Strong law of large numbers for supercritical super-diffusions

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1. \((L, \beta; D)\)-supercritical branching particle diffusion

- Scatter \(n\) ‘initial ancestors’ scattered in \(D \subseteq \mathbb{R}^d\) at positions \(x_1, \cdots, x_n\).
- Write \(\nu(x) = \sum_{i=1}^{n} \delta_{x_i}(dx)\) for the measure describing the initial state of the system.
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- From each point, issue an \(L\)-diffusion. Here we take

\[
L = \frac{1}{2} \nabla \cdot a(x) \nabla + b(x) \cdot \nabla \quad \text{on} \quad D.
\]

(absorption in \(\partial D\) allowed, \(a\) is a positive-definite matrix and \(b\) a vector, both are \(C^{1,\eta}(D)\) for some \(\eta \in (0, 1]\))
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- The resulting process is an (atomic) measure-valued Markov process $\{Z_t : t \geq 0\}$ where $Z_t(dx) = \sum_{i=1}^{N_t} \delta_{x_i(t)}(dx)$, where $\{x_1(t), \cdots, x_{N_t}(t)\}$ is the spatial configuration of the $N_t$ particles that are in existence at time $t$. 

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- The resulting process is an (atomic) measure-valued **Markov** process \(\{Z_t : t \geq 0\}\) where \(Z_t(dx) = \sum_{i=1}^{N_t} \delta_{x_i(t)}(dx)\), where \(\{x_1(t), \ldots, x_{N_t}(t)\}\) is the spatial configuration of the \(N_t\) particles that are in existence at time \(t\).

- We denote its law by \(\mathbb{P}_\nu\).
2. \((L, \beta; D)\)-supercritical branching particle diffusion

One way to characterise the evolution of the Markov process \(Z\) is to study its transition semi-group through

\[
\mathbb{E}_\nu[e^{-\langle f, Z_t \rangle}] = \prod_{i=1}^{n} v_f(x_i, t)
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where

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v_f(x, t) = \mathbb{E}_{\delta_x}[e^{-\langle f, Z_t \rangle}], \quad x \in D, t \geq 0
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- We get

\[
\frac{\partial}{\partial x} v_f(x, t) = Lv_f(x, t) + \beta(x)[v_f(x, t)^2 - v_f(x, t)] = 0, \quad x \in D, t \geq 0.
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with \(v_f(x, 0) = \exp\{-f(x)\}, \ x \in D\).
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- Can generalise this class of Markov processes and talk about measure-valued processes, such that the measure need not be atomic-valued.
3. \((L, \beta, \alpha; D)\)-superdiffusions

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- We will work with the definition of a superdiffusion on \(D \subseteq \mathbb{R}^d\), \(X = \{X_t : t \geq 0\}\) as a Markov process valued in the space of finite measures on \(D\), denoted by \(\mathcal{M}_F(D)\), with probabilities \(\{P_\mu : \mu \in \mathcal{M}_F(D)\}\), such that

\[
E_\mu[e^{-\langle f, X_t \rangle}] = \exp \left\{ \int_D u_f(x, t) \mu(dx) \right\},
\]

where

\[
\frac{\partial}{\partial t} u_f(x, t) = Lu_f(x, t) - \psi(u_f(x, t), x), \quad x \in D, t \geq 0
\]

with \(u_f(x, 0) = f(x), x \in D\) and

\[
\psi(\lambda, x) = -\beta(x)\lambda + \alpha(x)\lambda^2, \quad \lambda \in \mathbb{R}, x \in D,
\]

with \(\alpha, \beta \in C^\eta\) and \(\alpha \geq 0\).
4. Linear semigroups

For both \((L, \beta; D)\) branching particle diffusions and \((L, \beta, \alpha; D)\) superprocesses, the linear operator \(L + \beta\) plays a special role.

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\mathbb{E}_{\delta_x} [< f, Z_t >] = \mathbb{E}_{\delta_x} [< f, X_t >] = w_f(x, t),
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where

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\frac{\partial}{\partial x} w_f(x, t) = (L + \beta(x)) w_f(x, t), \quad x \in D, t \geq 0,
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- Spectral properties of \(L + \beta\) tell us something about spatial growth:

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\lambda_c = \lambda_c(L + \beta; D) = \inf\{\lambda : \exists h > 0 \text{ s.t.} (L + \beta - \lambda)h = 0\}
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- Local extinction is the event that a given (and it turns out subsequently all) compact domain(s), \(B \subset \subset D\) becomes empty: \(\exists T(\omega) < \infty\) such that \(X_{T+t}(B) = 0 \forall t \geq 0\). [Concept obviously still OK for \(Z\) as well]
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Theorem: (Englander-Pinsky '99, Englander-K '04) Local extinction iff \(\lambda_c \leq 0\). [Theorem doesn’t care if you talk about branching particle diffusions or superprocesses]
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  \item Linear semi-group suggests (and it is true) that

  \[ W_t^\phi(X) := e^{-\lambda_c t} \langle \phi, X_t \rangle, \quad t \geq 0 \]

  is a martingale: when $\lambda_c > 0$ (supercriticality) this martingale is uniformly integrable.
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- Change of measure and spine decomposition (see blackboard). For $\mu \in \mathcal{M}_F(D)$ such that $\langle \phi, \mu \rangle < \infty$,

$$\frac{dP^\phi_\mu}{dP_\mu} \bigg|_{\sigma(X_s: s \leq t)} = e^{-\lambda_c t} \frac{\langle \phi, X_t \rangle}{\langle \phi, \mu \rangle}$$
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- It turns out that the spine is a diffusion with generator $(L + \beta - \lambda_c)^{\phi}$: here we use the usual notation for Doob $h$-transform to a generator $A$ (with potential term)
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  A^h f = \frac{1}{h} A(h f).
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  $$A^hf = \frac{1}{h}A(hf).$$
- $\tilde{\phi}$ is the groundstate of the adjoint of $L + \beta - \lambda_c$ and the assumption $\langle \tilde{\phi}, \phi \rangle < \infty$ (and hence $\langle \tilde{\phi}, \phi \rangle = 1$) ensures that the spine is an ergodic diffusion with stationary distribution density $\tilde{\phi} \phi$. 


6. Laws of large numbers

We also see the spine by studying the linear semi-group, for "nice" $f$,

$$e^{-\lambda c t} E_{\mu} [\langle f, X_t \rangle] = \int_D \frac{f(y)}{\phi(y)} p^{(L+\beta-\lambda c)}(x, dy, t) \phi(x) \mu(dx) \xrightarrow{t \to \infty} \langle f, \tilde{\phi} \rangle \langle \phi, \mu \rangle$$
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- Assuming $\lambda_c > 0$ and $\langle \tilde{\phi}, \phi \rangle = 1$ the above limit as well as the fact that $W_\phi^\infty(X)$ is an UI limit is strongly suggestive that $\lambda_c$ is in fact the growth rate on compacta in the sense that a limit for

$$e^{-\lambda_c t} X_t(B) \quad \text{as} \quad t \to \infty$$

exists in some sense for all $B \subset \subset D$. (see blackboard)
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- A number of attempts have been made to address this, but only with weak convergence or strong convergence with restrictive conditions. [Englander-Turaev ’02, Fleischman-Swart ’03, Englander-Winter ’06, Liu-Ren-Song ’13]. But more success with branching particle diffusions where generic strong laws have been obtained [Englander-Harris-K ’10]
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- **Theorem:** Suppose that \( \lambda_c > 0 \), \( \langle \tilde{\phi}, \phi \rangle = 1 \), \( \| \alpha \phi \|_\infty < \infty \) and (Mystery Hypothesis), then, for all \( 0 \leq f \leq \phi \) and \( \mu \in \mathcal{M}_F(D) \) such that \( \langle \phi, \mu \rangle < \infty \) and \( \mu \in \mathcal{M}_F(D) \),

\[
\lim_{t \to \infty} e^{-\lambda_c t} \langle f, X_t \rangle = \langle f, \tilde{\phi} \rangle W^\phi_\infty(X) \quad \mathbb{P}_\mu\text{-a.s.}
\]
7. The skeleton

The event $\mathcal{E} := \{\exists t \geq 0 : X_t(D) = 0\}$ generates the rate function $\omega$ in the following sense:

$$P_\mu(\mathcal{E}) = e^{-\langle \omega, \mu \rangle}.$$
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- Moreover, the ground state of this $(L_0^\omega, \alpha\omega; D)$ branching diffusions is precisely $\varphi/\omega$ with eigenvalue $\lambda_c$ because
  $$L_0^\omega(\varphi/\omega) + \alpha\omega(\varphi/\omega) - \lambda_c(\varphi/\omega) = (L + \beta(x) - \lambda_c)\varphi = 0.$$
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  \[ L_0^\omega(\varphi/\omega) + \alpha\omega(\varphi/\omega) - \lambda_c(\varphi/\omega) = (L + \beta(x) - \lambda_c)\varphi = 0. \]
- One similarly shows that the ground state of the adjoint of $L_0^\omega + \alpha\omega$ is $\omega\tilde{\varphi}$.
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- If one changes measure on the skeleton process using the martingale \( W_t^{\phi/\omega}(Z) \), then a spine decomposition emerges (see blackboard) for which the spine has diffusion operator

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(L_0^\omega + \alpha\phi - \lambda_c)^{\phi/\omega} = (L + \beta - \lambda_c)^{\phi}
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- A SLLN for the skeleton would thus read: for $0 \leq f \leq \phi/\omega$

$$ \lim_{t \to \infty} e^{-\lambda_c t} \langle f, Z_t \rangle = \langle f, \omega \tilde{\phi} \rangle W_\infty^{\phi/\omega}(Z) \quad (\ast) $$
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  $$(L_0^\omega + \alpha \phi - \lambda_c)^{\phi/\omega} = (L + \beta - \lambda_c)^\phi$$

  and has stationary distribution $(\phi/\omega)(\omega \tilde{\phi}) = \phi \tilde{\phi}$.
- Remarkably one can prove that $W_\infty^{\phi}(X) = W_\infty^{\phi/\omega}(Z)$ almost surely.
- A SLLN for the skeleton would thus read: for $0 \leq f \leq \phi/\omega$

  $$\lim_{t \to \infty} e^{-\lambda c t} \langle f, Z_t \rangle = \langle f, \omega \tilde{\phi} \rangle W_\infty^{\phi/\omega}(Z)$$

  (*)&

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**Theorem:** Suppose that $\lambda_c > 0$, $\langle \tilde{\phi}, \phi \rangle = 1$, $||\alpha \phi||_\infty < \infty$ and (*) holds along all lattice sequences $\delta \mathbb{N}$, $\delta > 0$, then, for all $0 \leq f \leq \phi$ and $\mu \in \mathcal{M}_F(D)$ such that $\langle \phi, \mu \rangle < \infty$ and $\mu \in \mathcal{M}_F(D)$,

$$\lim_{t \to \infty} e^{-\lambda c t} \langle f, X_t \rangle = \langle f, \tilde{\phi} \rangle W_\infty^{\phi}(X) \quad \mathbb{P}_\mu\text{-a.s.}$$
9. Why the skeleton is a natural approach

First note, it suffices to prove that for $0 \leq f \leq \phi$

$$\liminf_{t \to \infty} e^{-\lambda_c t} \langle f, X_t \rangle \geq \langle f, \tilde{\phi} \rangle W^\phi_\infty (X)$$

Then consider the same liming with $f$ replaced by $\phi - f$: this give the limsup.
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(See blackboard)
10. Examples

- The mystery condition looks ugly, but it is easily verified thanks to SLLN for branching particle diffusions in (Englander, Harris, K. ’10).
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- **Example 1.** [Super-outward-OU process with constant branching] Suppose $D = \mathbb{R}^d,$

$$L = \frac{1}{2} \Delta + \gamma x \cdot \nabla,$$

$\beta$ is a constant valued in $(\gamma d, \infty)$ and $\alpha$ is uniformly bounded. Then,

$$\lambda_c = \beta - \gamma d, \quad \phi(x) = (\gamma / \pi)^{d/2} \exp\{-||x||^2\}, \quad \tilde{\phi}(x) = 1$$

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  All conditions, in particular (mystery condition), is automatically satisfied.

- **Example 2.** (Continuing unfinished work of Fleischmann & Swart ’03). [Super-Fisher-Wright diffusion] Suppose $D = (0, 1)$, $\beta > 1$ (constant) and $\alpha(x)$ uniformly bounded and \[ L = \frac{1}{2} x(1-x) \frac{d^2}{dx^2} \]
in which case \[ \lambda_c = \beta - 1, \quad \phi(x) = 6x(1-x), \quad \tilde{\phi}(x) = 1 \]