Optimal Stopping Theory and Lévy processes*

Erik Baurdoux

LSE/CIMAT
e.j.baurdoux@lse.ac.uk/baurdoux@cimat.mx

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Outline

- Secretary problem: finding best candidate
- Optimal stopping for Lévy processes
- American put
- Optimal prediction of the maximum of a Lévy process
  - in space
  - and in time

Optimal stopping

Examples:

- Secretary problem
- Quickest detection: minimise false alarm and delay.
- American options: when to exercise?

They all involve a decision to be made based on an observable, random process. Let X be a Lévy process $q \geq 0$ and g a positive function such that

$$V(x) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[e^{-q\tau} g(X_\tau)],$$

where $\mathbb{E}_x$ is the expectation operator when $X_0 = x$.

- Optimal time τ?
- Value function V?

Secretary problem; classical problem

One job, n candidates.

Job candidates present themselves one after the other. Candidates can be ranked and arrive in a random order. Either offer job to current candidate or reject (not allowed to return to a previously rejected candidate).

Strategy to follow to maximise probability of choosing overall best candidate?

Secretary problem

Clearly: not wise to choose the first candidate, as better are likely to come later.
Also not wise to wait too long, as best one already passed.Optimal stopping problem, as strategy depends on the observed "process".
Optimal stopping time (as n becomes large): Reject first n/e candidate and pick the first one after who is better than all the previous ones.
Probability of getting the best one: $1/e$

Idea of solution

Can be shown that optimal strategy of the form: ignore first $r-1$ candidates and picks the first one after that better than all previous.For such strategy, probability of picking overall best one

$$
\begin{aligned}
&= \sum_{i=r}^{n} \mathbb{P}(\text{best amongst first } i-1 \text{ is among first } r-1, i \text{ is best}) \\
&= \sum_{i=r}^{n} \mathbb{P}(\text{best amongst first } i-1 \text{ is among first } r-1 | i \text{ is best}) \times \frac{1}{n} \\
&= \quad \frac{1}{n} \sum_{i=r}^{n} \frac{r-1}{i-1}.
\end{aligned}
$$

With n large this is maximised by $r = n/e$.

Lévy process (Thanks Juan Carlos!)

Lévy process $\{X_t\}_{t\geq 0}$ has stationary, independent increments w.r.t. some $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$:
For $0 \leq s \leq t \leq u \leq v$

$$X_v - X_u \quad \text{and } X_t - X_s$$

are independent and for any $s, t \geq 0$

$$X_{t+s} - X_s \quad \text{and } X_t$$

have the same distribution.
Furthermore $X_0 = 0$ and X has càdlàg paths.
Examples are

- Brownian motion: increments $\mathcal{N}(0, t)$
- Poisson process: increments Poi(λt)
- Stable process: increments α-stable.

We shall consider real-valued Lévy processes.

Stopping times

A nonnegative random variable τ is a stopping time (for X) if for any $t \in \mathbb{R}_{\geq 0}$

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

We do not rule out τ for which $\mathbb{P}(\tau = \infty) > 0$.
Denote by $\mathcal{T}$ the set stopping times. "A stopping time does not depend on future values of X".

Examples:

- $t \in \mathbb{R}_{\geq 0}$.
- $\tau_A := \inf\{t : X_t \in A\}$ for A closed or open.

Not a stopping time:

$$\sup\{t : X_t \geq 3\}.$$

Strong Markov Property for LP

For fixed (non-random) $T > 0$, the process $\{Y_t\}_{t\geq 0}$

$$Y_t := X_{T+t} - X_T$$

is independent of $\mathcal{F}_T$ and has the same law as $\{X_t\}_{t\geq 0}$.
Similar result for a stopping time τ

$$Z_t := X_{\tau+t} - X_\tau \quad \text{on } \{\tau < \infty\}$$

is independent of $\mathcal{F}_\tau$ and has the same law as X. Here

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t\}.$$

Simple example of optimal stopping

Let B be a Brownian motion and g a smooth function that has a unique maximum at x^*. Let

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[g(B_\tau)].$$

Then $V(x) = g(x^*)$ for all x and

$$\tau^* = \inf\{t : B_t = x^*\}$$

is optimal.Note that

$$\tau^* = \inf\{t : B_t \in D\}$$

with $D = \{x^*\} = \{x \in \mathbb{R} : V(x) = g(x)\}$.
Also, $V'(x^*) = 0 = g'(x^*)$ (Smooth fit!)

General result

Consider

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[e^{-q\tau} g(X_\tau)].$$

Optimal stopping time exists (under mild conditions on g) and

$$\tau^* = \inf\{t : X_t \in D\}$$

with

$$D = \{x \in \mathbb{R} : V(x) = g(x)\}.$$

Independent of time due to strong Markov property and infinite horizon.
Aim: find D, τ^*?
Also, rule of thumb: smooth fit:

$$V'(x^*) = g'(x^*) \quad \text{for } x^* \in \partial D$$

depending on regularity of the process (more on this later!)

American options

European option with maturity T and pay-off function g has value function V

$$V(x) = \mathbb{E}[e^{-qT} g(X_T)]$$

where $q > 0$ discount rate. European put: $g(x) = \max(0, K - e^x)$. Here $K > 0$ is strike price.
Americans have more choice:

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-q\tau} g(X_\tau)].$$

What is V, and optimal time τ^*?General theory:

$$\tau^* = \inf\{t : X_t \in D\}$$

with

$$D = \{x \in \mathbb{R} : V(x) = g(x)\}.$$

Guess and verify

Lemma (Verification)

Let τ_0 be a stopping time and let

$$V_0(x) = \mathbb{E}[e^{-q\tau_0} g(X_{\tau_0})].$$

If $V_0(x) \geq g(x)$ for all x and if $e^{-qt} V_0(X_t)$ is a supermartingale, then τ_0 is optimal and $V_0(x) = V(x)$.

Proof: Note $V(x) \geq V_0(x)$. Now, for any stopping time τ

$$\begin{aligned} \mathbb{E}_x[e^{-q\tau} g(X_\tau)] &= \lim_{t\to\infty} \mathbb{E}_x[e^{-q(\tau\wedge t)} g(X_{\tau\wedge t})] \\ &\leq \lim_{t\to\infty} \mathbb{E}_x[e^{-q(\tau\wedge t)} V_0(X_{\tau\wedge t})] \\ &\leq V_0(x). \end{aligned}$$

Now take supremum over all $\tau \in \mathcal{T}$.

How to choose τ_0?

Free boundary problem

If V is sufficiently smooth then $e^{-qt}V(X_t)$ is a supermartingale when

$$(\mathcal{L} - q)V \leq 0$$

where

$$(\mathcal{L}f)(x) = bf'(x) + \frac{c}{2}f''(x) + \int_{\mathbb{R}} (f(x+y) - f(x) - yf'(x)1_{\{|y|<1\}})\nu(\mathrm{d}x).$$

Free boundary problem

$$\max((\mathcal{L} - q)V, g - V) = 0$$

and "sometimes" $g'(x) = V'(x)$ on $\partial\{x \in \mathbb{R} : V(x) = g(x)\}$.
Other approaches: viscosity solutions, linear programming, dual problem.

American put for Brownian motion

Theorem (Brownian case, $q = 1/2$)

An optimal stopping time for American put option is $\tau^* = \tau^-_{\log K/2}$ *and* $V(x) = \frac{K^2}{4} e^{-x}$.

Proof: The free boundary problem in this case is

$$\begin{aligned} V''(x) &= V(x) \quad \text{for } x \in D^c, \\ V(x) &= (K - e^x)^+ \quad \text{for } x \in D, \\ V(x) &\geq (K - e^x)^+ \quad \text{for } x \in \mathbb{R}, \\ V''(x) &\leq V(x) \quad \text{for } x \in \mathbb{R}, \\ V'(x) &= -e^x \quad \text{for } x \in \partial D. \end{aligned}$$

First equation gives $V(x) = ae^x + be^{-x}$ on D^c. Since $[\log K, \infty) \cap D = \emptyset$ and V bounded yields get $a = 0$. For $x \in \partial D$ we have $be^{-x} = V(x) = K - e^x$ and $-be^{-x} = V'(x) = -e^x$ which gives $\partial D = \{\log K/2\}$ and $b = K^2/4$. Other conditions are easily checked.

American put for general Lévy: direct approach

Direct approach can be summarised as follows:

- Show directly that τ^* must be of a certain form, e.g. $\tau_{x^*}^- = \inf\{t \geq 0 : X_t \leq x^*\}$ for some x^*.
- Calculate
$$V(x, x^*) = \mathbb{E}_x[e^{-q\tau_{x^*}^-} g(X_{\tau_{x^*}^-})].$$
- Find $V(x) = \sup_{x^*} V(x, x^*)$.

American put

American put:

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[e^{-q\tau} \max(K - e^{X_\tau}, 0)],$$

where $K > 0$ strike price.Optimal stopping time τ^* exists

$$\tau^* = \inf\{t \geq 0 : X_t \in D\}$$

with $D = \{y : V(y) = \max(K - e^y, 0)\}$. From $D \neq \emptyset$, $D \cap [\log K, \infty) = \emptyset$ and convexity of $V(\log x)$ it follows that $D = (-\infty, x^*]$ for some $x^* < \log K$.
Intuitively: sell when price is high.

A fluctuation identity

We know now optimal stopping time is of form $\tau^-_{x^*}$. Expression for

$$V(x,x^*) = \mathbb{E}_x\left[e^{-q\tau^-_{x^*}} \max\left(K - e^{X_{\tau^-_{x^*}}}, 0\right)\right]?$$

Denote $\underline{X}_t = \inf_{0\le s\le t} X_t$ and let $\mathbf{e}_q \sim \exp(q)$ independent of X.

Lemma

For $q,\beta \ge 0$

$$\mathbb{E}_x[e^{-q\tau^-_y + \beta X_{\tau^-_y}} 1_{\{\tau^-_y < \infty\}}] = e^{\beta x}\frac{\mathbb{E}[e^{\beta \underline{X}_{\mathbf{e}_q}} 1_{\{-\underline{X}_{\mathbf{e}_q} > x-y\}}]}{\mathbb{E}[e^{\beta \underline{X}_{\mathbf{e}_q}}]}.$$

Identity well known e.g. random walk case Darling et al. (1972 Ann. Math. Statist.)

Proof

Boils down to (case $x = 0$)

$$\begin{aligned}\mathbb{E}\left[e^{\beta \underline{X}_{\mathbf{e}_q}} 1_{\{\underline{X}_{\mathbf{e}_q} < y\}}\right] &= \mathbb{E}\left[e^{\beta \underline{X}_{\mathbf{e}_q}} 1_{\{\tau_y^- < \mathbf{e}_q\}}\right] \\ &= \mathbb{E}\left[e^{-q\tau_y^- + \beta X_{\tau_y^-}} \mathbb{E}\left[e^{\beta(\underline{X}_{\mathbf{e}_q} - X_{\tau_y^-})} \middle| \mathcal{F}_{\tau_y^-}\right]\right].\end{aligned}$$

On the event $\{\tau_y^- < \mathbf{e}_q\}$ and given $\mathcal{F}_{\tau_y^-}$

$$\begin{aligned}\underline{X}_{\mathbf{e}_q} - X_{\tau_y^-} &= \inf_{0 \le s \le \mathbf{e}_q} (X_s - X_{\tau_y^-}) \\ &= \inf_{\tau_y^- \le s \le \mathbf{e}_q} (X_s - X_{\tau_y^-}) \\ &\stackrel{\mathrm{d}}{=} \inf_{0 \le s \le \mathbf{e}_q - \tau_y^-} X_s \stackrel{\mathrm{d}}{=} \underline{X}_{\mathbf{e}_q} \quad \text{(this is a copy independent of } \mathcal{F}_{\tau_y^-}\text{)}\end{aligned}$$

Hence

$$\mathbb{E}\left[e^{\beta \underline{X}_{\mathbf{e}_q}} 1_{\{\underline{X}_{\mathbf{e}_q} < y\}}\right] = \mathbb{E}\left[e^{-q\tau_y^- + \beta X_{\tau_y^-}}\right] \mathbb{E}\left[e^{\beta \underline{X}_{\mathbf{e}_q}}\right]$$

Solution to American put

From this fluctuation identity we deduce that

$$V(x,x^*) = \frac{\mathbb{E}\left[\left(K\mathbb{E}[e^{\underline{X}_{\mathbf{e}_q}}] - e^{x+\underline{X}_{\mathbf{e}_q}}\right)1_{\{-\underline{X}_{\mathbf{e}_q} > x - x^*\}}\right]}{\mathbb{E}[e^{\underline{X}_{\mathbf{e}_q}}]}$$

Choose x^* to maximise this function. Can take $x = \log K$ for simplicity; maximise

$$\mathbb{E}\left[\left(\mathbb{E}[e^{\underline{X}_{\mathbf{e}_q}}] - e^{\underline{X}_{\mathbf{e}_q}}\right)1_{\{Ke^{\underline{X}_{\mathbf{e}_q}} < e^{x^*}\}}\right].$$

Choose x^* such that $\mathbb{E}[e^{\underline{X}_{\mathbf{e}_q}}] > e^{\underline{X}_{\mathbf{e}_q}}$ iff $Ke^{\underline{X}_{\mathbf{e}_q}} < e^{x^*}$.

Theorem (American put, general Lévy)

Optimal stopping time is $\tau^-_{x^*}$ with

$$x^* = \log\left(K\mathbb{E}[e^{\underline{X}_{\mathbf{e}_q}}]\right) < \log K.$$

Solution American put

Result first proved by Mordecki (random walk approximation); later Alili and Kyprianou (verification lemma).

Do we "know" $x^* = \log\left(K\mathbb{E}[e^{\underline{X}_{\mathbf{e}_q}}]\right)$?Involves Wiener–Hopf factor, known in more explicit form for special class of processes such as spectrally one-sided.

For Brownian motion and $q = 1/2$ we retrieve previous result $x^* = \log(K/2)$ because $-\underline{X}_{\mathbf{e}_q} \stackrel{\mathrm{d}}{=} \mathbf{e}_1$

Smooth/continuous fit

From the solution the following can be deduced:

- We have $V(x^*+) = K - e^{x^*+} = g(x^*)$. Continuous fit
- If $\mathbb{P}(\inf\{t > 0 : X_t < 0\} = 0) = 1$, then $V'(x^*+) = -e^{x^*} = g'(x^*)$. i.e. Regular downwards implies smooth fit.
- If $\mathbb{P}(\inf\{t > 0 : X_t < 0\} > 0) = 1$, then $V'(x^*+) > -e^{x^*} = g'(x^*)$. i.e. Irregular downwards implies no smooth fit.

Optimal prediction of the maximum for Lévy process

Optimal prediction: related to secretary problem, would like to "predict" maximum value of a process.
Our setting: X a Lévy process drifting to $-\infty$ Consider infinite horizon prediction problems

- in space
$$\inf_{\tau \in \mathcal{T}} \mathbb{E}[g(\overline{X}_\infty - X_\tau)]$$
- and in time
$$\inf_{\tau \in \mathcal{T}} \mathbb{E}[|\theta - \tau|],$$
where
$$\theta = \inf\{t > 0 : X_t = \overline{X}_\infty \text{ or } X_{t-} = \overline{X}_\infty\}.$$

Predicting ultimate maximum

Suppose X is a Lévy process drifting to $-\infty$ and consider:

$$\inf_{\tau} \mathbb{E}\left[g(\overline{X}_\infty - X_\tau)\right]$$

for some non-decreasing function g. Trivial solution:

Lemma

It is optimal to stop immediately, i.e. $\tau^* = 0$.

Proof is also surprisingly trivial, making use of similar argument as on next slides...

Predicting time of ultimate maximum, general Lévy

Now consider prediction "in time". Let F be distribution function of $\overline{X}_\infty$, then

Lemma

Assume that $\mathbb{E}[\theta] < \infty$. *The problem* $\inf_\tau \mathbb{E}[|\theta - \tau|]$ *is equivalent to*

$$\inf_\tau \mathbb{E}\left[\int_0^\tau \left(2F(\overline{X}_t - X_t)) - 1\right) \mathrm{d}t\right]$$

Follows from argument by Urusov (TPA 49 2005).

$$\begin{aligned} |\theta - \tau| &= \theta - \tau + 2(\tau - \theta)\mathbf{1}_{\{\theta \le \tau\}} \\ &= \theta - \tau + 2\int_0^\tau \mathbf{1}_{\{\theta \le t\}}\,\mathrm{d}t. \end{aligned}$$

Predicting time of ultimate maximum, general Lévy

It holds that

$$\mathbb{E}\left[\int_0^\tau \mathbf{1}_{\{\theta\leq t\}}\,\mathrm{d}t\right] \;=\; \mathbb{E}\left[\int_0^\tau \mathbb{P}\left(\theta \leq t \,|\, \mathcal{F}_t\right)\,\mathrm{d}t\right].$$

and

$$\begin{aligned}
\mathbb{P}\left(\theta \leq t \,|\, \mathcal{F}_t\right) &= \mathbb{P}\left(\sup_{s\geq t} X_s \leq \overline{X}_t \,\middle|\, \mathcal{F}_t\right) \\
&= \mathbb{P}\left(S + X_t \leq \overline{X}_t \;|\; \mathcal{F}_t\right) \\
&= F(\overline{X}_t - X_t),
\end{aligned}$$

where S denotes an independent copy of $\overline{X}_\infty$.

Solving the equivalent problem

Consider the optimal stopping problem

$$V(y) = \inf_{\tau} \mathbb{E}\left[\int_0^{\tau} \left(2F(Y_t^y) - 1\right) \, \mathrm{d}t\right]$$

where $Y_t^y = y \vee \overline{X}_t - X_t$. Distribution function F of $\overline{X}_\infty$ plays an important role, and its median.

Main result-regular downwards

Denote by m the median of $\overline{X}_\infty$. Suppose that X is regular downward and $\mathrm{m} > 0$.

Theorem (with Kees van Schaik, Acta Applicandae Mathematicae 2014)

There exists a $y^* \in (\mathrm{m}, \infty)$ *such that an optimal stopping time is given by*

$$\tau^* = \inf\{t \geq 0 \,|\, V(Y_t^y) = 0\} = \inf\{t \geq 0 \,|\, Y_t^y \geq y^*\}.$$

Furthermore, if F *is Lipschitz continuous on* $\mathbb{R}_{\geq 0}$ *then* V *is a non-decreasing, continuous function and* $V'_-(y^*) = V'_+(y^*) = 0$ *(smooth pasting).*

For irregular downward similar stopping time but no smooth pasting, just continuous fit $V_-(y^*) = 0$.

Idea of the proof

- Invoke general theory (Snell envelope) to conclude that optimal stopping time exists and

$$\tau^* = \inf\{t > 0 : V(Y_t^y) = 0\}.$$

- Show that V is non-decreasing. Also $V(0) < 0$ and $V(y) = 0$ for y large.
- This implies threshold strategy is optimal.
- Maximising over threshold leads to smooth or continuous fit according to regularity of the process.
- Note that $y^* > \mathrm{m}$: it only makes sense to stop after Y_t^y reaches m as otherwise the pay-off would be strictly positive.

Spectrally negative as corollary

Suppose X has no positive jumps, then $\overline{X}_\infty \sim \exp(\Phi(0))$ with $\Phi(0)$ unique solution to $e^{\psi(\lambda)} := \mathbb{E}[\exp(\lambda X_1)] = 1$ on $(0,\infty)$. Define for $q \geq 0$ the scale function $W^{(q)}$ as continuous function on $[0,\infty)$ such that

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\lambda)-q}. \quad \text{for } \psi(\lambda) > q.$$

When $q = 0$ we ignore superscript.

Corollary

When X is spectrally negative then y^* is the unique solution on $\mathbb{R}_{>0}$ to

$$\int_{[0,y]} (1 - 2e^{-\Phi(0)x}) W(\mathrm{d}x) = 0 \qquad \text{and}$$

$$V(y) = \int_y^{y^*} \left(2e^{-\Phi(0)x} - 1\right) W(x-y)\,\mathrm{d}x.$$

More explicit - Jump diffusion

Consider the jump-diffusion $X_t = \sigma B_t + \mu t - \sum_{i=1}^{N_t} Y_i$, where B is a Brownian motion, N is a Poisson process with intensity $\lambda > 0$, $(Y_i)_{i \geq 1}$ is a sequence of iid exponentially distributed random variables with parameter $\theta > 0$, $\sigma > 0$ and $\mu \in \mathbb{R}$.Then the Laplace exponent ψ is given by

$$\psi(z) = \frac{\sigma^2}{2} z^2 + \mu z - \frac{\lambda z}{\theta + z}$$

and scale function

$$W(x) = C_1 e^{\beta_1 x} + C_2 e^{\beta_3 x}$$

for certain constants C_i, β_i.

Jump diffusion-graph

We get the following value function:

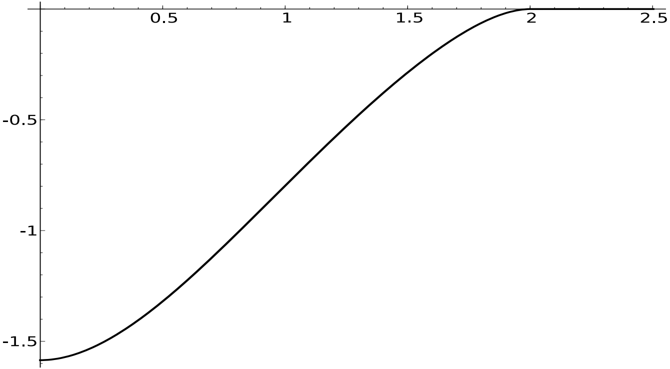

Figure : A plot of the value function, with $\sigma = \mu = 1/2$ and $\lambda = \theta = 1$.

Conclusion and related problems

Conclusion

- Optimal stopping: Find set $D = \{x : V(x) = g(x)\}$.
- Stopping time $\tau^* = \inf\{t : X_t \in D\} optimal$.
- Famous example: American put option: first passage time below some level.
- Verification lemma, free boundary problem, direct approach.
- Optimal prediction for Lévy processes drifting to $-\infty$.
- "time"-problem: first passage time of reflected process
- Direct method: Reduce to standard problem, use first principles to show optimal time is first passage time of reflected process and then deduce continuous/smooth fit

Related problems

- Finite maturity
- Other pay-offs: e.g. Russian option.
- Other random times, e.g. last passage time.
- Other processes: e.g. positive self-similar Markov processes.

Some references

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