Strong law of large numbers for supercritical super-diffusions

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1. $(L, \beta; D)$-supercritical branching particle diffusion

- Scatter a n 'initial ancestors' scattered in $D \subseteq \mathbb{R}^d$ at positions $x_1, \cdots, x_n$. Write $\nu(x) = \sum_{i=1}^n \delta_{x_i}(\mathrm{d}x)$ for the measure describing the initial state of the system.

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- From each point, issue an L-diffusion. Here we take

$$L = \frac{1}{2}\nabla \cdot a(x)\nabla + b(x) \cdot \nabla \qquad \text{on} \qquad D.$$

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- The resulting process is an (atomic) measure-valued **Markov** process $\{Z_t : t \geq 0\}$ where $Z_t(\mathrm{d}x) = \sum_{i=1}^{N_t} \delta_{x_i(t)}(\mathrm{d}x)$, where $\{x_1(t), \cdots, x_{N_t}(t)\}$ is the spatial configuration of the N_t particles that are in existence at time t.

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- We denote its law by $\mathbb{P}_\nu$.

2. $(L, \beta; D)$-supercritical branching particle diffusion

- One way to characterise the evolution of the Markov process Z is to study its transition semi-group through

$$\mathbb{E}_\nu[\mathrm{e}^{-\langle f, Z_t\rangle}] = \prod_{i=1}^{n} v_f(x_i, t)$$

where

$$v_f(x,t) = \mathbb{E}_{\delta_x}[\mathrm{e}^{-\langle f, Z_t\rangle}], \qquad x \in D, t \geq 0,$$

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- We get

$$\frac{\partial}{\partial x} v_f(x,t) = L v_f(x,t) + \beta(x)[v_f(x,t)^2 - v_f(x,t)] = 0, \qquad x \in D, t \geq 0.$$

with $v_f(x, 0) = \exp\{-f(x)\}$, $x \in D$.

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- Can generalise this class of Markov processes and talk about measure-valued processes, such that the measure need not be atomic-valued.

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- Can defined a superdiffusion through a process of approximation of branching particle diffusions (but we won't here).
- We will work with the definition of a superdiffusion on $D \subseteq \mathbb{R}^d$, $X = \{X_t : t \geq 0\}$ as a Markov process valued in the space of finite measures on D, denoted by $\mathcal{M}_F(D)$, with probabilities $\{\mathbf{P}_\mu : \mu \in \mathcal{M}_F(D)\}$, such that

$$\mathbf{E}_\mu[\mathrm{e}^{-\langle f, X_t\rangle}] = \exp\left\{\int_D u_f(x,t)\mu(\mathrm{d}x)\right\},$$

where

$$\frac{\partial}{\partial t}u_f(x,t) = Lu_f(x,t) - \psi(u_f(x,t), x), \qquad x \in D, t \geq 0$$

with $u_f(x,0) = f(x)$, $x \in D$ and

$$\psi(\lambda, x) = -\beta(x)\lambda + \alpha(x)\lambda^2, \qquad \lambda \in \mathbb{R}, x \in D,$$

with $\alpha, \beta \in C^\eta$ and $\alpha \geq 0$.

4. Linear semigroups

- For both $(L,\beta;D)$ branching particle diffusions and $(L,\beta,\alpha;D)$ superprocesses, the linear operator $L+\beta$ plays a special role.

$$\mathbb{E}_{\delta_x}[< f, Z_t >] = \mathbb{E}_{\delta_x}[< f, X_t >] = w_f(x,t),$$

where

$$\frac{\partial}{\partial x} w_f(x,t) = \big(L+\beta(x)\big) w_f(x,t), \qquad x \in D, t \geq 0,$$

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- Spectral properties of $L + \beta$ tell us something about spatial growth:

$$\lambda_c = \lambda_c(L + \beta; D) = \inf\{\lambda : \exists h > 0 \text{ s.t. } (L + \beta - \lambda)h = 0\}$$

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- *Local extinction* is the event that a given (and it turns out subsequently all) compact domain(s), $B \subset\subset D$ becomes empty: $\exists T(\omega) < \infty$ such that $X_{T+t}(B) = 0 \ \forall t \geq 0$. [Concept obviously still OK for Z as well]

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- **Theorem:** (Englander-Pinsky '99, Englander-K '04) Local extinction iff $\lambda_c \leq 0$. [Theorem doesn't care if you talk about branching particle diffusions or superprocesses]

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- Change of measure and spine decomposition (see blackboard). For $\mu \in \mathcal{M}_F(D)$ such that $\langle \phi, \mu \rangle < \infty$,

$$\left.\frac{\mathrm{d}\mathbf{P}_\mu^\phi}{\mathrm{d}\mathbf{P}_\mu}\right|_{\sigma(X_s : s \leq t)} = \mathrm{e}^{-\lambda_c t}\frac{\langle \phi, X_t \rangle}{\langle \phi, \mu \rangle}$$

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- It turns out that the spine is a diffusion with generator $(L + \beta - \lambda_c)^\phi$: here we use the usual notation for Doob h-transform to a generator A (with potential term)

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- $\widetilde{\phi}$ is the groundstate of the adjoint of $L+\beta-\lambda_c$ and the assumption $\langle\widetilde{\phi}, \phi\rangle < \infty$ (and hence $\langle\widetilde{\phi}, \phi\rangle = 1$) ensures that the spine is an ergodic diffusion with stationary distribution density $\widetilde{\phi}\phi$.

6. Laws of large numbers

- We also see the the spine by studying the linear semi-group, for "nice" f,

$$\mathrm{e}^{-\lambda_c t}\mathbf{E}_\mu[\langle f, X_t\rangle] = \int_D \frac{f(y)}{\phi(y)} p^{(L+\beta-\lambda_c)^\phi}(x, \mathrm{d}y, t)\phi(x)\mu(\mathrm{d}x) \stackrel{t\to\infty}{\to} \langle f, \widetilde{\phi}\rangle\langle \phi, \mu\rangle$$

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- Assuming $\lambda_c > 0$ and $\langle\widetilde{\phi}, \phi\rangle = 1$ the above limit as well as the fact that $W^\phi_\infty(X)$ is an UI limit is strongly suggestive that λ_c is in fact the growth rate on compacta in the sense that a limit for

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- A number of attempts have been made to address this, but only with weak convergence or strong convergence with restrictive conditions. [Englander-Turaev '02, Fleischman-Swart '03, Englander-Winter '06, Liu-Ren-Song '13]. But more success with branching particle diffusions where generic strong laws have been obtained [Englander-Harris-K '10]

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- **Theorem:** Suppose that $\lambda_c > 0$, $\langle \widetilde{\phi}, \phi\rangle = 1$, $||\alpha\phi||_\infty < \infty$ and **(Mystery Hypothesis)**, then, for all $0 \leq f \leq \phi$ and $\mu \in \mathcal{M}_F(D)$ such that $\langle \phi, \mu\rangle < \infty$ and $\mu \in \mathcal{M}_F(D)$,

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda_c t}\langle f, X_t\rangle = \langle f, \widetilde{\phi}\rangle W^\phi_\infty(X) \qquad \mathbf{P}_\mu\text{-a.s.}$$

7. The skeleton

- The event $\mathcal{E} := \{\exists t \geq 0 : X_t(D) = 0\}$ generates the rate function ω in the following sense:

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- Moreover, the ground state of this $(L_0^\omega, \alpha\omega; D)$ branching diffusions is precisely φ/ω with eigenvalue λ_c because

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$$L_0^\omega(\phi/\omega) + \alpha\omega(\phi/\omega) - \lambda_c(\phi/\omega) = (L + \beta(x) - \lambda_c)\phi = 0.$$
- One similarly shows that the ground state of the adjoint of $L_0^\omega + \alpha\omega$ is $\omega\widetilde{\phi}$.

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- A SLLN for the skeleton would thus read: for $0 \leq f \leq \phi/\omega$

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$$\lim_{t\to\infty} \mathrm{e}^{-\lambda_c t}\langle f, Z_t\rangle = \langle f, \omega\widetilde{\phi}\rangle W_\infty^{\phi/\omega}(Z) \qquad (*)$$

- **Theorem:** Suppose that $\lambda_c > 0$, $\langle \widetilde{\phi}, \phi\rangle = 1$, $||\alpha\phi||_\infty < \infty$ and **(*) holds along all lattice sequences $\delta\mathbb{N}$, $\delta > 0$,** , then, for all $0 \leq f \leq \phi$ and $\mu \in \mathcal{M}_F(D)$ such that $\langle \phi, \mu\rangle < \infty$ and $\mu \in \mathcal{M}_F(D)$,

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda_c t}\langle f, X_t\rangle = \langle f, \widetilde{\phi}\rangle W_\infty^\phi(X) \qquad \mathbf{P}_\mu\text{-a.s.}$$

9. Why the skeleton is a natural approach

- First note, it suffices to prove that for $0 \leq f \leq \phi$

$$\liminf_{t\to\infty} \mathrm{e}^{-\lambda_c t}\langle f, X_t\rangle \geq \langle f, \widetilde{\phi}\rangle W^{\phi}_{\infty}(X)$$

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10. Examples

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- **Example 1**. [Super-outward-OU process with constant branching] Suppose $D = \mathbb{R}^d$,

$$L = \frac{1}{2}\triangle + \gamma x \cdot \nabla,$$

β is a constant valued in $(\gamma d, \infty)$ and α is uniformly bounded. Then,

$$\lambda_c = \beta - \gamma d, \quad \phi(x) = (\gamma/\pi)^{d/2} \exp\{-||x||^2\}, \quad \widetilde{\phi}(x) = 1$$

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- **Example 2**. (Continuing unfinished work of Fleischmann & Swart '03). [Super-Fisher-Wright diffusion] Suppose $D = (0, 1)$, $\beta > 1$ (constant) and $\alpha(x)$ uniformly bounded and

$$L = \frac{1}{2}x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

in which case

$$\lambda_c = \beta - 1, \quad \phi(x) = 6x(1-x), \quad \widetilde{\phi}(x) = 1$$