Old and new examples of scale functions for spectrally negative Lévy processes

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[1]based on joint work with Friedrich Hubalek and Victor Rivero
"Old and new examples of scale functions for spectrally negative Lévy processes" (K. and Hubalek - preprint)
"Special, conjugate and complete scale functions for spectrally negative Lévy processes" (K. and Rivero - preprint).

Recollections

- $X = \{X_t : t \geq 0\}$ with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ will always denote a spectrally negative Lévy process (i.e. $\Pi(0,\infty) = 0$ and $-X$ is not a subordinator).

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- For $\theta \geq 0$ we may work with the Laplace exponent

$$\psi(\theta) := \log \mathbb{E}_0(e^{\theta X_1}).$$

- For each $q \geq 0$, the, so-called, q-scale function $W^{(q)} : \mathbb{R} \mapsto [0,\infty)$ defined by $W^{(q)}(x) = 0$ for $x < 0$ and otherwise continuous satisfying

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)dx = \frac{1}{\psi(\beta) - q}$$

for all β sufficiently large is fundamental to virtually **all** fluctuation identities concerning spectrally negative processes. For example the classical identity

$$\mathbb{E}_x(e^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+ < \tau_0^-)}) = \frac{W^{(q)}(x)}{W^{(q)}(a)}$$

for $q \geq 0$, $0 \leq x \leq a$.

Papers which (implicitly) use scale functions

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- Compound Poisson with negative exponentially distributed jumps of mean μ, arrival rate λ and positive drift c such that $c - \lambda/\mu > 0$.

$$W(x) = \frac{1}{c}\left(1 + \frac{\lambda}{c\mu - \lambda}(1 - e^{(\mu - c^{-1}\lambda)x})\right)$$

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- α-stable process with $\alpha \in (1, 2)$.

$$W(x) = x^{\alpha - 1}/\Gamma(\alpha)$$

Dig a little deeper ....

- Furrer (1998) studies ruin of an α-stable process with $\alpha \in (1,2)$ plus a drift ct and deduces that

$$W(x) = \frac{1}{c}(1 - E_{\alpha-1,1}(-cx^{\alpha-1}))$$

where

$$E_{\alpha-1,1}(z) = \sum_{k\geq 0} z^k/\Gamma(1+(\alpha-1)k)$$

is the two-parameter Mittag-Leffler function with indices $\alpha - 1$ and 1.

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- An unusual example from queuing theory due to Boxma and Cohen (1998). Let $\eta(x) = e^x \mathrm{erfc}(\sqrt{x})$ and consider a compound Poisson with rate λ satisfying $1 - \lambda > 0$, negative jumps with d.f. $F(x, \infty) = (2x + 1)\eta(x) - 2\sqrt{x/\pi}$ and unit positive drift. Then

$$W(x) = \frac{1}{1-\lambda}\left(1 - \frac{\lambda}{\nu_1 - \nu_2}(\nu_1\eta(x\nu_2^2) - \nu_2\eta(x\nu_1^2))\right).$$

where $\nu_{1,2} = 1 \pm \sqrt{\lambda}$.

- Asmussen in his book 'Ruin Probabilities' studies a compound Poisson with rate λ, negative jump of fixed size α and positive drift c. Then

$$W(x) = \frac{1}{c} \sum_{n=1}^{\lfloor x/\alpha \rfloor} e^{-\lambda(\alpha n - x)/c} \frac{1}{n!} \left(\frac{\lambda}{c} \right)^n (\alpha n - x)^n$$

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- Two new scale function occurring in study of Lévy-Lamperti processes [Chaumont, K. and Pardo (2007)]. The Lévy processes in question have unbounded variation processes with no Gaussian component and jump measure which is stable like (with stability parameter $\alpha \in (1,2)$) near the origin and has exponentially decaying tails. Their Laplace exponents are $\Gamma(\theta+\alpha)/[\Gamma(\theta)\Gamma(\alpha)]$ and $\Gamma(\theta-1+\alpha)/[\Gamma(\theta-1)\Gamma(\alpha)]$ and the respective scale functions are

$$W(x) = (1 - e^{-x})^{\alpha-1} \text{ and } W(x) = (1 - e^{-x})^{\alpha-1} e^{x}.$$

New examples: preliminaries

Henceforth we shall restrict ourselves to discussing the case of 0-scale functions for processes which do not drift to $-\infty$. [But with mild adaptation the the forthcoming methodology works for generating q-scale functions of oscillating LPs or LPs that drift to $-\infty$.]
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We make use of two simple facts.

- The definition of the scale function through its LT, integration by parts and the Wiener-Hopf factorization for spec. neg. processes imply that

$$\int_0^\infty e^{-\beta x} W'(x) dx = \frac{1}{\phi(\theta)}$$

where ϕ is the Laplace exponent of the descending ladder height process $H = \{H_t : t \geq 0\}$.

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- $$\int_0^\infty dt \cdot \mathbb{P}(H_t \in dx) = W'(x) dx.$$

A simple idea for generating scale functions

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- But then you need to know that a Lévy process exists for which your favourite H corresponds to its descending ladder height process.
- No problem - thanks to the Wiener-Hopf factorisation!

Parent process for given H

Suppose that H is a (killed) subordinator with Laplace exponent

$$\phi(\theta) = \kappa + \zeta\theta + \int_{(0,\infty)} (1 - e^{-\theta x})\Upsilon(dx)$$

such that Υ is absolutely continuous with monotone non-increasing density. Then there exists a spectrally negative Lévy process X, henceforth referred to as the **parent process**, such height process is precisely the process H. The Lévy triple (a, σ, Π) of the parent process is uniquely identified as follows. The Gaussian coefficient is given by $\sigma = \sqrt{2\zeta}$. The Lévy measure is given by

$$\Pi(-\infty, -x) = \frac{d\Upsilon(x)}{dx}. \quad (1)$$

Finally

$$a = \int_{(-\infty,-1)} x\Pi(dx) - \kappa. \quad (2)$$

Bounded and unbounded variation

- **When** $\Upsilon(0,\infty) < \infty$. The parent process is given by

$$X_t = (\kappa + \Upsilon(0,\infty))t + \sqrt{2\zeta}B_t - S_t \tag{3}$$

where $B = \{B_t : t \geq 0\}$ is a Brownian motion, $S = \{S_t : t \geq 0\}$ is an indpendent driftless subordinator with jump measure ν satisfying

$$\nu(x,\infty) = \frac{d\Upsilon}{dx}(x).$$

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- **When** $\Upsilon(0,\infty) = \infty$. The parent process X always has paths of unbounded variation.

Tempered stable descending ladder height

Introduce a new family of scale functions called **Gaussian-Tempered-Stable-Convolution class (GTSC)**. Choose a descending ladder height process H to have Laplace exponent:

$$\phi(\theta) = \kappa + \zeta\theta + c\Gamma(-\alpha)(\gamma^\alpha - (\gamma+\theta)^\alpha)$$

The associated Lévy measure is given by

$$\Upsilon(dx) = cx^{-\alpha-1}e^{-\gamma x}dx \qquad (x > 0).$$

Here the parameter regimes (partly to respect conditions on Υ) are:

$$\begin{gathered}\kappa \geq 0,\\ \zeta \geq 0\\ c > 0,\\ \alpha \in [-1, 1)\\ \gamma > 0.\end{gathered}$$

GTSC parent processes

The parent process with the tempered stable choice of H is characterized as follows:

$$\Pi(dx) = c\frac{\gamma}{|x|^{\alpha+1}}e^{-\gamma|x|}dx + c\frac{(\alpha+1)}{|x|^{\alpha+2}}e^{-\gamma|x|}dx \text{ for } x<0$$

ie jump part is the result of the independent sum of a tempered stable subordinator and another spectrally negative Lévy process, also belonging to the class of (generalized) tempered stable processes, but whose jump component has unbounded variation.

$$\sigma = \sqrt{2\zeta}$$

Instead of giving a we give the Laplace exponent of parent process

$$\psi(\theta) = \kappa\theta + \zeta\theta^2 + c\theta\Gamma(-\alpha)(\gamma^\alpha - (\gamma+\theta)^\alpha).$$

Take $\alpha = m/n \in \cap(0,1)$

Let $f(z) = \psi(z^n - \gamma)$. Because of the structure of the Laplace exponent a little algebra shows that

$$f(z) = c_0 + c_m z^m + c_n z^n + c_{m+n} z^{m+n} + c_{2n} z^{2n},$$

where

$$\begin{aligned} c_{2n} &= \zeta, \\ c_{n+m} &= -c\Gamma(-\alpha) \\ c_n &= \kappa - 2\zeta\gamma + c\Gamma(-\alpha)]\gamma^\alpha, \\ c_m &= c\Gamma(-\alpha)\gamma, \\ c_0 &= \zeta\gamma^2 - \gamma(\kappa - \zeta) - c\Gamma(-\alpha)\gamma^{\alpha+1}. \end{aligned}$$

Denote by $z_1, \ldots, z_\ell$ the distinct roots of $f(z)$ and by $m_1, \ldots, m_\ell$ their multiplicities and for $k = 1, 2, \ldots, \ell$ and $j = 1, \ldots, m_k$ let

$$A_{kj} = \frac{1}{(m_k - j)!} \frac{d^{m_k - j}}{dz^{m_k - j}} \left[\frac{(z - z_k)^{m_k}}{f(z)} \right]_{z = z_k}$$

Using Laplace inversion:

$$W(x) = \sum_{k=1}^{\ell} \sum_{j=0}^{m_k-1} A_{kj} \frac{1}{j!} e^{-\gamma x} x^{(j+1)/n-1} E^{(j)}_{\frac{1}{n},\frac{1}{n}}(z_k x^{\frac{1}{n}})$$

where

$$E^{(j)}_{\alpha,\beta}(x) = \frac{\partial^j}{\partial x^j} E_{\alpha,\beta}(x) = \sum_{n \geq k} \binom{n}{k} k! \frac{x^{n-k}}{\Gamma(n\alpha+\beta)}$$

and

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta+\alpha n)}$$

is the two-parameter Mittag-Leffler function.

Can perform similar analysis for the case $\alpha = -m/n \in [-1, 0)$

Special cases

- Take H as an inverse Gaussian subordinator: $\alpha = \frac{1}{2}, \kappa = \zeta = 0$, $\phi(\theta) = \delta((2\theta + \gamma^2)^{\frac{1}{2}} - \gamma)$ so that $\sigma = 0$ and

$$\Pi(dx) = \frac{3\delta}{2\sqrt{2\pi}}|x|^{-\frac{5}{2}} e^{-\frac{1}{2}\gamma^2|x|} + \frac{(\gamma^2/2 + \varphi)\delta}{\sqrt{2\pi}}|x|^{-\frac{3}{2}} e^{-\frac{1}{2}\gamma^2|x|} \text{ for } x < 0.$$

$$W(x) = \frac{1}{2\delta\gamma}\left[(1+\gamma^2 x)\mathrm{erfc}(-\gamma\sqrt{x/2}) + \gamma\sqrt{2x/\pi}\,e^{-\gamma^2/2x} - 1\right]$$

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- Take $\alpha \in (0,1), \kappa = \zeta = 0$: $\phi(\theta) = c\Gamma(-\alpha)(\gamma^\alpha - (\gamma + \theta)^\alpha)$ so that $\sigma = 0$ and

$$\Pi(dx) = c\frac{\gamma}{|x|^{\alpha+1}} e^{-\gamma|x|} dx + c\frac{(\alpha+1)}{|x|^{\alpha+2}} e^{-\gamma|x|} dx \text{ for } x < 0$$

$$W(x) = \int_0^x e^{-\gamma x} y^{\alpha-1} E_{\alpha,\alpha}(\gamma^\alpha y^\alpha) dy.$$

Computing potential density:

- Take H to be a Gamma process: $\alpha = \kappa = \zeta = 0$, $\phi(\theta) = -c\log(\gamma/(\gamma+\theta))$, so $\sigma = 0$

$$\Pi(dx) = c|x|^{-2}e^{-\gamma|x|}dx + c\gamma|x|^{-1}e^{-\gamma|x|}dx \text{ for } x < 0$$

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- Take H to be a compound Poisson with rate λ and gamma distributed jumps with parameters $\nu \in (0,1)$ and $\gamma > 0$, killed at rate $\kappa \geq 0$. Let $\rho = \lambda/(\lambda+\kappa)$

$$\Pi(dx) = \frac{\lambda(1-\nu)\gamma^{\nu}}{\Gamma(\nu)}|x|^{\nu-2}e^{-\gamma|x|}dx + \frac{\lambda\gamma^{\nu+1}}{\Gamma(\nu)}|x|^{\nu-1}e^{-\gamma|x|}dx \text{ for } x < 0$$

$$W(x) = \frac{1}{\lambda+\lambda} + \frac{\rho\gamma^{\nu}}{\lambda+\kappa}\int_0^x y^{\nu-1}e^{-\gamma y}E_{\nu,\nu}(\rho\gamma^{\nu}y^{\nu})dy.$$

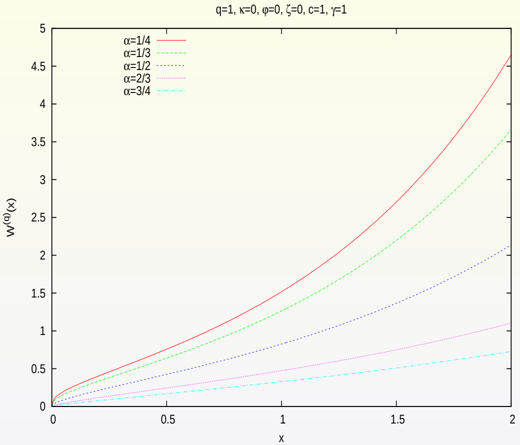

Figure: Scale functions $W^{(q)}(x)$ for a GTSC which oscillates: concavity/convexity proved by Ronnie Loeffen.

Special and conjugate scale functions

Instead of taking a tempered stable descending ladder height, take a **special Bernstein function**. That is to say, choose the Laplace exponent of the descending ladder height ϕ such that

$$\phi(\theta) = \kappa + \zeta\theta + \int_{(0,\infty)} (1 - e^{-\theta x})\Upsilon(dx) \quad \text{for } \theta \geq 0$$

with the assumption that Υ is absolutely continuous with a decreasing density and such that

$$\phi(\theta) = \frac{\theta}{\phi^*(\theta)} \quad \text{for } \theta \geq 0$$

where ϕ^* is also a Bernstein function (the conjugate to ϕ) which we shall write as

$$\phi^*(\theta) = \kappa^* + \zeta^*\theta + \int_{(0,\infty)} (1 - e^{-\theta x})\Upsilon^*(dx).$$

Special and conjugate scale functions ctd.

- Potential analysis of special Bernstein functions gives us an expression for the potential function associated to ϕ and hence an expression for the the **special scale function** whose parent process has Laplace exponent $\psi(\theta) = \theta\phi(\theta)$:

$$W(x) = \zeta^* + \kappa^* x + \int_0^x \Upsilon^*(y, \infty)\, dy$$

and W is a concave function.

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  and W is a concave function.

- If it so happens that Υ^* is absolutely continuous with non-increasing density, then we get the **conjugate scale function**

$$W^*(x) = \zeta + \kappa x + \int_0^x \Upsilon(y, \infty) dy.$$

  (also concave) whose parent process has Laplace exponent $\psi^*(\theta) = \theta\phi^*(\theta)$

Complete scale functions

- In particular if ϕ is a **complete Bernstein function** (ie it is a special Bernstein function such that Υ is absolutely continuous with a completely monotone density) then ϕ^* is completely monotone and both Υ and Υ^* have non-increasing densities.

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- Since most known examples of special Bernstein functions are complete Bernstein functions, the latter remarks suggest that for each known example of pairs of conjugate complete Bernstein functions one gets for free examples of (conjugate) parent processes and their scale functions.

Example

- Take $\phi(\theta) = a\theta^{\beta-\alpha} + b\theta^{\beta}$ where $0 < \alpha < \beta \leq 1$ and so parent process has Laplace exponent

$$\psi(\theta) = a\theta^{\beta-\alpha+1} + b\theta^{\beta+1},\ \theta \geq 0.$$

which is the sum of two independent spectrally negative stable processes.

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which is the sum of two independent spectrally negative stable processes.

- The conjugate parent process has Laplace exponent

$$\psi^*(\theta) = \frac{\theta^2}{a\theta^{\beta-\alpha} + b\theta^{\beta}}$$

which is an oscillating process, has no Gaussian component, and has Lévy measure given by

$$\Pi^*(-\infty, -x) = \frac{d^2}{dx^2}\left[\frac{1}{b}x^{\beta-1}E_{\alpha,\beta}(-ax^{\alpha}/b)\right]$$

- Finally the scale functions are given by

$$W(x) = \frac{1}{b}\int_0^x t^{\beta-1} E_{\alpha,\beta}(-at^{\alpha}/b)\,dt$$

and

$$W^*(x) = \frac{a}{\Gamma(2-\beta+\alpha)} x^{1-\beta+\alpha} + \frac{b}{\Gamma(2-\beta)} x^{1-\beta}, \; x \geq 0.$$

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- Note also that scale functions are continuous in the parameters of ψ and ψ^* and hence we may vary the parameters a, b, α, β to their extremes to get other examples. Eg

$$W^*(x) = b + \frac{a}{\Gamma(1-\alpha)} x^\alpha$$

is a scale function!

Moral of the story:

The Wiener-Hopf factorisation gives us a route for chanelling potential theory of subordinators into the theory of scale functions for spectrally negative Lévy processes.