Time Series Analysis in a nutshell

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Time series

Definition: A time series is a stochastic process $(X_t, t \in \mathcal{T})$.
The term is often also used for its (perhaps only partial) realisation $(x_t, t \in \mathcal{T}_0)$, where $\mathcal{T}_0 \subseteq \mathcal{T}$.

Remark

- In most cases $\mathcal{T}$ is an index set of consecutive time points; often $\mathcal{T} = \mathbb{Z}$, $\mathcal{T} = \mathbb{N}$, $\mathcal{T} = \mathbb{R}_0^+$, or $\mathcal{T} = \mathbb{R}$.

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- alternatively, (e.g. in geophysics) $\mathcal{T}$ can be a spatial index set (precipitation in Asia during a specific month),
  or $\mathcal{T}$ contains points of a surface (e.g. surface of the earth);
  $\mathcal{T}$ can even index time and space (wind fields across Europe)

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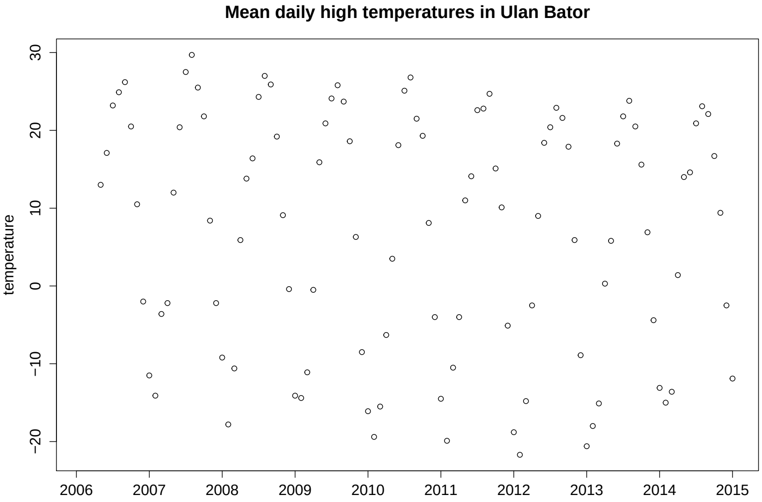
- **during this topic lecture $\mathcal{T} \subseteq \mathbb{Z}$, mostly $\mathcal{T} \subseteq \mathbb{N}$**
- Notation:
  - Time:

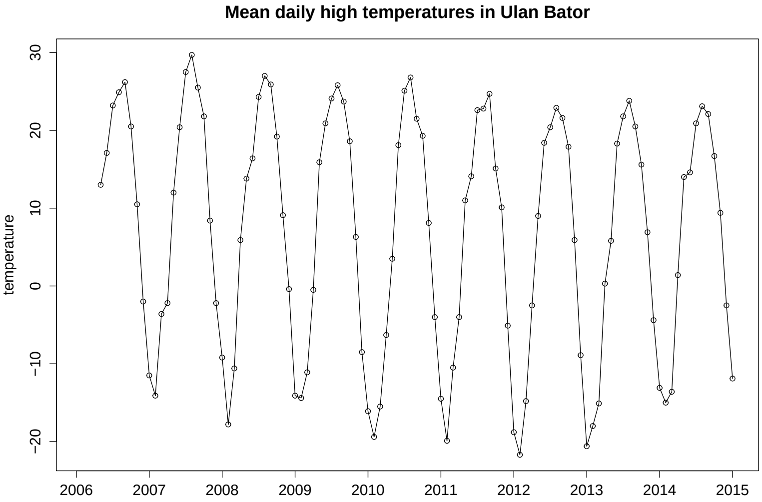
    discrete: $t_1 < t_2 < \ldots < t_n$ or
    continuous: $0 \leq t \leq T$, $t \geq 0$
    during this lecture **equidistant**: $t_j = \Delta j$ with $\Delta = 1$
  - Observations: $(x_{t_1}, \ldots, x_{t_n})$ or $(x_1, \ldots, x_n)$ or $(x_t, 0 \leq t \leq T)$
  - Process: $(X_{t_i})_{i \in \mathbb{N}}$ or $(X_i)_{i \in \mathbb{Z}}$ or $(X_t, 0 \leq t \leq T)$
- Time series can be real- or complex-valued and they can be multivariate.

A time series plot



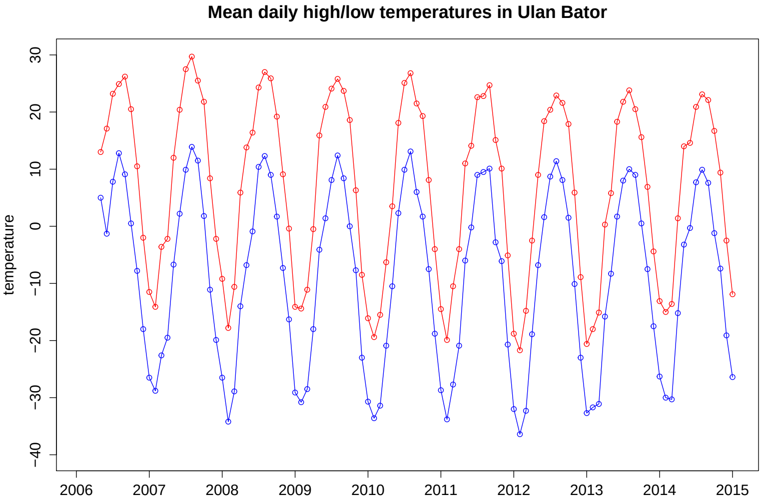
Source: WeatherSpark.com

Another time series plot



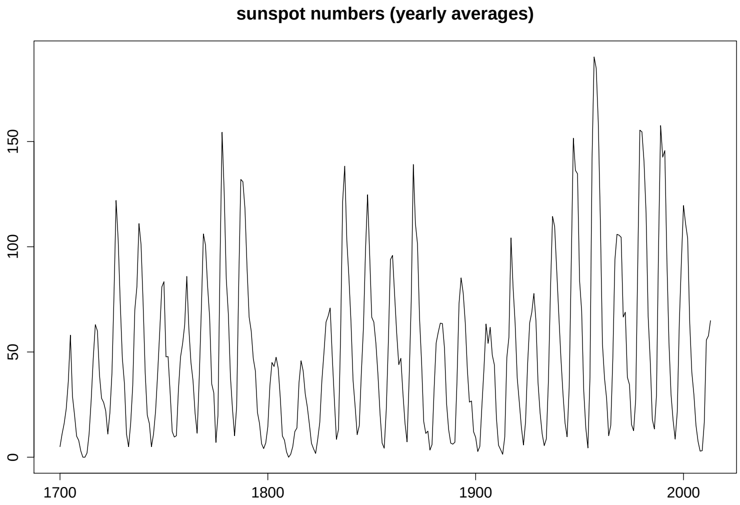
Source: WeatherSpark.com

Just out of curiosity



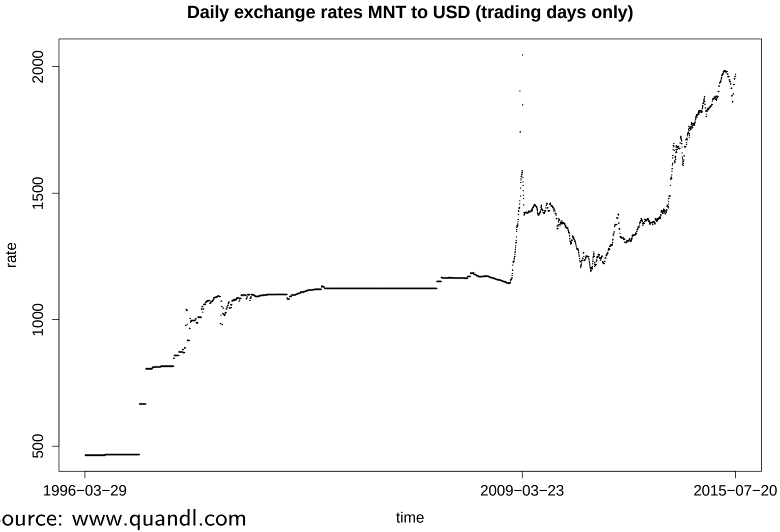
Source: www.WeatherSpark.com

A different series



Source: www.ips.gov.au/Educational/

Another time series



Source: www.quandl.com

Where are time series? - Examples

There exist for example

- **Financial time series:**
  Exchange rates, stock prices, interest rates, export numbers,...
- **Meteorological time series:**
  temperatures, precipitation per hour/day/week/month,...
- **Physical time series:**
  Sunspot numbers, measurements of an experiment, ...
- **Biological time series**
  genetic values of progeny,...
- **Medical time series**
  Patient data, clinical test data,...
- **Ecological time series:**
  pollutants (CO_2, Ozone, ...), water levels,...
- **Demographical time series:**
  population size, monthly income,...

Goals of time series analysis

general formulation:
understand dependencies over time

(a) describe / characterize
(b) model choice
(c) estimate parameters
(d) predict
(e) control

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Today we will only deal with:
(a): seasonal effects, trend, outliers, change points
(b): model dependence in time, which is characterized by the covariances between the random variables X_t

Deseasoning and detrending

Stationarity

Weak stationarity

Definition: A stochastic process $(X_t)_{t \in \mathcal{T}}$ is called *weakly stationary* if

(i) $E|X_t|^2 < \infty \quad \forall t \in \mathcal{T}$

(ii) $EX_t = \mu \quad \forall t \in \mathcal{T}$

(iii) $\text{Cov}\,(X_r, X_s) = \text{Cov}\,(X_{r+h}, X_{s+h})$
$\forall r, s, h: \; r, \; s, \; r+h, \; s+h \; \in \mathcal{T}$

Strict stationarity

Definition: A stochastic process $(X_t)_{t\in\mathcal{T}}$ is called *strictly (or strongly) stationary,* if

$$(X_{t_1}, ..., X_{t_n}) \stackrel{d}{=} (X_{t_1+h}, ..., X_{t_n+h})$$

for all $t_1, ..., t_n \in \mathcal{T}$, $n \in \mathbb{N}$, and h such that $t_1 + h, \ldots, t_n + h \in \mathcal{T}$.

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for all $t_1,...,t_n \in \mathcal{T}$, $n \in \mathbb{N}$, and h such that $t_1+h,\dots,t_n+h \in \mathcal{T}$.

In particular, $X_t \sim F$ for all $t \in \mathcal{T}$.

Stationarity

If $(X_t)_{t\in\mathcal{T}}$ is strictly stationary with finite variance, then $(X_t)_{t\in\mathcal{T}}$ is also weakly stationary. The converse is not true!

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Exercises:

(a) Find an example of a weakly stationary process, which is not strictly stationary.

(b) Show: If $(X_t)_{t\in\mathcal{T}}$ is a gaussian process and weakly stationary, then $(X_t)_{t\in\mathcal{T}}$ is also strictly stationary.

Stationarity: Interpretation

A stationary stochastic process is in a "stochastic equilibrium"; i.e.,

- sections of a sample path "look alike"
- fluctuations are purely random

A non-stationary stochastic process shows features like

- the mean level is not constant,
- the average size of the fluctuations is not constant,
- the type of dependence varies.

Deseasoning and detrending

Reduction of time series to stationary time series

Reduction of time series to stationary time series

The time series plot may help to detect, whether the time series contains deterministic components like

- trend component
- seasonal component
- change points (model change)
- outliers

Idealistic models (perhaps after appropriate transformation)

(i) $X_t = T_t + S_t + Y_t,\ t \in T,$ where

T_t: trend component (non-random)

S_t: seasonal component with period p (non-random)

(often $p = 4, 12, 52, 7, 365$)

Y_t: random fluctuations (stationary)

(ii) $X_t = T_t S_t + Y_t,\ t \in T$

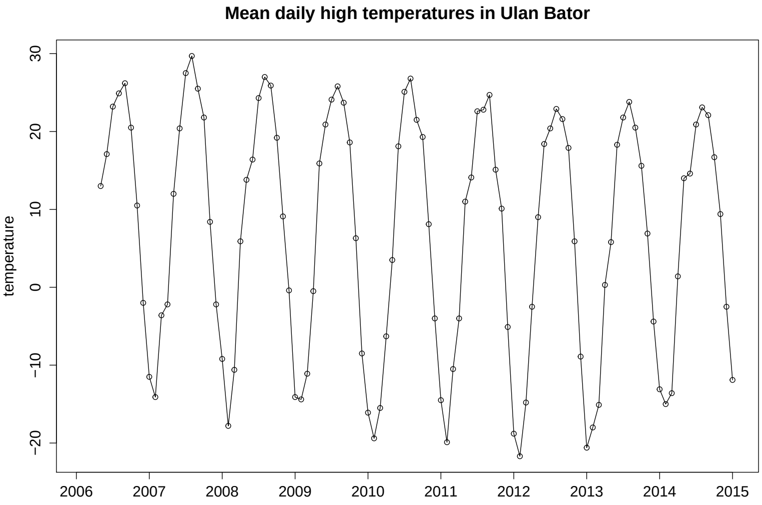
(iii) $\ln X_t - \ln X_{t-1} = \mu + Y_t, \quad t \in 0, 1, \ldots, T,$ where

μ: mean value

Y_t: random fluctuations (stationary)

(log-return model - often used for financial time series)

Example 1


Mean daily high temperatures in Ulan Bator
temperature
30
20
10
0
−10
−20
2006
2007
2008
2009
2010
2011
2012
2013
2014
2015

Estimate trend- and seasonal component in the model $X_t = T_t + S_t + Y_t$

W.l.o.g. assume that $\mathbb{E}Y_t = 0$ und $S_{t+p} = S_t$, $\sum_{j=1}^{p} S_j = 0$.

Assumption: monthly data $X_{j,k}$, $j = 1, \ldots, n$, $k = 1, \ldots, 12$; i.e.,

$$X_{j,k} = X_{k+12(j-1)}, \quad j = 1, \ldots, n, \quad k = 1, \ldots, 12.$$

Methode I: small trend method

Assumption: the trend component T_j is constant in year j.
A natural unbiased estimator is given by (since $\sum_{k=1}^{12} S_k = 0$)

$$\hat{T}_j = \frac{1}{12} \sum_{k=1}^{12} X_{j,k}.$$

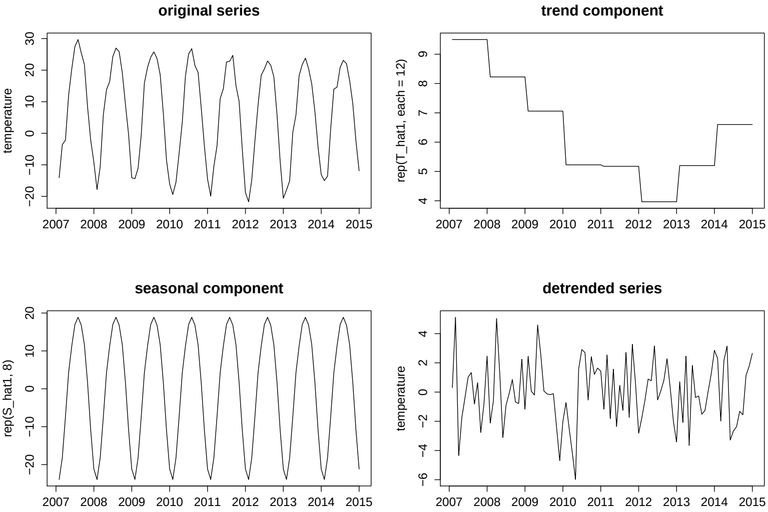
Then we estimate S_k as

$$\hat{S}_k = \frac{1}{n} \sum_{j=1}^{n} (X_{j,k} - \hat{T}_j),$$

then automatically $\sum_{k=1}^{12} \hat{S}_k = 0$. The residuals are

$$\hat{Y}_{j,k} = X_{j,k} - \hat{T}_j - \hat{S}_k, \quad j = 1, \ldots, n, \quad k = 1, \ldots, 12.$$

Example 1: Small trend method



Method II: MA-method

Step 1: First estimate the trend by a moving average, and the periodicity p determines the length of the moving part of the time series (window size).

$$
\begin{aligned}
p = 2q+1 \quad \text{odd:} \qquad \hat{T}_t &= \tfrac{1}{p} \sum_{j=-q}^{q} X_{t-j}, \\
p = 2q \quad \text{even:} \qquad \hat{T}_t &= \tfrac{1}{p} \left(\tfrac{1}{2} X_{t-q} + X_{t-q+1} \right. \\
&\quad \left. + \ldots + X_{t+q-1} + \tfrac{1}{2} X_{t+q} \right),
\end{aligned}
$$

for $q+1 \leq t \leq n-q$ (use one-sided MA's for $t \leq q$ and $t \geq n-q$).
It is also possible to use non-uniform weights.
This removes rapid fluctuations (high frequencies) from data: "low pass filter".

Method II: MA-method

Step 2: Estimate seasonal component

For each $k = 1, \dots, p$ calculate W_k as arithmetic mean of

$$X_{k+jp} - \hat{T}_{k+jp}, \quad q + 1 \leq k + jp \leq n - q \ (k \text{ fixed}, j \in \mathbb{Z}).$$

W_k would be a possible estimator for S_k, but $\sum_{k=1}^{p} W_k$ is not necessarily equal to 0. Hence, choose

$$\hat{S}_k = W_k - \frac{1}{p} \sum_{i=1}^{p} W_i, \quad k = 1, \dots, p,$$
$$\text{and } \hat{S}_k = \hat{S}_{k-p} \text{ for } k > p.$$

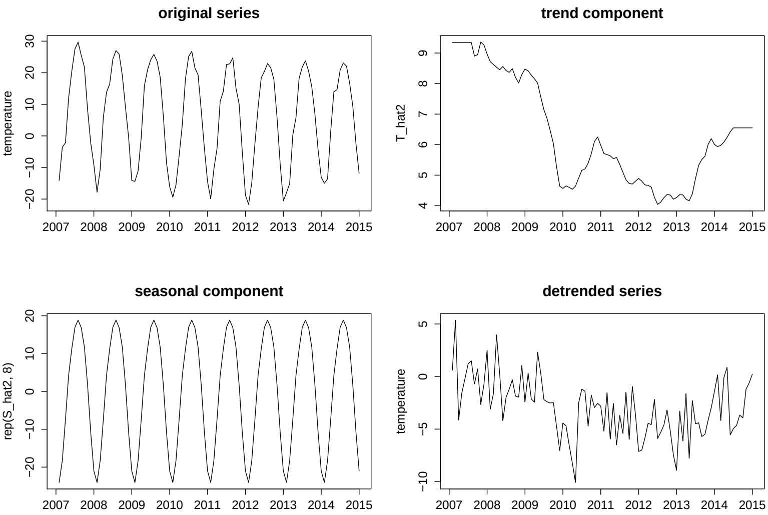
The deseasonalized data are then

$$D_t = X_t - \hat{S}_t, \quad t = 1, \dots, n.$$

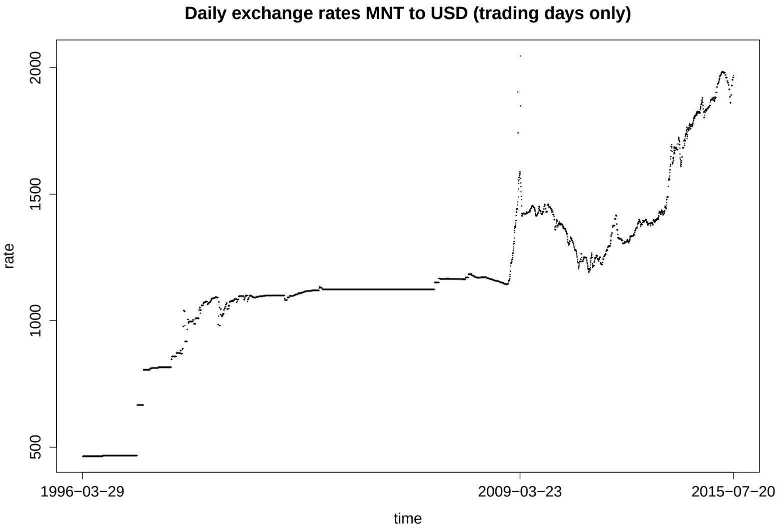
(Possibly reestimate a trend in the model without seasonal component.)

The residuals are $\hat{Y}_t := X_t - \hat{T}_t - \hat{S}_t$, $t = 1, \dots, n$.

Example 1: MA deseasoning



Example 2


Daily exchange rates MNT to USD (trading days only)
rate
500
1000
1500
2000
1996−03−29
2009−03−23
2015−07−20
time

Method I: Fitting a polynomial trend in $X_t = T_t + Y_t$

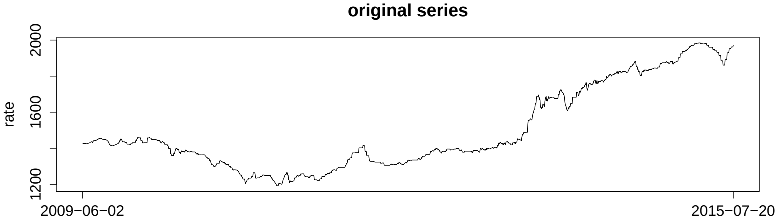
We may have reasons to assume that (T_t) is a polynomial in t:

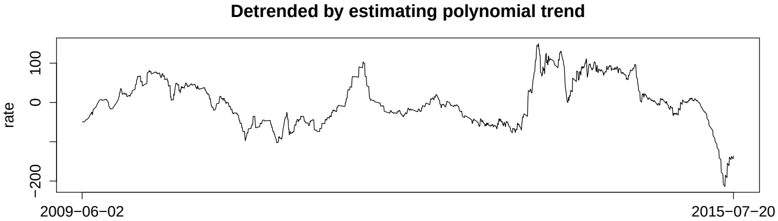
$$T_t = a_0 + a_1 t + a_2 t^2 + \cdots + a_r t^r$$

with $r \in \mathbb{N}_0$, $a_0, \ldots, a_r \in \mathbb{R}$.

- Fix r based on observed series (here $r = 2$)
- Estimate parameters by least squares estimation (LSE).

Example 2: log returns





Method II: Transforming the time series

Assumption: data X_j is of the form

$$\ln X_t - \ln X_{t-1} = \mu + Y_t, \quad t \in 0, 1, \ldots, T,$$

where

μ: mean value

Y_t: random fluctuations (stationary)

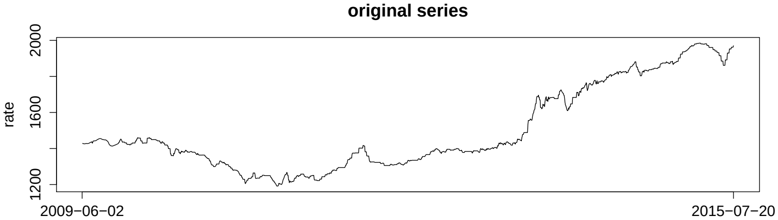
or even more specifically

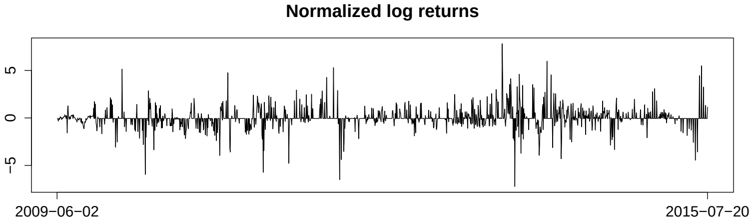
$Y_t \sim \mathcal{N}(0, \sigma^2)$

Then estimate μ and σ^2 and compute normalized log returns

$$Z_t := \hat{\sigma}^{-1}(\ln(X_t / X_{t-1}) - \hat{\mu}).$$

Example 2: log returns





Reduction of time series to stationary time series

Attention:

- Various other models/methods for time series reduction exist!
- Choosing the right model can simplify the analysis drastically. (and vice versa!)

Analyzing stationary time series

The autocovariance function

The autocovariance function

Definition: Let $(X_t)_{t\in\mathcal{T}}$ be a stochastic process with $\mathrm{Var}(X_t) < \infty$ for all $t \in \mathcal{T}$. Then

$$\gamma_X(r,s) = \mathrm{Cov}\,(X_r, X_s) = E\left[(X_r - EX_r)(X_s - EX_s)\right], \quad r, s, \in \mathcal{T}$$

is called *autocovariance function* of $(X_t)_{t\in\mathcal{T}}$.

For a weakly stationary stochastic process $(X_t)_{t\in\mathcal{T}}$ we have $\gamma_X(r,s) = \gamma_X(r-s,0)$ for all r, s. We define therefore

$$\gamma_X(h) = \gamma_X(h,0) = \mathrm{Cov}\,(X_{t+h}, X_t) \quad \forall t, h;$$

i.e., $\gamma_X(h)$ is the covariance between observations at a distance h (we say *between observations with lag h*).

The autocovariance function

Properties of ACF γ:

(i) $\gamma(0) \geq 0$

(ii) $|\gamma(h)| \leq \gamma(0)$

(iii) $\gamma(h) = \gamma(-h)$

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Exercise: Prove these properties.

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Exercise: Prove these properties.

Which functions can be ACFs?
($\rightarrow$ non-negative definite even functions)

The autocorrelation function

Definition: Let $(X_t)_{t\in\mathcal{T}}$ be a weakly stationary stochastic process with $\mathrm{Var}\,(X_t) < \infty$ for all $t \in \mathcal{T}$. Then

$$\rho_X(h) := \frac{\gamma_X(h)}{\gamma_X(0)}$$

is called *autocorrelation function (ACorrF)* of $(X_t)_{t\in\mathcal{T}}$.

Analyzing stationary time series

Common time series models

White noise

White noise $(Z_t) \sim \mathrm{WN}(0, \sigma^2)$

- $\mathbb{E} Z_t = 0 \ \forall\, t$
- $\gamma(h) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \end{cases}$

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Etymology: White noise is a random signal (or process) with a flat spectral density. The name is analogous to white light in which the spectral density of the light is distributed such that the eye's three color receptors (cones) are approximately equally stimulated.

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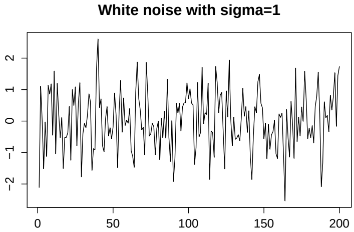

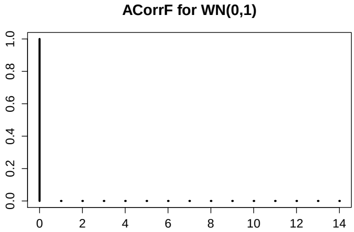

Moving average process of order q

MA(q)-process:

$$X_t := Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}, \quad t \in \mathbb{Z},$$

where $(Z_t) \sim \mathrm{WN}(0, \sigma_Z^2)$.

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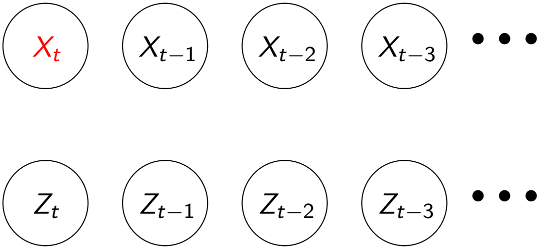
Example: MA(2)



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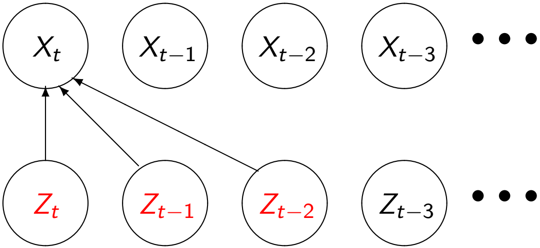


Moving average process of order q

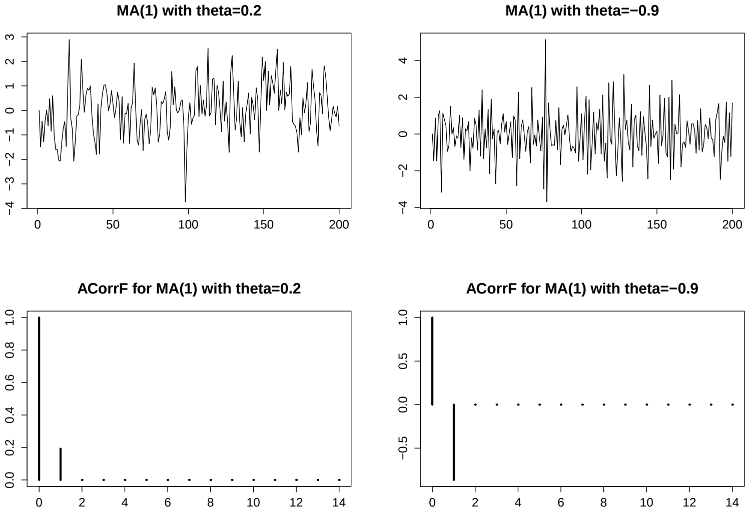
The process defined via

$$X_t = \sum_{j=0}^{q} \theta_j Z_{t-j}, \;\; t \in \mathbb{Z}, \quad (Z_j) \sim \text{WN}\left(0, \sigma_Z^2\right), \theta_0 = 1,$$

is weakly stationary with

$$\begin{aligned} EX_t &= 0, \qquad \text{Var}\, X_t = \sigma_Z^2 \sum_{j=0}^{q} \theta_j^2 \\ \gamma(h) &= \mathbb{E}\Big[\sum_{j=0}^{q} \theta_j Z_{t-j} \sum_{k=0}^{q} \theta_k Z_{t+h-k}\Big] = \begin{cases} 0 & h > q \\ \sigma_Z^2 \sum_{j=0}^{q} \theta_j \theta_{j+h}, & h = 0, \dots, q \\ \gamma(-h) & h < 0 \end{cases} \\ \varrho(h) &= \frac{\gamma(h)}{\gamma(0)} = \left(\sum_{j=0}^{q} \psi_j \psi_{j+h}\right)\left(\sum_{j=0}^{q} \psi_j^2\right)^{-1}, \qquad h = 1, \dots, q. \end{aligned}$$

Two MA(1) realisations



Autoregressive process of order p

AR(p)-process:

$$X_t := \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + Z_t, \quad t \in \mathbb{Z},$$

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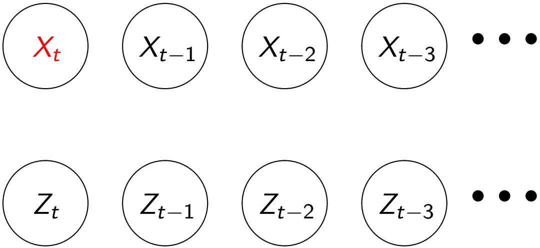

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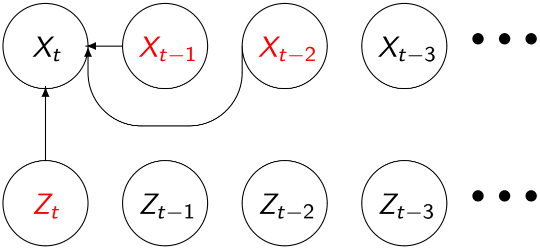

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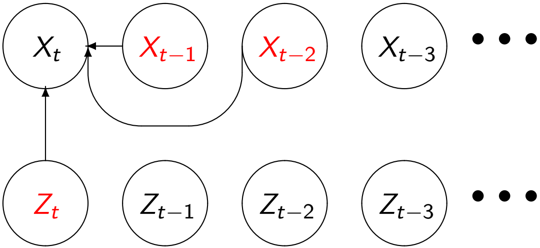


Note: AR(1) is a Markov process.

Autoregressive process of order 1

Question: Is there a stationary AR(1) process?

Autoregressive process of order 1

Question: Is there a stationary AR(1) process?

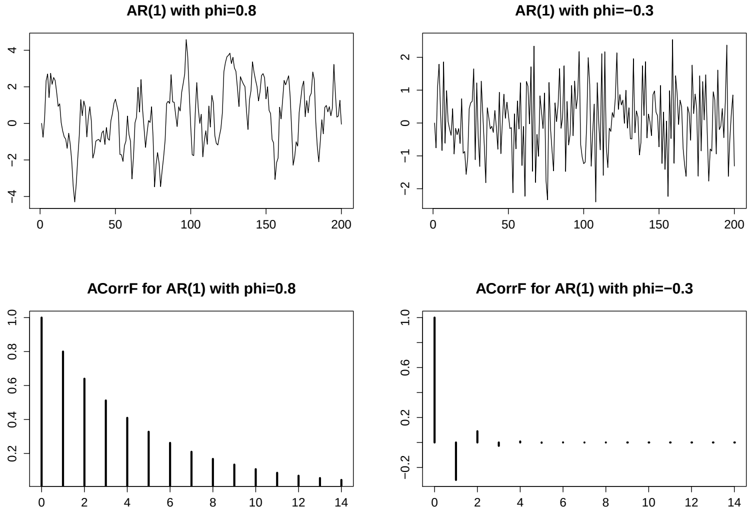
For $|\phi| < 1$ there exists a unique solution $(X_t)_{t\in\mathbb{Z}}$ to

$$X_t - \phi X_{t-1} = Z_t, \quad (Z_j) \sim \text{WN}\left(0, \sigma_Z^2\right)$$

that is weakly stationary with

$$\begin{aligned} \mathbb{E}X_t &= 0, \quad \text{Var}\, X_t = \mathbb{E}X_t^2 = \frac{\sigma_Z^2}{1-\phi^2} \\ \gamma(h) &= \frac{\phi^h}{1-\phi^2}\,\sigma_Z^2 = \phi^{|h|}\sigma_X^2, \quad h \in \mathbb{Z} \\ \Rightarrow \quad \rho(h) &= \phi^{|h|}, \quad h \in \mathbb{Z} \end{aligned}$$

Two AR(1) realisations



A combination of AR(p) and MA(q)

An ARMA(p, q)-process is a weakly stationary solution to:

$$X_t - \sum_{i=1}^{p} \phi_i X_{t-i} = \sum_{j=0}^{q} \theta_j Z_{t-j}, \quad t \in \mathbb{Z},$$

where $(Z_t) \sim \text{WN}(0, \sigma_Z^2)$, and $\theta_0 = 1$.

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Invoking the backshift operators and the polynomes

$$\begin{aligned} \Phi(z) &= 1 - \phi_1 z - \ldots - \phi_p z^p \\ \Theta(z) &= 1 + \theta_1 z + \ldots + \theta_q z^q \end{aligned}$$

we can write the ARMA(p, q)-process also as

$$\Phi(B) X_t = \Theta(B) Z_t.$$

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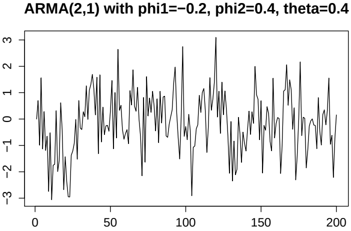
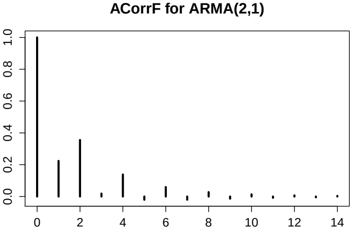
An ARMA(2,1) realisation:





Do ARMA(p, q) processes exist? Are they unique?

Assume that the characteristic polynomials Φ and Θ have no common zeros.
Then if $\Phi(z) \neq 0 \quad \forall z \in \mathbb{C},\ |z| \leq 1$ a unique weakly stationary solution to the ARMA equation

$$\Phi(B)X_t = \Theta(B)Z_t.$$

exists which is given by

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad t \in \mathbb{Z} \quad \text{with} \quad \Psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\Theta(z)}{\Phi(z)}, \quad |z| \leq 1.$$

Do ARMA(p, q) processes exist? Are they unique?

Assume that the characteristic polynomials $\Phi(z)$ and $\Theta(z)$ have common zeros, then there are two possibilities:

- none of the common zeros lies on the unit circle: cancel the common factor, then there remains a unique stationary solution of the ARMA equation with polynomials $\tilde{\Phi}$, $\tilde{\Theta}$ that have no common zero.
- At least one of the common zeros lies on the unit circle. Then there can be multiple stationary solutions.

Linear processes

Definition: The time series $(X_t)_{t\in\mathbb{Z}}$ is called *linear process*, if

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j},\ t \in \mathbb{Z},$$

for some parameters $(\psi_j)_{j\in\mathbb{Z}} \subset \mathbb{R}$ such that $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ and $(Z_t)_{t\in\mathbb{Z}} \sim \mathrm{WN}\left(0, \sigma_Z^2\right)$.

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All models that we have seen so far are linear.

Analyzing stationary time series

Estimating mean and ACF

Estimating the mean

Consider observations $X_1, ..., X_n$ of a real-valued stationary time series $(X_t)_{t\in\mathbb{Z}}$ with mean $\mathbb{E}X_t = \mu$ and ACF γ.

A natural unbiased estimator of μ is

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \;\; n \in \mathbb{N}.$$

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- If $\gamma(n) \to 0$ as $n \to \infty$, then $\mathrm{Var}\,\overline{X}_n = \mathbb{E}\left(\overline{X}_n - \mu\right)^2 \to 0$ as $n \to \infty$.
- If $\sum |\gamma(n)| < \infty$, then $\lim_{n\to\infty} n\mathrm{Var}\,\overline{X}_n = \sum_{h=-\infty}^{\infty} \gamma(h)$.

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Exercise: Proof this.

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A **Central Limit Theorem:** If $(X_t)_{t\in\mathbb{Z}}$ is linear,

i.e. $X_t = \mu + \sum\limits_{j=-\infty}^{\infty} \psi_j Z_{t-j}$, with $(Z_t)_{t\in\mathbb{Z}} \sim$ i.i.d. $\left(0, \sigma^2\right)$
and such that $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ and $\sum_{j=-\infty}^{\infty} \psi_j \neq 0$, then

$$\frac{\overline{X}_n - \mu}{\sqrt{v/n}} \overset{d}{\to} \mathcal{N}(0,1),$$

with $v = \sum \gamma(h) = \sigma^2 \left(\sum \psi_j\right)^2$.

Estimating the ACF

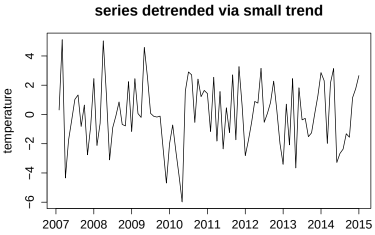
Consider observations $X_1, \ldots, X_n$ of a real-valued stationary time series $(X_t)_{t\in\mathbb{Z}}$ with mean $\mathbb{E}X_t = \mu$ and ACF γ.

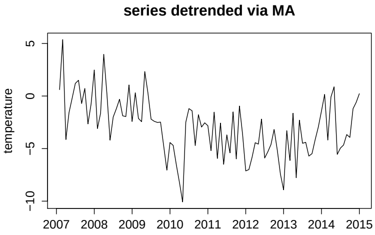
Classical estimators for γ and ρ are

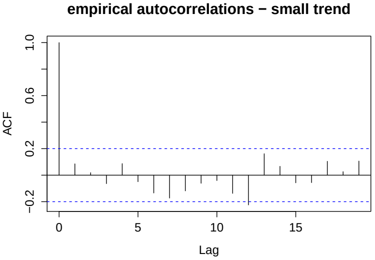
$$\begin{aligned}\hat{\gamma}(h) &= \frac{1}{n}\sum_{t=1}^{n-h}\left(X_t - \overline{X}_n\right)\left(X_{t+h} - \overline{X}_n\right),\ 0 \le h \le n-1,\\ \hat{\rho}(h) &= \hat{\gamma}(h)/\hat{\gamma}(0), \quad 0 \le h \le n-1\end{aligned}$$

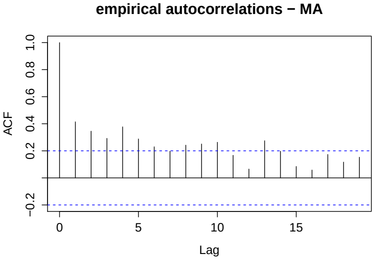
- $\hat{\gamma}(h)$ is in general biased, but asymptotically unbiased.
- Note: The estimate $\hat{\gamma}(h)$ is bad for h close to n.
- One can prove a CLT for $\hat{\gamma}$. ($\rightarrow$ Bartlett's formula)

Example 1: Mean daily high temperatures

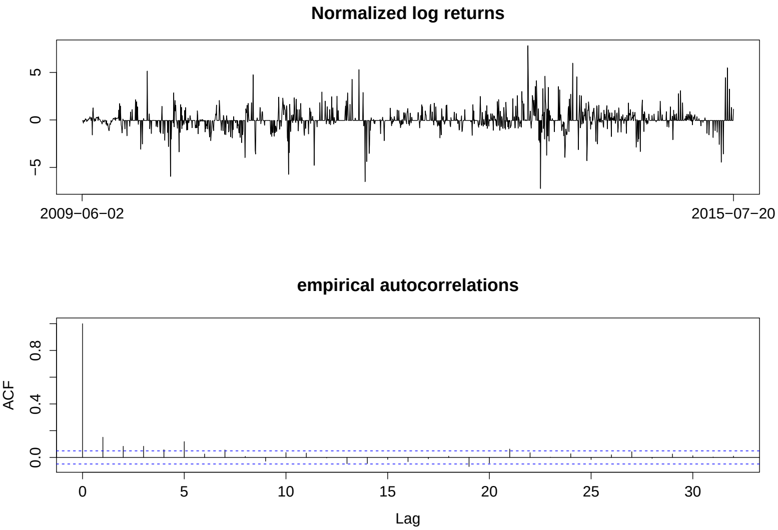








Example 2: Exchange rates



Analyzing stationary time series

Non-linear models: ARCH and GARCH

The ARCH - A nonlinear model

- 1982: Robert Engle proposes the ARCH (autoregressive conditionally heteroskedastic) time series to model the volatility of the UK inflation. The volatility is modeled as autoregressive process of previous observations:

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where

- $(\varepsilon_t)_{t\in\mathbb{N}}$ i.i.d. with $\mathbb{E}[\varepsilon_t] = 0$, $\mathrm{Var}(\varepsilon_t) = 1$ and
- ε_t independent of $\mathcal{F}_{t-1} = \sigma(Y_{t-k}, k = 1, 2, \ldots)$
- $\theta > 0$, $\alpha_i \geq 0$ with $\alpha_q > 0$.

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$\rightsquigarrow$ Nobel price 2003

The GARCH

- 1986: Bollerslev extends the ARCH model with an additional term of past volatilities and introduces the GARCH(p,q) (generalized ARCH) processes:

$$Y_t = \sigma_t \varepsilon_t \,, \quad \sigma_t^2 = \theta + \sum_{i=1}^{q} \alpha_i Y_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2, \quad t \in \mathbb{N},$$

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  Setting $\beta_i = 0$ yields the ARCH model.

The COGARCH

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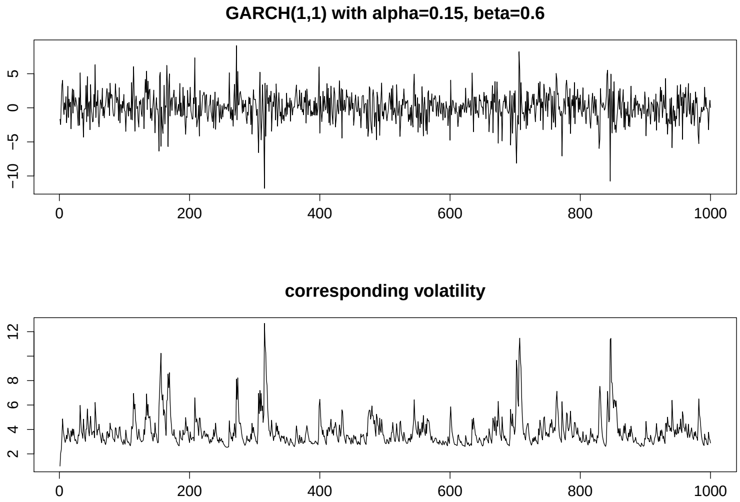
- 2004: Klüppelberg et al. propose a continuous-time GARCH(1,1) model: COGARCH(1,1).

A GARCH(1,1) realisation



Further reading

- Brockwell and Davis (1991)
  Time Series: Theory and Methods.
  2nd edition, Springer.
- Brockwell and Davis (2002)
  Introduction to Time Series and Forecasting.
  2nd edition, Springer.
- Box, Jenkins and Reinsel (2008)
  Time Series Analysis: Forecasting and Control.
  4th edition, Wiley.
- Andersen, Davis, Kreiss and Mikosch (Eds.) (2009)
  Handbook of Financial Time Series.
  Springer.
- Franq and Zakoian (2010)
  GARCH Models: Structure, Statistical Inference and Financial Applications.
  Wiley.
- ...