqrt{2\lambda_c} \frac{d}{dx},$$

where $\text{sgn}(x) := x/|x|$, $x \neq 0$. Fix an ϵ and let $\tau_{\pm b}$ and τ_0 denote the first hitting time (of a single Brownian particle) of $[-b, b]$ and of 0, respectively. We first show that as $t \rightarrow \infty$,

$$(5) \quad \sup_{b < |x| \leq t} P_x \left[\tau_{\pm b} > \frac{t(1 + \epsilon)}{\sqrt{2\lambda_c}} \right] \rightarrow 0.$$

Obviously, it is enough to show that for example

$$\mathbf{P}_t \left[\tau_0 > \frac{t(1 + \epsilon)}{\sqrt{2\lambda_c}} \right] \rightarrow 0$$

where \mathbf{P} corresponds to $\frac{1}{2}\Delta - \sqrt{2\lambda_c} \frac{d}{dx}$ on $[0, \infty)$. Indeed, if \mathcal{W} denotes standard Brownian motion starting at the origin with probability Q , then

$$\begin{aligned} \mathbf{P}_t \left[\tau_0 > \frac{t(1+\epsilon)}{\sqrt{2\lambda_c}} \right] &\leq \mathbf{P}_t \left[Y_{\frac{t(1+\epsilon)}{\sqrt{2\lambda_c}}} > 0 \right] = Q \left[t - \sqrt{2\lambda_c} \frac{t(1+\epsilon)}{\sqrt{2\lambda_c}} + \mathcal{W}_{\frac{t(1+\epsilon)}{\sqrt{2\lambda_c}}} > 0 \right] \\ &= Q \left[\mathcal{W}_{\frac{t(1+\epsilon)}{\sqrt{2\lambda_c}}} > \epsilon t \right] \rightarrow 0 \end{aligned}$$

(the last term tends to zero by the SLLN for \mathcal{W}).

We now claim that $\zeta(t) := \frac{t(1+2\epsilon)}{\sqrt{2\lambda_c}}$ satisfies

$$\lim_{t \rightarrow \infty} \sup_{|x| \leq t} \left| \frac{p(x, B, \zeta(t))}{\int_B \phi \tilde{\phi}(y) dy} - 1 \right| = 0.$$

(The condition $\zeta(a_t) = \mathcal{O}(t)$ is obviously satisfied.) By the ergodicity of $p(t, x, y)$, it is sufficient to show that ζ satisfies

$$\lim_{t \rightarrow \infty} \sup_{b < |x| \leq t} \left| \frac{p(x, B, \zeta(t))}{\int_B \phi \tilde{\phi}(y) dy} - 1 \right| = 0.$$

Let, for example $b < x < t$. By the strong Markov property at τ_b (the hitting time of b) and by (5),

$$\frac{p(x, B, \zeta(t))}{\int_B \phi \tilde{\phi}(y) dy} = \frac{p\left(b, B, \zeta(t) - \frac{t(1+\epsilon)}{\sqrt{2\lambda_c}}\right)}{\int_B \phi \tilde{\phi}(y) dy} P \left[\tau_b \leq \frac{t(1+\epsilon)}{\sqrt{2\lambda_c}} \right] + o(1),$$

uniformly in $b < x \leq t$.

Finally,

$$\lim_{t \rightarrow \infty} \frac{p\left(b, B, \zeta(t) - \frac{t(1+\epsilon)}{\sqrt{2\lambda_c}}\right)}{\int_B \phi \tilde{\phi}(y) dy} = 1$$

because $p(t, x, y)$ is an ergodic kernel and

$$\lim_{t \rightarrow \infty} \left[\zeta(t) - \frac{t(1+\epsilon)}{\sqrt{2\lambda_c}} \right] = \lim_{t \rightarrow \infty} \frac{t\epsilon}{\sqrt{2\lambda_c}} = \infty,$$

completing the proof of our claim about ζ . \diamond

Example 13 (Bounded domain). First note that when D is bounded, an important subset of $\mathcal{P}_p(D)$, $p > 1$ is formed by the operators $L + \beta$ which are uniformly elliptic on D with bounded coefficients which are smooth up to the boundary of D and with $\lambda_c > 0$. That is, in this case $L + \beta - \lambda_c$ is critical (see [P95], Section 4.7), and since ϕ and $\tilde{\phi}$ are Dirichlet eigenfunctions (are zero at the boundary of D), it is even product- p -critical for all $p > 1$. Theorem 7 thus applies.

Although in this case Y is not conservative in D , in fact even Theorem 6 will be applicable whenever (iii*) can be strengthened to the following uniform (in x) convergence on D :

$$(6) \quad \lim_{t \rightarrow \infty} \sup_{x \in D, y \in B} \left| \frac{p(x, y, \zeta(t))}{\phi \tilde{\phi}(y)} - 1 \right| = 0.$$

(Note that [AH77] has a similar global uniformity assumption — see the paragraph after (4).) Indeed, then the proof of Theorem 6 (which can be found later, in

Section 4) can be simplified, because the function a is not actually needed: D_{a_n} can be replaced by D for all $n \geq 1$.

As far as (6) is concerned, it is often relatively easy to check. For example, assume that $d = 1$ (the method can be extended for radially symmetric settings too) and so let $D = (r, s)$. Then the drift term of the spine is $b + a(\log \phi)'$. Now, if this is negative and bounded away from zero at $s - \epsilon < x < s$ and positive and bounded away from zero at $r < x < r + \epsilon$ with some $\epsilon \in (0, s - r)$, then (6) can be verified by a method similar to the one in the previous example. The above condition on the drift is not hard to check in a concrete example (note that since ϕ satisfies the Dirichlet boundary condition, $\log \phi$ tends to $-\infty$ at the boundary).

If we relax the regularity assumptions on $L + \beta$ then for example ϕ is not necessarily upper bounded, and so we are leaving the family of operators handled in [AH77] (see the four paragraphs preceding (4)); nevertheless our method still works as long as $L + \beta \in \mathcal{P}_p^*(D)$, $p > 1$ (for the SLLN) or $L + \beta \in \mathcal{P}_p(D)$, $p > 1$ (for the WLLN). \diamond

4. PROOFS

4.1. A spine approach. To establish the $L^p(P_{\delta_x})$ convergence of W^ϕ for $p > 1$ we appeal to a, by now, standard techniques that have been introduced to the literature by [LPP] and involves a change of measure inducing a ‘spine’ decomposition. Similar applications can be found in [A, BK, EK, HHc] to name but a few. See for example [Ev, E] as well as the discussion in [EK] for yet further references.

It is important to point out that we will need the spine decomposition not only to establish the L^p -convergence mentioned above but also in the key lemma (Lemma 17) in the proof of Theorem 6. In both cases, we found the spine method to be indispensable and we were not able to replace it by other L^p methods.

Before we can state our spine decomposition, we need to recall some facts concerning changes of measures for diffusions and Poisson processes.

Girsanov change of measure. Suppose that Y is adapted to some filtration $\{\mathcal{G}_t : t \geq 0\}$. Under the change of measure

$$(7) \quad \frac{d\mathbb{P}_x^\phi}{d\mathbb{P}_x} \Big|_{\mathcal{G}_t} = \frac{\phi(Y_t)}{\phi(x)} e^{-\int_0^t (\lambda_c - \beta(Y_s)) ds}$$

the process (Y, \mathbb{P}_x^ϕ) corresponds to the h -transformed ($h = \phi$) generator $(L + \beta - \lambda_c)^\phi = L + a\phi^{-1}\nabla\phi \cdot \nabla$. Note now in particular that since $L + \beta \in \mathcal{P}_1(D)$, it follows that (Y, \mathbb{P}_x^ϕ) is an ergodic diffusion with transition density $p(x, y, t)$ and an invariant density $\phi\phi$.

Change of measure for Poisson processes. Suppose that given a non-negative continuous function $g(t), t \geq 0$, the Poisson process (n, \mathbb{L}^g) where $n = \{\{\sigma_i : i = 1, \dots, n_t\} : t \geq 0\}$ has instantaneous rate $g(t)$. Further, assume that n is adapted to $\{\mathcal{G}_t : t \geq 0\}$. Then under the change of measure

$$\frac{d\mathbb{L}^{2g}}{d\mathbb{L}^g} \Big|_{\mathcal{G}_t} = 2^{n_t} \exp \left\{ - \int_0^t g(s) ds \right\}$$

the process (n, \mathbb{L}^{2g}) is also a Poisson process with rate $2g$. See Chapter 3 in [JS].

Theorem 14 (The spine construction). *Let $\{\mathcal{F}_t : t \geq 0\}$ be the natural filtration generated by X . Define the change of measure*

$$\left. \frac{d\tilde{P}_{\delta_x}}{dP_{\delta_x}} \right|_{\mathcal{F}_t} = e^{-\lambda_c t} \frac{\langle \phi, X_t \rangle}{\phi(x)} = \frac{W_t^\phi}{\phi(x)}.$$

Then, under \tilde{P}_{δ_x} , X can be constructed as follows:

- a single particle, $Y = \{Y_t\}_{t \geq 0}$, referred to as the spine, initially starts at x and moves as a diffusion corresponding to the h -transformed operator $L + a\phi^{-1}\nabla\phi \cdot \nabla$;
- the spine undergoes fission into two particles at an accelerated rate $2\beta(Y_t)$ at time t , one of which is selected uniformly at random to continue the spine motion Y ;
- the remaining child gives rise to an independent copy of a P -branching diffusion started at its space-time point of creation.

An similar construction for BBM was established in Chauvin and Rouault [CR]. See [EK] (Theorem 5) or [HHa], for example, on how to prove it.

Remark 15 (The Spine decomposition). Theorem 14 says that $(X, \tilde{P}_{\delta_x})$ has the same law as a process constructed in the following way. A (Y, \mathbb{P}_x^ϕ) -diffusion is initiated along which $(L, \beta; D)$ -branching processes immigrate at space-time points $\{(Y_{\sigma_i}, \sigma_i) : i \geq 1\}$ where, given Y , $n = \{\{\sigma_i : i = 1, \dots, n_t\} : t \geq 0\}$ is a Poisson process with law $\mathbb{L}^{2\beta(Y)}$. It will often be *very* useful to think of $(X, \tilde{P}_{\delta_x})$ as being constructed in this richer way and it will be convenient to define the natural filtration of the spine and the birth process along the spine as $\mathcal{G}_t := \sigma(Y_s, n_s : s \leq t)$. Note that using the ‘spine construction’ of (X, \tilde{P}) as discussed in Remark 15, we can write

$$W_t^\phi = e^{-\lambda_c t} \phi(Y_t) + \sum_{i=1}^{n_t} e^{-\lambda_c \sigma_i} W_i,$$

where, conditional on the spine filtration \mathcal{G}_t , W_i is an independent copy of the martingale W_t^ϕ started from position Y_{σ_i} and run for time $t - \sigma_i$ where σ_i is the i^{th} fission time along the spine for $i = 1, \dots, n_t$. Remembering that particles off the spine behave the same as if under the original measure P and that the martingale property gives $P_{\delta_x}(W_t^\phi) = \phi(x)$, we then have the so called ‘*spine decomposition*’:

$$(8) \quad \tilde{E} \left(W_t^\phi \middle| \mathcal{G}_t \right) e^{-\lambda_c t} \phi(Y_t) + \sum_{i=1}^{n_t} e^{-\lambda_c \sigma_i} \phi(Y_{\sigma_i})$$

The L^p -convergence of the martingale. The a.s. convergence of W^ϕ can be complemented with the following result.

Lemma 16. *Assume that $L + \beta$ belongs to $\mathcal{P}_p(D)$ and that $\langle \beta\phi^p, \tilde{\phi} \rangle < \infty$ for some $p \in (1, 2]$. Then, for $x \in D$, W^ϕ is an $L^p(P_{\delta_x})$ -convergent martingale. Moreover, if $\sup_D \beta < \infty$, then the same conclusion holds if we only assume that $p > 1$.*

Proof. Pick p so that $q = p - 1 \in (0, 1]$, $\langle \phi^p, \tilde{\phi} \rangle < \infty$ and $\langle \beta\phi^p, \tilde{\phi} \rangle < \infty$. (If $K := \sup_D \beta < \infty$ and we only assume that $p > 1$, then $\langle \phi^p, \tilde{\phi} \rangle = \langle \phi^{p-1}, \phi\tilde{\phi} \rangle < \infty$ implies $\langle \phi^r, \tilde{\phi} \rangle = \langle \phi^{r-1}, \phi\tilde{\phi} \rangle < \infty$ and $\langle \beta\phi^r, \tilde{\phi} \rangle \leq K \langle \phi^{r-1}, \phi\tilde{\phi} \rangle < \infty$ for all $r \in (0, p)$, and so we can assume that in fact $p \in (1, 2]$.)

We adopt an approach similar to the one in [HHa, HHb]. position Y_{σ_i} and along the spine for $i = 1, \dots, n_t$. original measure P then have the so called Using the conditional form of Jensen's inequality, the spine decomposition (8) and that $(u + v)^q \leq u^q + v^q$ for $u, v > 0$ when $q \in (0, 1)$, we find

$$\begin{aligned} \phi(x)^{-1} E_{\delta_x} \left[(W_t^\phi)^p \right] &= \tilde{E}_{\delta_x} \left[(W_t^\phi)^q \right] = \tilde{E}_{\delta_x} \left\{ \tilde{E} \left[(W_t^\phi)^q \mid \mathcal{G}_t \right] \right\} \\ &\leq \tilde{E}_{\delta_x} \left\{ \left[\tilde{E} \left(W_t^\phi \mid \mathcal{G}_t \right) \right]^q \right\} \\ &\leq \mathbb{E}_x^\phi \mathbb{L}^{2\beta(Y)} \left(e^{-\lambda_c q t} \phi(Y_t)^q + \sum_{i=1}^{n_t} e^{-\lambda_c q \sigma_i} \phi(Y_{\sigma_i})^q \right) \\ &= e^{-\lambda_c q t} \mathbb{E}_x^\phi [\phi(Y_t)^q] + \mathbb{E}_x^\phi \left[\int_0^t e^{-\lambda_c q s} 2\beta(Y_s) \phi(Y_s)^q ds \right]. \end{aligned}$$

Call the two expressions on the right hand side the *spine term*, $A(x, t)$, and the *sum term*, $B(x, t)$, respectively. Since Y has generator $L + a\phi^{-1}(\nabla\phi) \cdot \nabla$ and $\langle \phi, \tilde{\phi} \rangle = 1$, Y is ergodic and $\mathbb{E}_x^\phi(f(Y_t)) \rightarrow \langle f\phi, \tilde{\phi} \rangle$. Then

$$\lim_{t \uparrow \infty} e^{\lambda_c q t} A(t, x) = \lim_{t \uparrow \infty} \mathbb{E}_x^\phi(\phi(Y_t)^q) = \langle \phi^p, \tilde{\phi} \rangle < \infty$$

for all $x \in D$. For the sum term note that $\lim_{s \uparrow \infty} \mathbb{E}_x^\phi(\beta(Y_s)\phi(Y_s)^q) = \langle \beta\phi^p, \tilde{\phi} \rangle < \infty$ and so $\lim_{t \uparrow \infty} B(t, x) < \infty$ for all $x \in D$. By Doob's inequality, W^ϕ is therefore an L^p -convergent (uniformly integrable) martingale, as required. \square

4.2. Proof of Theorem 6 along lattice times. The statement that $E_{\delta_x}(W_\infty^\phi) = 1$ follows from Lemma 16. The rest of the proof will be based on the following key lemma.

Lemma 17. *Fix $\delta > 0$ and let $B \subset \subset D$. Define*

$$U_t = e^{-\lambda_c t} \langle \phi |_B, X_t \rangle,$$

where $\phi|_B(x) = \phi(x)\mathbf{1}_{(x \in B)}$. Then for any non-decreasing sequence $\{m_n\}_{n \geq 1}$,

$$\lim_{n \uparrow \infty} |U_{(m_n+n)\delta} - E(U_{(m_n+n)\delta} | \mathcal{F}_{n\delta})| = 0, \quad P_{\delta_x} - a.s.$$

Proof. We will suppress the dependence in n in our notation below and simply write m instead m_n . Suppose that $\{X_i : i = 1, \dots, N_{n\delta}\}$ describes the configuration of particles at time $n\delta$. Note that we may always write

$$(9) \quad U_{(m+n)\delta} = \sum_{i=1}^{N_{n\delta}} e^{-n\delta\lambda_c} U_{m\delta}^{(i)}$$

where given $\mathcal{F}_{n\delta}$, the collection $\{U_{m\delta}^{(i)} : i = 1, \dots, N_{n\delta}\}$ are mutually independent and equal in distribution to $U_{m\delta}$ under $P_{\delta_{X_i}}$ respectively.

By the Borel-Cantelli lemma, it is sufficient to prove that for $x \in D$ and for all $\epsilon > 0$,

$$\sum_{n \geq 1} P_{\delta_x} (|U_{(m+n)\delta} - E(U_{(m+n)\delta} | \mathcal{F}_{n\delta})| > \epsilon) < \infty.$$

To this end we first note that,

$$P_{\delta_x} (|U_{(m+n)\delta} - E(U_{(m+n)\delta} | \mathcal{F}_{n\delta})| > \epsilon) \leq \frac{1}{\epsilon^p} E_{\delta_x} (|U_{(m+n)\delta} - E(U_{(m+n)\delta} | \mathcal{F}_{n\delta})|^p).$$

Now recall the following very useful result, for example see [B] or [CH]: if $p \in (1, 2)$ and X_i are independent random variables with $E(X_i) = 0$ (or they are martingale differences), then

$$(10) \quad E \left| \sum_{i=1}^n X_i \right|^p \leq 2^p \sum_{i=1}^n E |X_i|^p.$$

Jensen's inequality also implies that for each $n \geq 1$, $|\sum_{i=1}^n u_i|^p \leq n^{p-1} \sum_{i=1}^n (|u_i|^p)$ and, in particular, $|u+v|^p \leq 2^{p-1}(|u|^p + |v|^p)$.

Note that

$$U_{s+t} - E(U_{s+t} | \mathcal{F}_t) = \sum_{i=1}^{N_t} e^{-\lambda_c t} \left(U_s^{(i)} - E(U_s^{(i)} | \mathcal{F}_t) \right)$$

where conditional on \mathcal{F}_t , $Z_i := U_s^{(i)} - E(U_s^{(i)} | \mathcal{F}_t)$ are independent with $E(Z_i) = 0$. Thus, by (10) and Jensen,

$$\begin{aligned} & E (|U_{s+t} - E(U_{s+t} | \mathcal{F}_t)|^p | \mathcal{F}_t) \\ & \leq 2^p e^{-p\lambda_c t} \sum_{i=1}^{N_t} E \left(|U_s^{(i)} - E(U_s^{(i)} | \mathcal{F}_t)|^p \middle| \mathcal{F}_t \right) \\ & \leq 2^p e^{-p\lambda_c t} \sum_{i=1}^{N_t} E \left(2^{p-1} \left(|U_s^{(i)}|^p + |E(U_s^{(i)} | \mathcal{F}_t)|^p \right) \middle| \mathcal{F}_t \right) \\ & \leq 2^p e^{-p\lambda_c t} \sum_{i=1}^{N_t} 2^{p-1} E \left(|U_s^{(i)}|^p + E(|U_s^{(i)}|^p | \mathcal{F}_t) \middle| \mathcal{F}_t \right) \\ & \leq 2^{2p} e^{-p\lambda_c t} \sum_{i=1}^{N_t} E \left(|U_s^{(i)}|^p \middle| \mathcal{F}_t \right) \end{aligned}$$

Then, as a consequence of the previous estimate, we have that

$$(11) \quad \begin{aligned} & \sum_{n \geq 1} E_{\delta_x} (|U_{(m+n)\delta} - E(U_{(m+n)\delta} | \mathcal{F}_{n\delta})|^p) \\ & \leq 2^{2p} \sum_{n \geq 1} e^{-\lambda_c n \delta p} E_{\delta_x} \left(\sum_{i=1}^{N_{n\delta}} E_{\delta_{X_i}} [(U_{m\delta})^p] \right) \end{aligned}$$

Recalling the definition of the terms $A(x, t)$ and $B(x, t)$ from the proof of Theorem 16(ii) and trivially noting that $U_t \leq W_t^\phi$, we have

$$\begin{aligned} & \sum_{n \geq 1} E_{\delta_x} (|U_{(m+n)\delta} - E(U_{(m+n)\delta} | \mathcal{F}_{n\delta})|^p) \\ & \leq 2^{2p} \sum_{n \geq 1} e^{-\lambda_c n \delta p} E_{\delta_x} \left(\sum_{i=1}^{N_{n\delta}} E_{\delta_{X_i}} [(W_{m\delta}^\phi)^p] \right) \\ & \leq 2^{2p} \sum_{n \geq 1} E_{\delta_x} \left(\sum_{i=1}^{N_{n\delta}} e^{-p\lambda_c n \delta} \phi(X_i) (A(X_i, m\delta) + B(X_i, m\delta)) \right) \\ & = 2^{2p} \sum_{n \geq 1} \phi(x) e^{-q\lambda_c \delta n} \mathbb{E}_x^\phi (A(Y_{n\delta}, m\delta) + B(Y_{n\delta}, m\delta)) \end{aligned}$$

where we have used the ‘one-particle picture’ (equation (2)) and the spine change of measure at (7). Since the spine Y is Markovian and ergodic under \mathbb{P}_x^ϕ , we know that

$$\mathbb{E}_x^\phi [A(Y_{n\delta}, m\delta)] = e^{-\lambda_c q m\delta} \mathbb{E}_x^\phi (\phi(Y_{(m+n)\delta})^q).$$

Denoting $m_\infty := \lim_{n \rightarrow \infty} m_n$, the latter converges to $e^{-q\lambda_c m_\infty \delta} \langle \phi^p, \tilde{\phi} \rangle$ (which will be zero if $m_\infty = \infty$) as $n \uparrow \infty$. Recall the assumption that $\langle \beta\phi^p, \tilde{\phi} \rangle < \infty$. Similarly as before, we have that

$$\mathbb{E}_x^\phi [B(Y_{n\delta}, m\delta)] = 2 \int_0^{m\delta} e^{-\lambda_c q s} \mathbb{E}_x^\phi (\beta(Y_{s+n\delta}) \phi(Y_{s+n\delta})^q) ds$$

which has a finite limit equal to $2 \int_0^{m_\infty \delta} e^{-\lambda_c s} \langle \beta\phi^p, \tilde{\phi} \rangle ds$ as $n \uparrow \infty$. These facts are enough to conclude that the last sum remains finite to complete the Borel-Cantelli argument. \square

We now complete the proof of Theorem 6 along lattice times. Assume that $L + \beta \in \mathcal{P}_p^*$ for some $p > 1$. Recall now that $I(B) := \int_B \phi(y) \tilde{\phi}(y) dy < 1$ and note that, similarly to (9),

$$\begin{aligned} & E(U_{t+s} | \mathcal{F}_t) \\ &= \sum_{i=1}^{N_t} e^{-\lambda_c t} \phi(X_i) p(X_i, B, s) = \sum_{i=1}^{N_t} e^{-\lambda_c(t+s)} \int_B \phi(y) q(X_i, y, s) dy \\ &= \sum_{i=1}^{N_t} e^{-\lambda_c t} \phi(X_i) I(B) + \sum_{i=1}^{N_t} e^{-\lambda_c(t+s)} \int_B \phi(y) Q(X_i, y, s) dy \\ &= I(B) W_t^\phi + \sum_{i=1}^{N_t} e^{-\lambda_c(t+s)} \int_B \phi(y) Q(X_i, y, s) dy =: I(B) W_t^\phi + \Theta(t, s). \end{aligned}$$

Let us replace now t by $n\delta$ and s by $m_n\delta$, where

$$m_n := \zeta(a_{n\delta})/\delta,$$

and ζ is the function appearing in the definition of \mathcal{P}_p^* . (Although we do not need it yet, we note that, according to (iv) in Definition 4, one has $m_n \leq Kn$ as $n \rightarrow \infty$, where $K > 0$ does not depend on δ .) Then

$$E(U_{(n+m_n)\delta} | \mathcal{F}_{n\delta}) = I(B) W_{n\delta}^\phi + \Theta(n\delta, m_n\delta) =: I(B) W_{n\delta}^\phi + \Upsilon(n).$$

Define the event

$$A_n := \{\text{supp}(X_{n\delta}) \not\subset D_{a_{n\delta}}\}.$$

Because of the choice of m_n and since $I(B) < 1$, we have

$$\begin{aligned} & |\Upsilon(n)| \leq \\ & \sum_{i=1}^{N_{n\delta}} e^{-\lambda_c n\delta} \phi(X_i) e^{-\lambda_c m_n\delta} \alpha_{m_n\delta} + |\Upsilon(n)| \mathbf{1}_{A_n} = e^{-\lambda_c m_n\delta} \alpha_{m_n\delta} W_{n\delta}^\phi + |\Upsilon(n)| \mathbf{1}_{A_n}. \end{aligned}$$

Since, according to Definition 4(iv), $\lim_{n \rightarrow \infty} \mathbf{1}_{A_n} = 0$, P -a.s., therefore

$$\limsup_{n \uparrow \infty} |\Upsilon(n)| \leq \lim_{n \uparrow \infty} e^{-\lambda_c m_n\delta} \alpha_{m_n\delta} W_{n\delta}^\phi = 0 \quad P_{\delta_x} - a.s.,$$

and so

$$(12) \quad \lim_{n \uparrow \infty} \left| E_{\delta_x}(U_{(n+m_n)\delta} | \mathcal{F}_{n\delta}) - \langle \phi|_B, \tilde{\phi} dx \rangle W_\infty^\phi \right| = 0 \quad P_{\delta_x} - a.s.$$

Since $\text{Span}\{\phi|_B, B \subset \subset D\}$ is dense in C_c^+ , the result for lattice times follows by standard arguments along with Lemma 17. \square

4.3. Replacing lattice times with continuous time. The following lemma is enough to conclude the convergence in Theorem 6 (see the remark after the lemma). It upgrades convergence along lattice times to the full sequence of times and is based on the idea to be found in Lemma 8 of [AH76].

Lemma 18. *Suppose that for some $p > 1$, $\langle \phi^p, \tilde{\phi} \rangle < \infty$ and for all $\delta > 0$ it is true that for all $g \in C_c^+(D)$ and $x \in D$*

$$\lim_{n \uparrow \infty} e^{-\lambda_c n \delta} \langle g, X_{n\delta} \rangle = \langle g, \tilde{\phi} \rangle W_\infty^\phi \quad P_{\delta_x} - a.s.,$$

then the same result holds when $n\delta$ is replaced by t and $\lim_{n \uparrow \infty}$ by $\lim_{t \uparrow \infty}$.

Remark 19. Recall that we assumed that $\zeta(a_t) = \mathcal{O}(t)$ as $t \rightarrow \infty$, and so referring to the previous subsection, $m_n = \zeta(a_{n\delta})/\delta \leq Kn$ with some $K > 0$ which does not depend on δ . In fact, by possibly further increasing the function a , we can actually take $\zeta(a_t) = Kt$ and $m_n = Kn$. Then, from the previous subsection we already know that

$$\lim_{n \uparrow \infty} e^{-\lambda_c (K+1)n\delta} \langle g, X_{(K+1)n\delta} \rangle = \langle g, \tilde{\phi} \rangle W_\infty^\phi \quad P_{\delta_x} - a.s.$$

Thus the assumption in Lemma 18 is indeed satisfied (write $\delta' := \delta(K+1)$). \diamond

Proof. First suppose that $B \subset \subset D$ and for each $x \in D$ and $\epsilon > 0$, define

$$B^\epsilon(x) = \{y \in B : \phi(y) > (1 + \epsilon)^{-1} \phi(x)\}.$$

Note in particular that $x \in B^\epsilon(x)$ if and only if $x \in B$. Next define for each $\delta > 0$

$$\Xi_B^{\delta, \epsilon}(x) = \mathbf{1}_{\{\text{supp}(X_t) \subset B^\epsilon(x) \text{ for all } t \in [0, \delta]\}}$$

and let $\xi_B^{\delta, \epsilon}(x) = E_{\delta_x}(\Xi_B^{\delta, \epsilon}(x))$. An important feature of the latter quantity in the forthcoming proof is that $\xi_B^{\delta, \epsilon}(x) \rightarrow \mathbf{1}_B(x)$ as $\delta \downarrow 0$. With this notation we now note the crucial estimate

$$e^{-\lambda_c t} \langle \phi|_B, X_t \rangle \geq \frac{e^{-\delta}}{(1 + \epsilon)} \sum_{i=1}^{N_{n\delta}} e^{-\lambda_c n \delta} \phi(X_i) \Xi_B^{\delta, \epsilon}(X_i).$$

Note that the sum on the right hand side is of the form (9) where now $U_{(m+n)\delta}$ is played by the right hand side above and $U_{m\delta}^{(i)}$ is played by the role of $\phi(X_i) \Xi_B^{\delta, \epsilon}(X_i)$. Similar L^p estimates to those found in Lemma 17 show us that the estimate (11) is still valid in the setting here and hence

$$\begin{aligned} & \sum_{n \geq 1} E_{\delta_x} \left(|U_{(m+n)\delta} - E(U_{(m+n)\delta} | \mathcal{F}_{n\delta})|^p \right) \\ & \leq 2^{2p} \sum_{n \geq 1} e^{-\lambda_c n \delta p} E_{\delta_x} \left(\sum_{i=1}^{N_{n\delta}} \phi(X_i)^p \xi_B^{\delta, \epsilon}(X_i) \right). \end{aligned}$$

However, with $q = p - 1$, the righthand side can again be upper estimated by

$$2^{2p} \sum_{n \geq 1} e^{-\lambda_c n \delta p} E_{\delta_x} \langle \phi^p, X_{n\delta} \rangle = 2^{2p} \sum_{n \geq 1} e^{-\lambda_c n \delta q} \mathbb{E}_x^\phi (\phi(Y_{n\delta})^q) < \infty$$

where the equality follows by equation (2), and the fact that the final sum is finite, follows by the ergodicity of \mathbb{P}_x^ϕ and the assumption that $\langle \phi^p, \tilde{\phi} \rangle < \infty$.

We may now appeal to the Borel-Cantelli Lemma to deduce that

$$\lim_{n \uparrow \infty} \left| \sum_{i=1}^{N_{n\delta}} e^{-\lambda_c n \delta} \phi(X_i) \Xi_B^{\delta, \epsilon}(X_i) - e^{-\lambda_c n \delta} \langle \phi \xi_B^{\delta, \epsilon}, X_{n\delta} \rangle \right| = 0$$

P_{δ_x} -almost surely and hence, using the fact that the Strong Law of Large Numbers has been proved already for $n\delta$ -sequences,

$$\liminf_{t \uparrow \infty} e^{-\lambda_c t} \langle \phi|_B, X_t \rangle \geq \frac{e^{-\delta}}{(1 + \epsilon)} \langle \phi \xi_B^{\delta, \epsilon}, \tilde{\phi} \rangle W_\infty^\phi.$$

Taking $\delta \downarrow 0$ reveals that $\langle \phi \xi_B^{\delta, \epsilon}, \tilde{\phi} \rangle \rightarrow \langle \phi|_B, \tilde{\phi} \rangle$ in the lower estimate above, and hence subsequently taking $\epsilon \downarrow 0$ gives us

$$\liminf_{t \uparrow \infty} e^{-\lambda_c t} \langle \phi|_B, X_t \rangle \geq \langle \phi|_B, \tilde{\phi} \rangle W_\infty^\phi.$$

Recall that this estimate was computed for the case that $B \subset\subset D$. Suppose now that $B \subseteq D$ (not necessarily bounded). Then there exists an increasing sequence of compactly embedded domains in B , say $\{B_n : n \geq 1\}$, such that $\bigcup_{n \geq 1} B_n = B$. We may then note that for each $n \geq 1$

$$\liminf_{t \uparrow \infty} e^{-\lambda_c t} \langle \phi|_B, X_t \rangle \geq \liminf_{t \uparrow \infty} e^{-\lambda_c t} \langle \phi|_{B_n}, X_t \rangle \geq \langle \phi|_{B_n}, \tilde{\phi} \rangle W_\infty^\phi,$$

and hence, as n is arbitrary,

$$\liminf_{t \uparrow \infty} e^{-\lambda_c t} \langle \phi|_B, X_t \rangle \geq \langle \phi|_B, \tilde{\phi} \rangle W_\infty^\phi, \quad P_{\delta_x} - a.s.$$

Now that we have a tight lower estimate for the liminf for arbitrary Borel $B \subseteq D$, we shall look at the limsup, also for arbitrary Borel $B \subseteq D$. Using the normalization $\langle \phi, \tilde{\phi} \rangle = 1$, one has

$$\limsup_{t \uparrow \infty} e^{-\lambda_c t} \langle \phi|_B, X_t \rangle = W_\infty^\phi - \liminf_{t \uparrow \infty} e^{-\lambda_c t} \langle \phi|_{D \setminus B}, X_t \rangle \leq \langle \phi|_B, \tilde{\phi} \rangle W_\infty^\phi, \quad P_{\delta_x} - a.s.$$

This, together with the liminf result, yields

$$\lim_{t \uparrow \infty} e^{-\lambda_c t} \langle \phi|_B, X_t \rangle = \langle \phi|_B, \tilde{\phi} \rangle W_\infty^\phi, \quad P_{\delta_x} - a.s.$$

Then, just like for lattice times, a straightforward measure theoretic consideration shows that $\phi|_B$ can be replaced by an arbitrary test function $g \in C_c^+(D)$, completing the proof. \square

4.4. Proof of Theorem 7.

Proof. The last part of the theorem is merely a consequence of Lemma 16. For any $g \in C_c^+(D)$ define for each $x \in D$ the function $h_s(x) = \mathbb{E}_x^\phi(g(\xi_s))$, and note

that, uniformly in x and s , the function $h_s(x)$ is bounded. Now define $U_t[g] = e^{-\lambda c t} \langle g\phi, X_t \rangle$ and observe that, just as in Theorem 6, one has

$$U_{t+s}[g] = \sum_{i=1}^{N_t} e^{-\lambda c t} U_s^{(i)}[g],$$

where by (2),

$$E(U_s^{(i)}[g]|\mathcal{F}_t) = \phi(X_i(t))h_s(X_i(t)).$$

Next, note from the Markov property at t and the proof¹ of Theorem 6 that for fixed s and $\epsilon > 0$

$$\lim_{t \uparrow \infty} P_{\delta_x} (|U_{t+s}[g] - E(U_{t+s}[g]|\mathcal{F}_t)| > \epsilon) = 0.$$

By the Markov inequality, for each $\epsilon > 0$,

$$P_{\delta_x} \left(\left| E(U_{t+s}[g]|\mathcal{F}_t) - \langle \phi g, \tilde{\phi} \rangle W_t^\phi \right| > \epsilon \right) \leq \frac{1}{\epsilon} E_{\delta_x} \left| E(U_{t+s}[g]|\mathcal{F}_t) - \langle \phi g, \tilde{\phi} \rangle W_t^\phi \right|.$$

However, making use of the many-to-one identity, the right hand side above respects the following inequalities,

$$\begin{aligned} & E_{\delta_x} \left| E(U_{t+s}[g]|\mathcal{F}_t) - \langle \phi g, \tilde{\phi} \rangle W_t^\phi \right| \\ & \leq E_{\delta_x} \left(\sum_{i=1}^{N_t} e^{-\lambda c t} \phi(X_i(t)) |h_s(X_i(t)) - \langle \phi g, \tilde{\phi} \rangle| \right) \\ & = \phi(x) \mathbb{E}_x^\phi |h_s(\xi_t) - \langle \phi g, \tilde{\phi} \rangle|. \end{aligned}$$

Hence taking limits as $t \uparrow \infty$, and using ergodicity of the spine as well as the uniform boundedness of $h_s(x)$, we have

$$\lim_{t \uparrow \infty} P_{\delta_x} \left(\left| E(U_{t+s}[g]|\mathcal{F}_t) - \langle \phi g, \tilde{\phi} \rangle W_t^\phi \right| > \epsilon \right) \leq \frac{\phi(x)}{\epsilon} \langle |h_s - \langle \phi g, \tilde{\phi} \rangle|, \phi \tilde{\phi} \rangle.$$

Finally, noting that $\lim_{s \uparrow \infty} h_s(x) = \langle \phi g, \tilde{\phi} \rangle$ and again the uniform boundedness of $h_s(x)$, we have by dominated convergence that

$$\lim_{s \uparrow \infty} \lim_{t \uparrow \infty} P_{\delta_x} \left(\left| E(U_{t+s}[g]|\mathcal{F}_t) - \langle \phi g, \tilde{\phi} \rangle W_t^\phi \right| > \epsilon \right) \leq \frac{\phi(x)}{\epsilon} \langle \lim_{s \uparrow \infty} |h_s - \langle \phi g, \tilde{\phi} \rangle|, \phi \tilde{\phi} \rangle = 0.$$

To finish, we need an epsilon-delta argument. First note that, under the given conditions, the martingale W^ϕ converges in the L^p norm and hence, in particular, converges in probability to its limit. Next, fix $\epsilon > 0$. Then

$$\begin{aligned} & P_{\delta_x} \left(\left| U_{t+s}[g] - \langle \phi g, \tilde{\phi} \rangle W_\infty^\phi \right| > \epsilon \right) \\ & \leq P_{\delta_x} \left(|U_{t+s}[g] - E(U_{t+s}[g]|\mathcal{F}_t)| + \left| E(U_{t+s}[g]|\mathcal{F}_t) - \langle \phi g, \tilde{\phi} \rangle W_t^\phi \right| \right. \\ & \quad \left. + \langle \phi g, \tilde{\phi} \rangle \left| W_t^\phi - W_\infty^\phi \right| > \epsilon \right) \\ & \leq P_{\delta_x} \left(|U_{t+s}[g] - E(U_{t+s}[g]|\mathcal{F}_t)| > \frac{\epsilon}{3} \right) + P_{\delta_x} \left(\left| E(U_{t+s}[g]|\mathcal{F}_t) - \langle \phi g, \tilde{\phi} \rangle W_\infty^\phi \right| > \frac{\epsilon}{3} \right) \\ & \quad + P_{\delta_x} \left(\langle \phi g, \tilde{\phi} \rangle \left| W_t^\phi - W_\infty^\phi \right| > \frac{\epsilon}{3} \right). \end{aligned}$$

¹Note that even though U_t is defined differently, we still have martingale differences and the key upper estimate of $U_t \leq \text{const} \cdot W_t^\phi$ still holds.

Now for each $\delta > 0$, by choosing s sufficiently large, we can take limits on the left hand side above to deduce

$$\lim_{t \uparrow \infty} P_{\delta_x} \left(\left| U_{t+s}[g] - \langle \phi g, \tilde{\phi} \rangle W_{\infty}^{\phi} \right| > \epsilon \right) < \delta.$$

So, in fact

$$\lim_{t \uparrow \infty} P_{\delta_x} \left(\left| U_t[g] - \langle \phi g, \tilde{\phi} \rangle W_{\infty}^{\phi} \right| > \epsilon \right) = 0.$$

In particular, taking $g = \kappa/\phi$ for any $\kappa \in C_c^+(D)$, yields the convergence (in probability) of $\exp\{-\lambda_c t\}X_t$ in the vague topology of measures. \square

Acknowledgement. This research has been completed during the first author's visit to Bath and was funded by an EPSRC grant (EP/E05448X/1). The generous support of EPSRC is gratefully acknowledged. The first author also expresses his thanks to the University of Bath for the hospitality during his stay.

REFERENCES

- [AH76] Asmussen, S. and Hering, H. (1976) Strong limit theorems for general supercritical branching processes with applications to branching diffusions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 36, no. 3, 195–212.
- [AH77] Asmussen, S. and Hering, H. (1977) Strong limit theorems for supercritical immigration-branching processes. *Math. Scand.* 39, no. 2, 327–342.
- [A] Athreya, K. (2000) Change of Measures for Markov Chains and the LlogL Theorem for Branching Processes. *Bernoulli* **6(2)** 323–338.
- [B] Biggins, J. (1992) Uniform convergence in the branching random walk, *Ann. Probab.*, **20**, 137–151.
- [BK] Biggins, J.D. and Kyprianou A.E. (2001) Measure change in multitype branching. *Preprint*
- [CH] Champneys, A, Harris, S.C., Toland, J., Warren, J. and Williams, D. (1995) Algebra, analysis and probability for a coupled system of reaction-diffusion equations. *Phil. Trans. R. Soc. Lond.*, **350**, 69–112.
- [CR] Chauvin, B. and Rouault, A. (1988) KPP equation and supercritical branching Brownian motion in the subcritical speed area. Application to spatial trees. *Probab. Theory Related Fields*, 80, 299–314.
- [D] Dawson, D. A. (1993) *Measure-Valued Markov Processes*. Ecole d'Eté Probabilités de Saint Flour XXI., LNM **1541**, 1–260.
- [Du] Durrett, R. (1995) *Probability: Theory and Examples*. 2nd Edition. Duxbury press.
- [Dy] Dynkin, E. B. (1994) An introduction to branching measure-valued processes. CRM Monograph Series, 6. American Mathematical Society, Providence, RI, 134 pp.
- [E] Engländer, J. (2007) Law of large numbers for superdiffusions: the non-ergodic case, preprint.
- [EP] Engländer, J. and Pinsky, R. (1999) On the construction and support properties of measure-valued diffusions on $D \subseteq R^d$ with spatially dependent branching, *Ann. Probab.* 27(2), 684–730.
- [EK] Engländer, J. and Kyprianou, A. (2003) Local extinction versus local exponential growth for spatial branching processes, *Ann. Probab.* 32(1A), 78–99.
- [ET] Engländer, J. and Turaev, D. (2002) A scaling limit theorem for a class of superdiffusions. *Ann. Probab.* 30(2) 683–722.
- [EW] Engländer, J. and Winter, A. (2006) Law of large numbers for a class of superdiffusions. *Ann. Inst. H. Poincaré B*, 42(2), 171–185.
- [E] Etheridge, A. (2000) An introduction to superprocesses. AMS lecture notes.
- [Ev] Evans, S.N. (1993) Two representations of a superprocess. *Proc. Royal. Soc. Edin.* 123A 959–971.
- [GHH] Git, Y., Harris, J. W. and Harris, S. C. (2007) Exponential growth rates in a typed branching diffusion. *Ann. Appl. Probab.* 17(2), 609–653.
- [H] Harris, S. C. (2000) Convergence of a “Gibbs-Boltzman” random measure for a typed branching diffusion. *Séminaire de Probabilités XXXIV. Lecture Notes in Math.* 1729, 239–256. Springer, Berlin.

- [HHa] Hardy, R. and Harris, S. C. (2004) A new formulation of the spine approach. Preprint available at <http://arxiv.org/abs/math.PR/0611054>.
- [HHb] Hardy, R. and Harris, S. C. (2004) Spine proofs for \mathcal{L}^p -convergence of branching-diffusion martingales. Preprint available at <http://arxiv.org/abs/math.PR/0611056>.
- [HHc] Hardy, R. and Harris, S. C. (2006) A conceptual approach to a path result for branching Brownian motion. *Stoc. Proc. Appl.* 116(12), 1992–2013.
- [JS] Jacod, J., Shiryaev, A. N. (2003) Limit theorems for stochastic processes. Second edition. Grundlehren der Mathematischen Wissenschaften, 288. Springer-Verlag.
- [LPP] Lyons, R., Pemantle R. and Peres, Y. (1995) Conceptual proofs of $L \log L$ criteria for mean behaviour of branching processes. *Ann. Probab.* 23, 1125–1138.
- [P95] Pinsky, R. G. (1995) Positive Harmonic Functions and Diffusion. Cambridge University Press.
- [P96] Pinsky, R. G. (1996) Transience, recurrence and local extinction properties of the support for supercritical finite measure-valued diffusions. *Ann. Probab.* 24(1), 237–267.
- [W] Watanabe, S. (1968) A limit theorem of branching processes and continuous state branching processes. *J. Math. Kyoto Univ.* 8, 141–167.

DEPARTMENT OF STATISTICS AND APPLIED PROBABILITY, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106-3110, USA

E-mail address: englander@pstat.ucsb.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF BATH, BATH, BA2 7AY, UK

E-mail address: S.C.Harris@bath.ac.uk

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF BATH, BATH, BA2 7AY, UK

E-mail address: a.kyprianou@bath.ac.uk