Self-Similar Markov Processes (and SDEs)

Leif Döring, Mannheim University, Germany

7. August 2015

Self-similar Markov processes (ssMp)

Definition

A strong Markov process $(X_t : t \geq 0)$ on $\mathbb{R}$ with RCLL paths, with probabilities $\mathbb{P}_x$, $x \in \mathbb{R}$, is a ssMp if there exists an index $\alpha \in (0, \infty)$ such that, for all $c > 0$ and $x \in \mathbb{R}$,

$$(cX_{tc^{-\alpha}} : t \geq 0) \text{ under } \mathbb{P}_x$$

is equal in law to

$$(X_t : t \geq 0) \text{ under } \mathbb{P}_{cx}.$$

Definition

pssMp if sample paths are positve and absorbed at the origin.

This is a small tour through some Markov process theory along the example of self-similar processes.

We discuss

$$\text{results for ssMps} \quad \overset{\textit{Examples}}{\longleftrightarrow} \quad \text{SDEs}$$

Note: Goal of these lectures is an SDE point of view (inspired by work of Maria-Emilia, Amaury, Zenghu & friends) rather than the stable process point of view of Kyp & friends.

Warning: We do NOT arrive at the most general results for ssMps.

Our journey is the destination!

Our journey goes through ideas from stochastic calculus and many examples towards one particular result for ssMps.

Content

- Examples and Brownian SDEs
- Lamperti's representation for pssMps and generators
- Examples
- Lamperti SDE and Jump Diffusions

Content

Several Examples of ssMps

- Brownian motion (B_t) is a ssMp with index 2
- stopped Brownian motion $(B_t 1_{(T_0 > t)})$ is a pssMp with index 2
- Bessel processes of dimension δ - $Bes(\delta)$ - i.e. solutions of

$$dX_t = \frac{\delta - 1}{2} \frac{1}{X_t}\, dt + dB_t$$

  give pssMp with index 2.
- squared-Bessel processes of dimension δ - $Bes^2(\delta)$ - i.e. solutions of

$$dX_t = \delta\, dt + 2\sqrt{X_t} dB_t$$

  give pssMps with index 1.
- Brownian motion conditioned to be positive $(B^\uparrow)$ is a pssMp with index 2

Several Examples of ssMps

- Brownian motion (B_t) is a ssMp with index 2
- stopped Brownian motion $(B_t 1_{(T_0>t)})$ is a pssMp with index 2
- Bessel processes of dimension δ - $Bes(\delta)$ - i.e. solutions of

$$dX_t = \frac{\delta - 1}{2}\frac{1}{X_t}\,dt + dB_t$$

  give pssMp with index 2.
- squared-Bessel processes of dimension δ - $Bes^2(\delta)$ - i.e. solutions of

$$dX_t = \delta\,dt + 2\sqrt{X_t}dB_t$$

  give pssMps with index 1.
- Brownian motion conditioned to be positive $(B^{\uparrow})$ is a pssMp with index 2

Several Examples of ssMps

- Brownian motion (B_t) is a ssMp with index 2
- stopped Brownian motion $(B_t 1_{(T_0>t)})$ is a pssMp with index 2
- Bessel processes of dimension δ - $Bes(\delta)$ - i.e. solutions of

$$dX_t = \frac{\delta - 1}{2} \frac{1}{X_t} \, dt + dB_t$$

  give pssMp with index 2.
- squared-Bessel processes of dimension δ - $Bes^2(\delta)$ - i.e. solutions of

$$dX_t = \delta \, dt + 2\sqrt{X_t} dB_t$$

  give pssMps with index 1.
- Brownian motion conditioned to be positive $(B^{\uparrow})$ is a pssMp with index 2

Several Examples of ssMps

- Brownian motion (B_t) is a ssMp with index 2
- stopped Brownian motion $(B_t 1_{(T_0>t)})$ is a pssMp with index 2
- Bessel processes of dimension δ - $Bes(\delta)$ - i.e. solutions of

$$dX_t = \frac{\delta - 1}{2}\frac{1}{X_t}\,dt + dB_t$$

  give pssMp with index 2.
- squared-Bessel processes of dimension δ - $Bes^2(\delta)$ - i.e. solutions of

$$dX_t = \delta\,dt + 2\sqrt{X_t}dB_t$$

  give pssMps with index 1.
- Brownian motion conditioned to be positive $(B^{\uparrow})$ is a pssMp with index 2

Several Examples of ssMps

- Brownian motion (B_t) is a ssMp with index 2
- stopped Brownian motion $(B_t 1_{(T_0>t)})$ is a pssMp with index 2
- Bessel processes of dimension δ - $Bes(\delta)$ - i.e. solutions of

$$dX_t = \frac{\delta - 1}{2}\frac{1}{X_t}\,dt + dB_t$$

  give pssMp with index 2.
- squared-Bessel processes of dimension δ - $Bes^2(\delta)$ - i.e. solutions of

$$dX_t = \delta\,dt + 2\sqrt{X_t}dB_t$$

  give pssMps with index 1.
- Brownian motion conditioned to be positive $(B^{\uparrow})$ is a pssMp with index 2

How to check Self-Similarity?

There is no general approach!

- For (B_t) show that scaled process is also a BM.
- For $(B_t 1_{(T_0>t)})$ consider the joint process $(B_t, \inf_{s\leq t} B_t)$.
- Show the process is "limit" of self-similar processes.
- For the SDE examples use SDEs Theory.

Self-Similarity for $Bes^2(\delta)$

$$\begin{aligned} cX_{tc^{-1}} &= c\left(X_0 + \delta tc^{-1} + \int_0^{tc^{-1}} 2\sqrt{X_s}\,dB_s\right) \\ &= cX_0 + \delta t + c\int_0^t 2\sqrt{X_{sc^{-1}}}\,d(B_{sc^{-1}}) \\ &= cX_0 + \delta t + \int_0^t 2\sqrt{cX_{sc^{-1}}}\,d(\sqrt{c}B_{sc^{-1}}) \\ &=: cX_0 + \delta t + \int_0^t 2\sqrt{cX_{sc^{-1}}}\,dW_t. \end{aligned}$$

Hence, (X_t) and $(cX_{tc^{-1}})$ both satisfy the same SDE

$$dX_t = \delta dt + 2\sqrt{X_t}dB_t$$

driven by some Brownian motions.

Why does this imply $Bes^2(\delta)$ is ssMp? → Need some SDE theory.

Self-Similarity for $Bes^2(\delta)$

$$\begin{aligned} cX_{tc^{-1}} &= c\left(X_0 + \delta tc^{-1} + \int_0^{tc^{-1}} 2\sqrt{X_s}\, dB_s\right) \\ &= cX_0 + \delta t + c\int_0^t 2\sqrt{X_{sc^{-1}}}\, d(B_{sc^{-1}}) \\ &= cX_0 + \delta t + \int_0^t 2\sqrt{cX_{sc^{-1}}}\, d(\sqrt{c}B_{sc^{-1}}) \\ &=: cX_0 + \delta t + \int_0^t 2\sqrt{cX_{sc^{-1}}}\, dW_t. \end{aligned}$$

Hence, (X_t) and $(cX_{tc^{-1}})$ both satisfy the same SDE

$$dX_t = \delta dt + 2\sqrt{X_t} dB_t$$

driven by some Brownian motions.

Why does this imply $Bes^2(\delta)$ is ssMp? → Need some SDE theory.

Self-Similarity for $Bes^2(\delta)$

$$
\begin{aligned}
cX_{tc^{-1}} &= c\left(X_0 + \delta tc^{-1} + \int_0^{tc^{-1}} 2\sqrt{X_s}\, dB_s\right) \\
&= cX_0 + \delta t + c\int_0^t 2\sqrt{X_{sc^{-1}}}\, d(B_{sc^{-1}}) \\
&= cX_0 + \delta t + \int_0^t 2\sqrt{cX_{sc^{-1}}}\, d(\sqrt{c}B_{sc^{-1}}) \\
&=: cX_0 + \delta t + \int_0^t 2\sqrt{cX_{sc^{-1}}}\, dW_t.
\end{aligned}
$$

Hence, (X_t) and $(cX_{tc^{-1}})$ both satisfy the same SDE

$$
dX_t = \delta dt + 2\sqrt{X_t} dB_t
$$

driven by some Brownian motions.

Why does this imply $Bes^2(\delta)$ is ssMp? $\rightarrow$ Need some SDE theory.

1dim SDE Theory

Consider the 1dim SDE

$$dX_t = a(X_t)dt + \sigma(X_t)dB_t, \quad X_0 \in \mathbb{R},$$

driven by a BM.

Notation (Solutions)

- A (weak) solution is a stochastic process satisfying almost surely the integrated version

$$X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t \sigma(X_s)dB_s, \quad X_0 \in \mathbb{R}.$$

- A solution is called strong if it is adapted to the filtration generated by the driving noise (B_t).

Reference e.g. Karatzas/Shreve

1dim SDE Theory

Question: Which SDEs can you solve explicitly?

Roughly everything that comes from Itō's formula calculations.

Exercise: Play around with the exponential function to solve

$$dX_t = aX_t dt + \sigma X_t dB_t.$$

Example: Which SDE is solved by $X_t = B_t^3$
$\rightarrow$ blackboard?

1dim SDE Theory

Consider the SDE

$$dX_t = a(X_t)dt + \sigma(X_t)dB_t, \quad X_0 \in \mathbb{R}.$$

Notation (Uniqueness)

- We say weak uniqueness holds if any two weak solutions have the same law.
- We say pathwise uniqueness holds if any two weak solutions are indistinguishable.

Example: Tanaka's SDE

$$dX_t = sign(X_t)dB_t$$

has a weak solution, has no strong solution, weak uniqueness holds, pathwise uniqueness is wrong. *sign* is a bad function!

$\rightarrow$ blackboard

1dim SDE Theory

Consider the SDE

$$dX_t = a(X_t)dt + \sigma(X_t)dB_t, \quad X_0 \in \mathbb{R}.$$

Notation (Uniqueness)

- We say weak uniqueness holds if any two weak solutions have the same law.
- We say pathwise uniqueness holds if any two weak solutions are indistinguishable.

Example: Tanaka's SDE

$$dX_t = sign(X_t)dB_t$$

has a weak solution, has no strong solution, weak uniqueness holds, pathwise uniqueness is wrong. *sign* is a bad function!

$\rightarrow$ blackboard

1dim SDE Theory

Theorem (Itō)

If a and σ are Lipschitz, then there is a unique strong solution.

Proof: Fixpoint theorem in good process space $\rightarrow$ constructive.

Problem: No interesting function is globally Lipschitz.

Theorem

If a and σ are locally Lipschitz and grow at most linearly, then there is a unique strong solution.

Problem: Many interesting functions are not locally Lipschitz.

Theorem (Strook/Varadhan)

If a and σ are bounded and continuous, then there is a weak solution.

1dim SDE Theory

Theorem (Itō)

If a and σ are Lipschitz, then there is a unique strong solution.

Proof: Fixpoint theorem in good process space $\rightarrow$ constructive.

Problem: No interesting function is globally Lipschitz.

Theorem

If a and σ are locally Lipschitz and grow at most linearly, then there is a unique strong solution.

Problem: Many interesting functions are not locally Lipschitz.

Theorem (Strook/Varadhan)

If a and σ are bounded and continuous, then there is a weak solution.

1dim SDE Theory

Theorem (Itō)

If a and σ are Lipschitz, then there is a unique strong solution.

Proof: Fixpoint theorem in good process space $\rightarrow$ constructive.

Problem: No interesting function is globally Lipschitz.

Theorem

If a and σ are locally Lipschitz and grow at most linearly, then there is a unique strong solution.

Problem: Many interesting functions are not locally Lipschitz.

Theorem (Strook/Varadhan)

If a and σ are bounded and continuous, then there is a weak solution.

1dim SDE Theory

Theorem

Pathwise uniqueness implies weak uniqueness.

Theorem

Weak existence and pathwise uniqueness imply strong existence.

Theorem

Weak uniqueness implies strong Markov and Feller properties.

Theorem (Yamada/Watanabe - Brownian case)

If a is locally Lipschitz and σ is locally $\frac{1}{2}$-Hölder, then pathwise uniqueness holds.

$\rightarrow$ blackboard

Note: Apply same strategy whenever you have an Itō formula!

1dim SDE Theory

Theorem

Pathwise uniqueness implies weak uniqueness.

Theorem

Weak existence and pathwise uniqueness imply strong existence.

Theorem

Weak uniqueness implies strong Markov and Feller properties.

Theorem (Yamada/Watanabe - Brownian case)

If a is locally Lipschitz and σ is locally $\frac{1}{2}$-Hölder, then pathwise uniqueness holds.

$\rightarrow$ blackboard

Note: Apply same strategy whenever you have an Itō formula!

1dim SDE Theory

Theorem

Pathwise uniqueness implies weak uniqueness.

Theorem

Weak existence and pathwise uniqueness imply strong existence.

Theorem

Weak uniqueness implies strong Markov and Feller properties.

Theorem (Yamada/Watanabe - Brownian case)

If a is locally Lipschitz and σ is locally $\frac{1}{2}$-Hölder, then pathwise uniqueness holds.

$\rightarrow$ blackboard

Note: Apply same strategy whenever you have an Itō formula!

1dim SDE Theory

Theorem

Pathwise uniqueness implies weak uniqueness.

Theorem

Weak existence and pathwise uniqueness imply strong existence.

Theorem

Weak uniqueness implies strong Markov and Feller properties.

Theorem (Yamada/Watanabe - Brownian case)

If a is locally Lipschitz and σ is locally $\frac{1}{2}$-Hölder, then pathwise uniqueness holds.

$\rightarrow$ blackboard

Note: Apply same strategy whenever you have an Itō formula!

1dim SDE Theory

Remarks:

- To prove pathwise uniqueness there is a strategy!
- There is no general strategy to prove weak uniqueness!

This is a strange problem: Only know how to proceed in the harder case.

Note: All results (in law) extend to general stochastic equations (Kurtz).

Note: Pathwise uniqueness results differ for different noise; proofs usually same strategy but ugly.

1dim SDE Theory

Remarks:

- To prove pathwise uniqueness there is a strategy!
- There is no general strategy to prove weak uniqueness!

This is a strange problem: Only know how to proceed in the harder case.

Note: All results (in law) extend to general stochastic equations (Kurtz).

Note: Pathwise uniqueness results differ for different noise; proofs usually same strategy but ugly.

1dim SDE Theory

Remarks:

- To prove pathwise uniqueness there is a strategy!
- There is no general strategy to prove weak uniqueness!

This is a strange problem: Only know how to proceed in the harder case.

Note: All results (in law) extend to general stochastic equations (Kurtz).

Note: Pathwise uniqueness results differ for different noise; proofs usually same strategy but ugly.

Question

How would you construct a positive strong solution for

$$dX_t = \delta dt + 2\sqrt{X_t} dB_t, \quad X_0 = 0,$$

for $\delta > 0$? For $\delta \leq 0$?

Note that

- $a \equiv \delta$ is Lipschitz
- $\sigma(x) = 2\sqrt{x}$ is $\frac{1}{2}$-Hölder

so pathwise uniqueness holds.

A Counterexample

The SDE

$$dX_t = |X_t|^\beta dB_t, \qquad X_0 = 0,$$

has precisely one solution $X_t \equiv 0$ if $\beta \geq \frac{1}{2}$.

For $\beta < \frac{1}{2}$ there are infinitely many solutions.

The equation has only one solution $X \in \mathcal{S}$ where

$$\mathcal{S} = \left\{ (X_t)_{t\geq 0} : \int_0^\infty 1_{(X_s=0)}\, ds = 0 \text{ a.s.} \right\}$$

Very hard and due to Bass/Burdy/Chen.

A Counterexample

The SDE

$$dX_t = |X_t|^\beta dB_t, \qquad X_0 = 0,$$

has precisely one solution $X_t \equiv 0$ if $\beta \geq \frac{1}{2}$.

For $\beta < \frac{1}{2}$ there are infinitely many solutions.

The equation has only one solution $X \in \mathcal{S}$ where

$$\mathcal{S} = \left\{ (X_t)_{t\geq 0} : \int_0^\infty 1_{(X_s=0)}\, ds = 0 \text{ a.s.} \right\}$$

Very hard and due to Bass/Burdy/Chen.

A Counterexample

The SDE

$$dX_t = |X_t|^\beta dB_t, \qquad X_0 = 0,$$

has precisely one solution $X_t \equiv 0$ if $\beta \geq \frac{1}{2}$.

For $\beta < \frac{1}{2}$ there are infinitely many solutions.

The equation has only one solution $X \in \mathcal{S}$ where

$$\mathcal{S} = \left\{ (X_t)_{t\geq 0} : \int_0^\infty 1_{(X_s=0)}\, ds = 0 \text{ a.s.} \right\}$$

Very hard and due to Bass/Burdy/Chen.

Another Counterexample

The SDE

$$dX_t = 3X_t^{1/3}dt + 3X_t^{2/3}dB_t, \quad X_0 = 0, \tag{1}$$

has infinitely many solutions (both real and non-negative).

But: The SDE has only one positve solution in $\mathcal{S}$.
$\rightarrow$ blackboard

Question: Can you relate all solutions to the solutions in $\mathcal{S}$?

Another Counterexample

The SDE

$$dX_t = 3X_t^{1/3}dt + 3X_t^{2/3}dB_t, \quad X_0 = 0, \tag{1}$$

has infinitely many solutions (both real and non-negative).

<u>But:</u> The SDE has only one positve solution in $\mathcal{S}$.
$\rightarrow$ blackboard

<u>Question:</u> Can you relate all solutions to the solutions in $\mathcal{S}$?

A Consequence to Self-Similarity

Uniqueness holds for

$$dX_t = \delta dt + 2\sqrt{X_t} dB_t,$$

hence, solutions are strong Markov and $(X_t) \stackrel{\mathcal{L}}{=} (cX_{tc^{-1}})$, so $Bes^2(\delta)$ is a ssMp with index 1.

Remark: Same argument shows that interesting positive solution to SDE (1) defines a ssMp. Or, use

Lemma

In general, suppose (X_t) is a pssMp with index α, then (X_t^α) is a pssMp with index 1.

Proof: Set $Y = X^\alpha$, then

$$\left(cY_{tc^{-1}}\right)_{t\geq 0} = \left((c^{1/\alpha}X_{tc^{-1}})^\alpha\right)_{t\geq 0} = \left((c^{1/\alpha}X_{t(c^{1/\alpha})^{-\alpha}})^\alpha\right)_{t\geq 0} = \left(Y_t\right)_{t\geq 0}$$

A Consequence to Self-Similarity

Uniqueness holds for

$$dX_t = \delta dt + 2\sqrt{X_t}dB_t,$$

hence, solutions are strong Markov and $(X_t) \stackrel{\mathcal{L}}{=} (cX_{tc^{-1}})$, so $Bes^2(\delta)$ is a ssMp with index 1.

Remark: Same argument shows that interesting positive solution to SDE (1) defines a ssMp. Or, use

Lemma

In general, suppose (X_t) is a pssMp with index α, then (X_t^α) is a pssMp with index 1.

Proof: Set $Y = X^\alpha$, then

$$\big(cY_{tc^{-1}}\big)_{t\geq 0} = \big((c^{1/\alpha}X_{tc^{-1}})^\alpha\big)_{t\geq 0} = \big((c^{1/\alpha}X_{t(c^{1/\alpha})^{-\alpha}})^\alpha\big)_{t\geq 0} = \big(Y_t\big)_{t\geq 0}$$

A Consequence to Self-Similarity

Uniqueness holds for

$$dX_t = \delta dt + 2\sqrt{X_t} dB_t,$$

hence, solutions are strong Markov and $(X_t) \stackrel{\mathcal{L}}{=} (cX_{tc^{-1}})$, so $Bes^2(\delta)$ is a ssMp with index 1.

<u>Remark:</u> Same argument shows that interesting positive solution to SDE (1) defines a ssMp. Or, use

Lemma

In general, suppose (X_t) is a pssMp with index α, then (X_t^α) is a pssMp with index 1.

Proof: Set $Y = X^\alpha$, then

$$\big(cY_{tc^{-1}}\big)_{t\geq 0} = \big((c^{1/\alpha} X_{tc^{-1}})^\alpha\big)_{t\geq 0} = \big((c^{1/\alpha} X_{t(c^{1/\alpha})^{-\alpha}})^\alpha\big)_{t\geq 0} = \big(Y_t\big)_{t\geq 0}$$

A Consequence to Self-Similarity

Uniqueness holds for

$$dX_t = \delta dt + 2\sqrt{X_t} dB_t,$$

hence, solutions are strong Markov and $(X_t) \stackrel{\mathcal{L}}{=} (cX_{tc^{-1}})$, so $Bes^2(\delta)$ is a ssMp with index 1.

Remark: Same argument shows that interesting positive solution to SDE (1) defines a ssMp. Or, use

Lemma

In general, suppose (X_t) is a pssMp with index α, then (X_t^α) is a pssMp with index 1.

Proof: Set $Y = X^\alpha$, then

$$\big(cY_{tc^{-1}}\big)_{t\geq 0} = \big((c^{1/\alpha}X_{tc^{-1}})^\alpha\big)_{t\geq 0} = \big((c^{1/\alpha}X_{t(c^{1/\alpha})^{-\alpha}})^\alpha\big)_{t\geq 0} = \big(Y_t\big)_{t\geq 0}$$

To remember for later

Solutions to

$$dX_t = \delta dt + 2\sqrt{X_t} dB_t,$$

for a ssMp that is NOT absorbed at zero if only if $\delta > 0$. Recall, a pssMp is by definition absorbed at 0.

A discontinuous ssMp

Definition

A Lévy process (X_t) is called (strictly) α-stable if it is also a self-similar Markov process.

- Theorem: $\alpha \in (0,2]$. [$\alpha = 2 \to$ BM, exclude this.]
- Theorem: Characteristic exponent $\Psi(\theta) := -\log \mathbb{E}(e^{i\theta X_1})$ satisfies

$$\Psi(\theta) = |\theta|^\alpha (e^{\pi i \alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(\theta>0)} + e^{-\pi i \alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(\theta<0)}), \qquad \theta \in \mathbb{R}.$$

where $\rho = P_0(X_t \geq 0)$.

- Theorem: Assume jumps in both directions, then

$$\Pi(dx) = \left(\frac{\Gamma(1+\alpha)}{\pi} \frac{1}{|x|^{1+\alpha}} \left(\sin(\pi\alpha\rho) 1_{\{x>0\}} + \sin(\pi\alpha\hat{\rho}) 1_{\{x<0\}} \right) \right) dx$$

A discontinuous ssMp

Definition

A Lévy process (X_t) is called (strictly) α-stable if it is also a self-similar Markov process.

- Theorem: $\alpha \in (0,2]$. [$\alpha = 2 \to$ BM, exclude this.]
- Theorem: Characteristic exponent $\Psi(\theta) := -\log \mathbb{E}(e^{i\theta X_1})$ satisfies

$$\Psi(\theta) = |\theta|^\alpha (e^{\pi i \alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(\theta>0)} + e^{-\pi i \alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(\theta<0)}), \qquad \theta \in \mathbb{R}.$$

  where $\rho = P_0(X_t \geq 0)$.
- Theorem: Assume jumps in both directions, then

$$\Pi(dx) = \left(\frac{\Gamma(1+\alpha)}{\pi} \frac{1}{|x|^{1+\alpha}} \left(\sin(\pi\alpha\rho) 1_{\{x>0\}} + \sin(\pi\alpha\hat{\rho}) 1_{\{x<0\}} \right) \right) dx$$

A discontinuous ssMp

Definition

A Lévy process (X_t) is called (strictly) α-stable if it is also a self-similar Markov process.

- Theorem: $\alpha \in (0,2]$. [$\alpha = 2 \to$ BM, exclude this.]
- Theorem: Characteristic exponent $\Psi(\theta) := -\log \mathbb{E}(e^{i\theta X_1})$ satisfies

$$\Psi(\theta) = |\theta|^\alpha (e^{\pi i \alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(\theta>0)} + e^{-\pi i \alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(\theta<0)}), \qquad \theta \in \mathbb{R}.$$

where $\rho = P_0(X_t \geq 0)$.

- Theorem: Assume jumps in both directions, then

$$\Pi(dx) = \left(\frac{\Gamma(1+\alpha)}{\pi} \frac{1}{|x|^{1+\alpha}} \left(\sin(\pi\alpha\rho) 1_{\{x>0\}} + \sin(\pi\alpha\hat{\rho}) 1_{\{x<0\}} \right) \right) dx$$

A discontinuous ssMp

Definition

A Lévy process (X_t) is called (strictly) α-stable if it is also a self-similar Markov process.

- Theorem: $\alpha \in (0,2]$. [$\alpha = 2 \to$ BM, exclude this.]
- Theorem: Characteristic exponent $\Psi(\theta) := -\log \mathbb{E}(e^{i\theta X_1})$ satisfies

$$\Psi(\theta) = |\theta|^\alpha (e^{\pi i \alpha (\frac{1}{2} - \rho)} \mathbf{1}_{(\theta > 0)} + e^{-\pi i \alpha (\frac{1}{2} - \rho)} \mathbf{1}_{(\theta < 0)}), \qquad \theta \in \mathbb{R}.$$

where $\rho = P_0(X_t \geq 0)$.

- Theorem: Assume jumps in both directions, then

$$\Pi(dx) = \left(\frac{\Gamma(1+\alpha)}{\pi} \frac{1}{|x|^{1+\alpha}} \left(\sin(\pi\alpha\rho) 1_{\{x>0\}} + \sin(\pi\alpha\hat{\rho}) 1_{\{x<0\}} \right) \right) dx$$

Content

- Examples and Brownian SDEs
- Lamperti's representation for pssMps and generators
- Examples
- Lamperti SDE and Jump Diffusions

Notation

- Let (ξ_t) a Lévy process which is killed and sent to the cemetery state $-\infty$ at an independent and exponentially distributed random time with rate in $q \in [0, \infty)$.
- Sometimes write $\xi^{(x)}$ if started in x, but always $\xi = \xi^{(0)}$.
- Define the integrated exponential Lévy process

$$I_t = \int_0^t e^{\alpha \xi_s} ds, \qquad t \geq 0,$$

and its limit $I_\infty := \lim_{t \uparrow \infty} I_t$.
- Define the inverse of the increasing process I:

$$\varphi(t) = \inf\{s > 0 : I_s > t\}, \qquad t \geq 0.$$

As usual, we work with the convention $\inf \emptyset = \infty$.

Notation

- Let (ξ_t) a Lévy process which is killed and sent to the cemetery state $-\infty$ at an independent and exponentially distributed random time with rate in $q \in [0, \infty)$.
- Sometimes write $\xi^{(x)}$ if started in x, but always $\xi = \xi^{(0)}$.
- Define the integrated exponential Lévy process

$$I_t = \int_0^t e^{\alpha \xi_s} ds, \qquad t \geq 0,$$

and its limit $I_\infty := \lim_{t \uparrow \infty} I_t$.

- Define the inverse of the increasing process I:

$$\varphi(t) = \inf\{s > 0 : I_s > t\}, \qquad t \geq 0.$$

As usual, we work with the convention $\inf \emptyset = \infty$.

Notation

- Let (ξ_t) a Lévy process which is killed and sent to the cemetery state $-\infty$ at an independent and exponentially distributed random time with rate in $q \in [0, \infty)$.
- Sometimes write $\xi^{(x)}$ if started in x, but always $\xi = \xi^{(0)}$.
- Define the integrated exponential Lévy process

$$I_t = \int_0^t e^{\alpha \xi_s} ds, \qquad t \geq 0,$$

and its limit $I_\infty := \lim_{t \uparrow \infty} I_t$.

- Define the inverse of the increasing process I:

$$\varphi(t) = \inf\{s > 0 : I_s > t\}, \qquad t \geq 0.$$

As usual, we work with the convention $\inf \emptyset = \infty$.

Notation

- Let (ξ_t) a Lévy process which is killed and sent to the cemetery state $-\infty$ at an independent and exponentially distributed random time with rate in $q \in [0, \infty)$.
- Sometimes write $\xi^{(x)}$ if started in x, but always $\xi = \xi^{(0)}$.
- Define the integrated exponential Lévy process

$$I_t = \int_0^t \mathrm{e}^{\alpha \xi_s} \mathrm{d}s, \qquad t \geq 0,$$

  and its limit $I_\infty := \lim_{t \uparrow \infty} I_t$.
- Define the inverse of the increasing process I:

$$\varphi(t) = \inf\{s > 0 : I_s > t\}, \qquad t \geq 0.$$

  As usual, we work with the convention $\inf \emptyset = \infty$.

Lamperti transform for POSITIVE ssMp

Theorem (Part (i))

If $X^{(x)}$, $x > 0$, is a pssMp with index α, then it can be represented as follows. For $x > 0$,

$$X_t^{(x)} = x \exp\{\xi_{\varphi(x^{-\alpha} t)}\}, \qquad t \le T_0,$$

and ξ is a (possibly killed) Lévy process.

Furthermore, $\zeta^{(x)} = x^\alpha I_\infty$, where $\zeta^{(x)} = \inf\{t > 0 : X_t^{(x)} \le 0\}$.

Note: Using $\xi^{(\log x)} = \xi + \log x$, one can also write

$$X_t^{(x)} = \exp\{\xi^{(\log x)}_{\varphi(t)}\}, \qquad t \le T_0.$$

Note: First version more common, but second version shows better what happens.

Lamperti transform for POSITIVE ssMp

Theorem (Part (i))

If $X^{(x)}$, $x > 0$, is a pssMp with index α, then it can be represented as follows. For $x > 0$,

$$X_t^{(x)} = x \exp\{\xi_{\varphi(x^{-\alpha} t)}\}, \qquad t \leq T_0,$$

and ξ is a (possibly killed) Lévy process.

Furthermore, $\zeta^{(x)} = x^\alpha I_\infty$, where $\zeta^{(x)} = \inf\{t > 0 : X_t^{(x)} \leq 0\}$.

Note: Using $\xi^{(\log x)} = \xi + \log x$, one can also write

$$X_t^{(x)} = \exp\{\xi_{\varphi(t)}^{(\log x)}\}, \qquad t \leq T_0.$$

Note: First version more common, but second version shows better what happens.

Lamperti transform for POSITIVE ssMp

Theorem (Part (ii))

Conversely, suppose ξ is a given (possibly killed) Lévy process. For each $x > 0$, define

$$X_t^{(x)} = x \exp\{\xi_{\varphi(x^{-\alpha} t)}\} \mathbf{1}_{(t < x^\alpha I_\infty)}, \qquad t \geq 0.$$

Then $X^{(x)}$ defines a pssMp, up to its absorption time at the origin.

For a Lévy process ξ either

(0) ξ is killed

(a) $\lim_{t\uparrow\infty} \xi_t = +\infty$ a.s.

(b) $\lim_{t\uparrow\infty} \xi_t = -\infty$ a.s.

(c) $\limsup_{t\uparrow\infty} \xi_t = \infty$, $\liminf_{t\uparrow\infty} \xi_t = -\infty$ a.s.

If $E[\xi_1] < \infty$, then law of large numbers is $\lim_{t\to\infty} \frac{\xi_t}{t} = E[\xi_1]$ a.s.

Definition

We say

(0) ξ is killed

(a) ξ drifts to $+\infty$

(b) ξ drifts to $-\infty$

(c) ξ oscillates

Example: $\xi_t = at + \sigma B_t$

Lamperti transform for POSITIVE ssMp

Consequence for pssMps

For all $x > 0$ we have

(1) $\zeta^{(x)} = \infty$ a.s. iff ξ drifts to $+\infty$ or oscillates,

(2) $\zeta^{(x)} < \infty$ and $X^{(x)}_{\zeta^{(x)}-} = 0$ a.s. iff ξ drifts to $-\infty$,

(3) $\zeta^{(x)} < \infty$ and $X^{(x)}_{\zeta^{(x)}-} > 0$ a.s. iff ξ is killed.

$\rightarrow$ blackboard drawings

Summary

$$(X, \mathrm{P}_x)_{x>0} \text{ pssMp} \quad \leftrightarrow \quad (\xi, \mathbb{P}_y)_{y \in \mathbb{R}} \text{ killed Lévy}$$

$$X_t = \exp(\xi_{S(t)}), \qquad \xi_s = \log(X_{T(s)}),$$

S a random time-change $\qquad$ T a random time-change

$$\left.\begin{array}{c} X \text{ never hits zero} \\ X \text{ hits zero continuously} \\ X \text{ hits zero by a jump} \\ X \text{ has continuous paths} \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \xi \to \infty \text{ or } \xi \text{ oscillates} \\ \xi \to -\infty \\ \xi \text{ is killed} \\ \xi \text{ has continuous paths} \end{array}\right.$$

Example

We know $Bes^2(\delta)$

$$dX_t = \delta dt + 2\sqrt{X_t}dB_t$$

is self-similar so it is a pssMp up to T_0. I am telling you that $Bes^2(\delta)$ hits zero (continuously) if and only if $\delta < 2$.

Questions: Can you guess (without calculating) the corresponding Lévy process?

Generator Theory

Recall: The generator of a Markov process (more precisely Feller) on $\mathcal{X}$ is the operator

$$\mathcal{A}f(x) = \lim_{t\to 0} \frac{E^x[f(X_t)] - f(x)}{t}, \quad x \in \mathcal{X},$$

defined on the domain $\mathcal{D}(\mathcal{A}) = \{f \in C_b : \mathcal{A}f(x) \text{ exists in } C_b\}$.

Note: It is normal to know the action $\mathcal{A}$ but not the full domain $\mathcal{D}(\mathcal{A})$.
BUT: Domain is very important!

Generator Theory

Dynkin Formula and it's inverse

(1) If $(A, \mathcal{D}(\mathcal{A}))$ is the generator of (X_t) and $f \in \mathcal{D}(\mathcal{A})$, then

$$M_t = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\,ds, \quad t \geq 0,$$

is a martingale.

(2) If $f \in C_b$ and there is $g \in C_b$ with

$$M_t = f(X_t) - f(X_0) - \int_0^t g(X_s)\,ds, \quad t \geq 0,$$

is a martingale, then $f \in \mathcal{D}(\mathcal{A})$ and $g = \mathcal{A}f$.

Generator Theory

Dynkin Formula and it's inverse

(1) If $(A, \mathcal{D}(\mathcal{A}))$ is the generator of (X_t) and $f \in \mathcal{D}(\mathcal{A})$, then

$$M_t = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\,ds, \quad t \geq 0,$$

is a martingale.

(2) If $f \in C_b$ and there is $g \in C_b$ with

$$M_t = f(X_t) - f(X_0) - \int_0^t g(X_s)\,ds, \quad t \geq 0,$$

is a martingale, then $f \in \mathcal{D}(\mathcal{A})$ and $g = \mathcal{A}f$.

Generator Theory

Example: $(\frac{1}{2}\Delta, C_0(\mathbb{R}))$ generates Brownian motion B and

$$\left(\frac{1}{2}\Delta, C_0(\mathbb{R}_+) \cap \{f : f(0) = 0\}\right)$$

generates Brownian motion absorbed at zero $B^\dagger$:

$$\begin{aligned}
\mathcal{A}^\dagger f(x) &= \lim_{t\to 0} \frac{E^x[f(B^\dagger_t)] - f(x)}{t} \\
&= \lim_{t\to 0} \frac{E^x[f(B^\dagger_t)1_{(t<T_0)}] + E^x[f(B^\dagger_t)1_{(t\geq T_0)}] - f(x)}{t} \\
&= \mathcal{A}f(x) + f(0)\frac{C}{x^2},
\end{aligned}$$

using the asymptotic $\lim_{t\to 0} \frac{P^x[T_0\leq t]}{t} = \frac{C}{x^2}$. Hence, convergence takes place in C_b iff $f(0) = 0$ and action of $\mathcal{A}^\dagger$ is determined by $\mathcal{A}$.

Generator Theory

- For solutions of $dX_t = a(X_t)dt + \sigma(X_t)dB_t$ the generator acts as

$$\mathcal{A}f(x) = a(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x), \quad x \in \mathbb{R},$$

because (Itō formula)

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)a(X_s)ds + \int_0^t f'(X_s)\sigma(X_s)dB_s + \frac{1}{2}\int_0^t f''(X_s)\sigma^2(X_s)ds, \quad t \geq 0.$$

- For a Lévy process with triplet (a, σ^2, Π) the generator acts as

$$\mathcal{A}f(x) = af'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_{\mathbb{R}} \left(f(x+u) - f(x) - f'(x)u1_{|u|\leq 1}\right)\Pi(du), \quad x \in \mathbb{R}.$$

Generator Theory

- For solutions of $dX_t = a(X_t)dt + \sigma(X_t)dB_t$ the generator acts as

$$\mathcal{A}f(x) = a(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x), \quad x \in \mathbb{R},$$

because (Itō formula)

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)a(X_s)ds + \int_0^t f'(X_s)\sigma(X_s)dB_s + \frac{1}{2}\int_0^t f''(X_s)\sigma^2(X_s)ds, \quad t \geq 0.$$

- For a Lévy process with triplet (a, σ^2, Π) the generator acts as

$$\mathcal{A}f(x) = af'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_{\mathbb{R}} \big(f(x+u) - f(x) - f'(x)u1_{|u|\leq 1}\big)\Pi(du), \quad x \in \mathbb{R}.$$

Generator Theory

Many calculations can be performed that explain some transformations such as time-change and Doob's h-transform.

For h-transforms the formula $\mathcal{A}^h f(x) = \frac{1}{h(x)} \mathcal{A} fh(x)$ holds.

Some might know that $(B_t^\uparrow)$, BM conditioned to be positive is an h-transform of $(B_t^\dagger)$ with $h(x) = x$.

Pluging-in gives

$$\begin{aligned}\mathcal{A}^\uparrow f(x) &= \frac{1}{h(x)} \frac{1}{2} \frac{d^2}{dx^2} fh(x)\\ &= \frac{1}{h(x)} \frac{1}{2} (f''(x)h(x) + f(x)h''(x) + 2f'(x)h'(x))\\ &= \frac{1}{2} f''(x) + \frac{1}{x} f'(x)\end{aligned}$$

$\rightarrow$ $(B_t^\uparrow)$ is $Bes(3)$-process, self-similar with index 1. Stable case harder!

Generator Theory

Many calculations can be performed that explain some transformations such as time-change and Doob's h-transform.

For h-transforms the formula $\mathcal{A}^h f(x) = \frac{1}{h(x)}\mathcal{A} fh(x)$ holds.

Some might know that $(B_t^{\uparrow})$, BM conditioned to be positive is an h-transform of $(B_t^{\dagger})$ with $h(x) = x$.

Pluging-in gives

$$\begin{aligned}\mathcal{A}^{\uparrow} f(x) &= \frac{1}{h(x)}\frac{1}{2}\frac{d^2}{dx^2} fh(x)\\ &= \frac{1}{h(x)}\frac{1}{2}(f''(x)h(x) + f(x)h''(x) + 2f'(x)h'(x))\\ &= \frac{1}{2}f''(x) + \frac{1}{x}f'(x)\end{aligned}$$

$\rightarrow$ $(B_t^{\uparrow})$ is $Bes(3)$-process, self-similar with index 1. Stable case harder!

Generator Theory

Many calculations can be performed that explain some transformations such as time-change and Doob's h-transform.

For h-transforms the formula $\mathcal{A}^h f(x) = \frac{1}{h(x)}\mathcal{A}fh(x)$ holds.

Some might know that $(B_t^{\uparrow})$, BM conditioned to be positive is an h-transform of $(B_t^{\dagger})$ with $h(x) = x$.

Pluging-in gives

$$\begin{aligned}\mathcal{A}^{\uparrow} f(x) &= \frac{1}{h(x)}\frac{1}{2}\frac{d^2}{dx^2}fh(x)\\ &= \frac{1}{h(x)}\frac{1}{2}(f''(x)h(x) + f(x)h''(x) + 2f'(x)h'(x))\\ &= \frac{1}{2}f''(x) + \frac{1}{x}f'(x)\end{aligned}$$

$\rightarrow$ $(B_t^{\uparrow})$ is $Bes(3)$-process, self-similar with index 1. Stable case harder!

Generator Theory

Many calculations can be performed that explain some transformations such as time-change and Doob's h-transform.

For h-transforms the formula $\mathcal{A}^h f(x) = \frac{1}{h(x)} \mathcal{A} fh(x)$ holds.

Some might know that $(B_t^{\uparrow})$, BM conditioned to be positive is an h-transform of $(B_t^{\dagger})$ with $h(x) = x$.

Pluging-in gives

$$\begin{aligned}\mathcal{A}^{\uparrow} f(x) &= \frac{1}{h(x)} \frac{1}{2} \frac{d^2}{dx^2} fh(x) \\ &= \frac{1}{h(x)} \frac{1}{2} (f''(x)h(x) + f(x)h''(x) + 2f'(x)h'(x)) \\ &= \frac{1}{2} f''(x) + \frac{1}{x} f'(x)\end{aligned}$$

$\rightarrow$ $(B_t^{\uparrow})$ is $Bes(3)$-process, self-similar with index 1. Stable case harder!

Generator Theory

Time-Change: If X and $\tilde{X}$ are Markov processes with generators $\mathcal{A}$ and $\tilde{\mathcal{A}}$ acting as

$$\mathcal{A}f(x) = \beta(x)\tilde{\mathcal{A}}f(x), \quad x \in \mathcal{X},$$

for a measurable function $\beta : \mathcal{X} \to \mathbb{R}$, then

$$X_t = \tilde{X}_{(\int_0^t \beta^{-1}(\tilde{X}_s)\,ds)^{-1}}, \quad t \geq 0,$$

if

$$\inf\left\{t : \int_0^t \beta^{-1}(\tilde{X}_s)\,ds = \infty\right\} = \inf\{t : \beta(\tilde{X}_t) = 0\}.$$

Theorem due to Volkonskii. (Proof: Martingale problem, change variables).

Note: Multiplication in generator changes only speed not directions.

Generator Theory

Time-Change: If X and $\tilde{X}$ are Markov processes with generators $\mathcal{A}$ and $\tilde{\mathcal{A}}$ acting as

$$\mathcal{A}f(x) = \beta(x)\tilde{\mathcal{A}}f(x), \quad x \in \mathcal{X},$$

for a measurable function $\beta : \mathcal{X} \to \mathbb{R}$, then

$$X_t = \tilde{X}_{(\int_0^t \beta^{-1}(\tilde{X}_s)\,ds)^{-1}}, \quad t \geq 0,$$

if

$$\inf\left\{t : \int_0^t \beta^{-1}(\tilde{X}_s)\,ds = \infty\right\} = \inf\{t : \beta(\tilde{X}_t) = 0\}.$$

Theorem due to Volkonskii. (Proof: Martingale problem, change variables).

Note: Multiplication in generator changes only speed not directions.

Generator Theory

Time-Change: If X and $\tilde{X}$ are Markov processes with generators $\mathcal{A}$ and $\tilde{\mathcal{A}}$ acting as

$$\mathcal{A}f(x) = \beta(x)\tilde{\mathcal{A}}f(x), \quad x \in \mathcal{X},$$

for a measurable function $\beta : \mathcal{X} \to \mathbb{R}$, then

$$X_t = \tilde{X}_{(\int_0^t \beta^{-1}(\tilde{X}_s)\, ds)^{-1}}, \quad t \geq 0,$$

if

$$\inf\left\{t : \int_0^t \beta^{-1}(\tilde{X}_s)\, ds = \infty\right\} = \inf\{t : \beta(\tilde{X}_t) = 0\}.$$

Theorem due to Volkonskii. (Proof: Martingale problem, change variables).

Note: Multiplication in generator changes only speed not directions.

Generator Theory

Fun example: SABR model (stochastic α, β, ρ model with $\beta < 1$)

$$\begin{cases} dX_t = \sigma_t X_t^{\beta} dB_t \\ d\sigma_t = \alpha\, \sigma_t dW_t \end{cases}$$

Suppose B and W are independent even though the ρ in the name stands for their correlation.

Question: Any idea for the limit $\lim_{t\to\infty} X_t$?

Hint: Generator is

$$\mathcal{A}f(x,y) = y^2 \left(x^{2\beta} \frac{1}{2} f_{xx}(x,y) + \frac{1}{2} f_{yy}(x,y) \right).$$

Lamperti's representation, revisited

Theorem (Lamperti), continuous case

The action of the generator for a continuous pssMp is

$$\mathcal{A}f(x) = \frac{1}{x^{\alpha}} \left[\left(a + \frac{\sigma^2}{2} \right) x\, f'(x) + \sigma\, x^2\, f''(x) \right]$$

and the corresponding Lévy process is $\xi_t = at + \sigma B_t$.

Why? Righthand side is

$$\mathcal{A}f(x) = \frac{1}{x^{\alpha}} \mathcal{A}_{e^{\text{BM with drift}}} f(x),$$

where $\mathcal{A}_{e^{\text{BM with drift}}}$ is the generator of $e^{\text{BM with drift}}$ and you know which SDE it solves.

Lamperti's representation, revisited

Theorem (Lamperti), for $E[e^{\xi_1}] < \infty$

The action of the generator for a pssMp is

$$\mathcal{A}f(x) = \frac{1}{x^\alpha}\Bigg[\log E[e^{\xi_1}]\, x\, f'(x) + \frac{\sigma^2}{2} x^2 f''(x) + \int_{-\infty}^{\infty} \big[f(e^u x) - f(x) - f'(x)(e^u - 1)x 1_{|u|\le 1}\big]\Pi(du)\Bigg]$$

and the corresponding Lévy process has triplet (a, σ^2, Π).

Why? Righthand side is

$$\mathcal{A}f(x) = \frac{1}{x^\alpha}\mathcal{A}_{e^\xi} f(x),$$

where A_{e^ξ} is the generator of e^ξ.

General Remarks 1

There are three transformations for Markov processes (SDEs in particular) and we know what happens:

- change space (Itō formula)
- change time (Volkonskii)
- reverse time (h-transform)

Keep this in mind if you want to analyze a process !!!

General Remarks 1

There are three transformations for Markov processes (SDEs in particular) and we know what happens:

- change space (Itō formula)
- change time (Volkonskii)
- reverse time (h-transform)

Keep this in mind if you want to analyze a process !!!

General Remarks 2

For pssMps (and other processes such as CSBPs) there are three equivalent ways of thinking:

- time-change
- generator
- SDE

All have advantages and disadvantages. Advantages are

- time-change can be good to analyze asymptotics
- generator good for quick calculations
- SDE good because you have Itō formula and local times for instance (for full power use illegal functions!)

Content

- Examples and Brownian SDEs
- Lamperti's representation for pssMps and generators
- Examples
- Lamperti SDE and Jump Diffusions

Continuous pssMp Examples (no killing)

Recall

$$\mathcal{A}f(x) = \left(a + \frac{\sigma^2}{2}\right) x^{1-\alpha} f'(x) + \frac{\sigma^2}{2} x^{2-\alpha} f''(x),$$

so, setting $\delta = a + \frac{\sigma^2}{2}$, all pssMps with continuous paths and index α are solutions (up to T_0) to

$$dX_t = \delta X_t^{1-\alpha} dt + \sigma X_t^{1-\alpha/2} dB_t, \qquad X_0 > 0, \tag{2}$$

for some $\delta \in \mathbb{R}, \sigma > 0$.

Corollary: Solutions to the SDE (2) hit zero in finite time a.s. if $\delta < \frac{\sigma^2}{2}$. Otherwise, almost surely zero is not hit.

Continuous pssMp Examples (no killing)

Example $\alpha = 1$: With $\sigma = 2$ meet again $Bes^2(\delta)$:

$$dX_t = \delta dt + 2\sqrt{X_t} dB_t$$

and (comparing generators)

$$\xi_t = (\delta - 2)t + 2B_t.$$

Hence, due the consequence of Lamperti's representation zero is hit in finite time iff $\delta < 2$.

Stable process killed on entry to $(-\infty, 0)$

Theorem (Chaumont/Caballero)

For the pssMp constructed by killing a stable process on first entry to $(-\infty, 0)$, the underlying Lévy process, ξ^*, that appears through the Lamperti transform has characteristic exponent given by

$$-\log E(e^{iz\xi_1^*}) = \frac{\Gamma(\alpha - iz)}{\Gamma(\alpha\hat{\rho} - iz)} \frac{\Gamma(1 + iz)}{\Gamma(1 - \alpha\hat{\rho} + iz)}, \qquad z \in \mathbb{R}.$$

The radial part of a stable process

- Suppose that X is a symmetric stable process,
- We know that $|X|$ is a pssMp.

Theorem (Chaumont/Caballero)

Suppose that the underlying Lévy process for $|X|$ is written $\xi^{\odot}$, then it characteristic exponent is given by

$$-\log E(e^{iz\xi_1^{\odot}}) = 2^{\alpha} \frac{\Gamma(\frac{1}{2}(-iz+\alpha))}{\Gamma(-\frac{1}{2}iz)} \frac{\Gamma(\frac{1}{2}(iz+1))}{\Gamma(\frac{1}{2}(iz+1-\alpha))}, \qquad z \in \mathbb{R}.$$

Content

- Examples and Brownian SDEs
- Lamperti's representation for pssMps and generators
- Examples
- Lamperti SDE and Jump Diffusions

Extending pssMps to 0

- Recurrent Case (continuous exit)

  $\rightarrow$ blackboard
- Transient Case

  $\rightarrow$ blackboard

Next: simple proof for special case of spec negative pssMps, assume $\alpha = 1$.

Lévy Jump SDEs

A Lévy SDE is

$$dX_t = a(X_t)dt + \sigma(X_{t-})dL_t$$

driven by a Lévy process is an abbreviation for

$$X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t \sigma(X_{s-})dL_s, \quad t \geq 0.$$

Theory and results mostly analogous to Brownian theory (apart from pathwise uniqueness), similar Itō construction of stochastic integral.

Example: If (L_t) is spec pos α-stable, then pathwise uniqueness holds if a is Lipschitz and σ is $(1 - \frac{1}{\alpha})$-Hölder (Li/Mytnik).

Jump Diffusions

We want more general equations:

$$X_t = X_0 + \int_0^t a(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s + \int_0^t \int_U c(X_{s-},u)(\mathcal{N}-\mathcal{N}')(ds,du) + \int_0^t \int_V d(X_{s-},u)\mathcal{M}(ds,du)$$

where

- $\mathcal{N}$ PPP on $[0,\infty) \times U$ with intensity $\mathcal{N}'(ds,du) = ds\nu(du)$ and ν is σ-finite
- $\mathcal{M}$ PPP on $[0,\infty) \times V$ with intensity $\mathcal{M}'(ds,du) = ds\mu(du)$ and μ is finite

Jump Diffusions

$$\begin{aligned}
&\int_0^t \int_U c(X_{s-}, u)(\mathcal{N} - \mathcal{N}')(ds, du) \\
&\overset{L^2}{:=} \lim_{\varepsilon \to 0} \int_0^t \int_{U_\varepsilon} c(X_{s-}, u)(\mathcal{N} - \mathcal{N}')(ds, du) \\
&:= \lim_{\varepsilon \to 0} \Big(\int_0^t \int_{U_\varepsilon} c(X_{s-}, u)\mathcal{N}(ds, du) - \int_0^t \int_{U_\varepsilon} c(X_{s-}, u)\mathcal{N}'(ds, du) \Big) \\
&= \lim_{\varepsilon \to 0} \Big(\sum_{x \in \mathcal{N}([0,t] \times U_\varepsilon)} c(X_{s-}, x) - \int_0^t \int_{U_\varepsilon} c(X_{s-}, u)\, \nu(du)\, ds \Big).
\end{aligned}$$

Warning: In general both limits can be infinite but the compensated integral converges under suitable conditions.

Note: If limiting compensator integral is finite, then jump integral is finite and integral is difference of jump integral and compensator integral.

Jump Diffusions

Example 1:

Lévy processes in Lévy-Itō form:

- $U = [-1, 1]$, $\mathcal{N}'(ds, du) = ds\,\Pi(du)$,
- $V = [-1, 1]^c$, $\mathcal{M} = \mathcal{N}$,
- $a(x) = a$, $\sigma(x) = \sigma$, $c(x, u) = u$, $d(x, u) = u$.

Example 2:

- $U = [-1, 1]$, $\mathcal{N}'(ds, du) = ds\,\Pi(du)$,
- $V = [-1, 1]^c$, $\mathcal{M} = \mathcal{N}$,
- $c(x, u) = c(x)u$, $d(x, u) = d(x)u$.

Note: Lévy SDEs are special jump SDEs: Jumps always take the form $d(X_{t-})\Delta L_t$ just as a Brownian integral gives $\sigma(X_t)\Delta B_t$.

General jump diffusions have jumps $d(X_{t-}, \Delta L_t)$ which is more flexible for modelling.

Itō Formula

With X as above and $f \in C^2$, we get

$$
\begin{aligned}
& f(X_t) - f(X_0) \\
= & \int_0^t f'(X_s)a(X_s)\,ds + \int_0^t f'(X_s)\sigma(X_s)\,dB_s + \frac{1}{2}\int_0^t f''(X_s)\sigma^2(X_s)\,ds \\
& + \int_0^t \int_U \big[f(X_{s-} + c(X_{s-},u)) - f(X_{s-})\big](\mathcal{N} - \mathcal{N}')(ds,du) \\
& + \int_0^t \int_V \big[f(X_{s-} + d(X_{s-},u)) - f(X_{s-})\big]\mathcal{M}(ds,du) \\
& + \int_0^t \int_U \big[f(X_s + c(X_s,u)) - f(X_s) - f'(X_s)c(X_s,u))\big]\mathcal{N}'(ds,du).
\end{aligned}
$$

Special case: Lévy for $a = \sigma = const$ and $d(x,u) = c(x,u) = u$ and $\mathcal{N} = \mathcal{M}$.

Itō Formula

With X as above and $f \in C^2$, we get

$$
\begin{aligned}
& f(X_t) - f(X_0) \\
= & \int_0^t f'(X_s)a(X_s)\,ds + \int_0^t f'(X_s)\sigma(X_s)\,dB_s + \frac{1}{2}\int_0^t f''(X_s)\sigma^2(X_s)\,ds \\
& + \int_0^t \int_U \big[f(X_{s-} + c(X_{s-},u)) - f(X_{s-})\big](\mathcal{N} - \mathcal{N}')(ds,du) \\
& + \int_0^t \int_V \big[f(X_{s-} + d(X_{s-},u)) - f(X_{s-})\big]\mathcal{M}(ds,du) \\
& + \int_0^t \int_U \big[f(X_s + c(X_s,u)) - f(X_s) - f'(X_s)c(X_s,u))\big]\mathcal{N}'(ds,du).
\end{aligned}
$$

Special case: Lévy for $a = \sigma = const$ and $d(x,u) = c(x,u) = u$ and $\mathcal{N} = \mathcal{M}$.

If f is bounded, then

$$\int_0^t \int_V \left| f(X_{s-} + c(X_{s-}, u)) - f(X_{s-}) \right| \mu(du)\, ds$$
$$\leq 2||f||_\infty t \int_V \mu(du) < \infty$$

so adding and substracting compensation for $\mathcal{M}$ gives

$$M_t = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\, ds$$

is a martingale, where

$$\begin{aligned}\mathcal{A}f(x) = a(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) \\ + \int_U \left[f(x + c(x,u)) - f(x) - f'(x)c(x,u) \right] \nu(du) \\ + \int_V \left[f(x + d(x,u)) - f(x) \right] \mu(du).\end{aligned}$$

Consequence: Have generator action for jump diffusions, Lévy processes.

If f is bounded, then

$$\int_0^t \int_V \left| f(X_{s-} + c(X_{s-}, u)) - f(X_{s-}) \right| \mu(du)\, ds$$
$$\leq 2||f||_\infty t \int_V \mu(du) < \infty$$

so adding and substracting compensation for $\mathcal{M}$ gives

$$M_t = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\, ds$$

is a martingale, where

$$\mathcal{A}f(x) = a(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)$$
$$+ \int_U \big[f(x + c(x,u)) - f(x) - f'(x)c(x,u)\big]\nu(du)$$
$$+ \int_V \big[f(x + d(x,u)) - f(x)\big]\mu(du).$$

Consequence: Have generator action for jump diffusions, Lévy processes.

If f is bounded, then

$$\int_0^t \int_V \left| f(X_{s-} + c(X_{s-}, u)) - f(X_{s-}) \right| \mu(du)\, ds$$
$$\leq 2||f||_\infty t \int_V \mu(du) < \infty$$

so adding and substracting compensation for $\mathcal{M}$ gives

$$M_t = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\, ds$$

is a martingale, where

$$\mathcal{A}f(x) = a(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)$$
$$+ \int_U \left[f(x + c(x,u)) - f(x) - f'(x)c(x,u) \right] \nu(du)$$
$$+ \int_V \left[f(x + d(x,u)) - f(x) \right] \mu(du).$$

Consequence: Have generator action for jump diffusions, Lévy processes.

Jump Diffusions

Remark: All general SDE theorems hold equally for jump diffusions. Only uniqueness results need adjustment.

Remark: There are some pathwise uniqueness results, essentially same proof as for BM (Itō formula with $\phi_n(\cdot) \to |\cdot|$). More difficult because of unfriendly jump Itō formula.

Jump Diffusions and Time-Change

Suppose solution $\tilde{X}$ of a jump diffusion has generator $\tilde{\mathcal{A}}$. How to produce time-change X with generator $\mathcal{A} = \beta\tilde{\mathcal{A}}$?

We know how to change drift and diffusion, but what to do with the jumps? $\rightarrow$ add extra component in PPP!

$$X_t = X_0 + \int_0^t \beta(X_s)a(X_s)\,ds + \int_0^t \sqrt{\beta(X_s)}\,\sigma(X_s)\,dB_s$$
$$+ \int_0^t \int_0^{\beta(X_{s-})} \int_U c(X_{s-}, u)(\mathcal{N} - \mathcal{N}')(ds, dr, du)$$
$$+ \int_0^t \int_0^{\beta(X_{s-})} \int_V d(X_{s-}, u)\mathcal{M}(ds, dr, du),$$

where

- $\mathcal{N}$ PPP on $[0,\infty) \times [0,\infty) \times U$ with $\mathcal{N}'(ds, dr, du) = ds\,dr\,\nu(du)$
- $\mathcal{M}$ PPP on $[0,\infty) \times [0,\infty) \times V$ with $\mathcal{M}'(ds, dr, du) = ds\,dr\,\mu(du)$

Exercise: Calculate generator for X with Itō formula to confirm $\mathcal{A} = \beta\tilde{\mathcal{A}}$.

Jump Diffusions and Time-Change

Suppose solution $\tilde{X}$ of a jump diffusion has generator $\tilde{\mathcal{A}}$. How to produce time-change X with generator $\mathcal{A} = \beta\tilde{\mathcal{A}}$?

We know how to change drift and diffusion, but what to do with the jumps? $\rightarrow$ add extra component in PPP!

$$X_t = X_0 + \int_0^t \beta(X_s)a(X_s)\,ds + \int_0^t \sqrt{\beta(X_s)}\,\sigma(X_s)\,dB_s$$
$$+ \int_0^t \int_0^{\beta(X_{s-})} \int_U c(X_{s-},u)(\mathcal{N}-\mathcal{N}')(ds,dr,du)$$
$$+ \int_0^t \int_0^{\beta(X_{s-})} \int_V d(X_{s-},u)\mathcal{M}(ds,dr,du),$$

where

- $\mathcal{N}$ PPP on $[0,\infty)\times[0,\infty)\times U$ with $\mathcal{N}'(ds,dr,du) = ds\,dr\,\nu(du)$
- $\mathcal{M}$ PPP on $[0,\infty)\times[0,\infty)\times V$ with $\mathcal{M}'(ds,dr,du) = ds\,dr\,\mu(du)$

Exercise: Calculate generator for X with Itō formula to confirm $\mathcal{A} = \beta\tilde{\mathcal{A}}$.

Jump Diffusions and Time-Change

Suppose solution $\tilde{X}$ of a jump diffusion has generator $\tilde{\mathcal{A}}$. How to produce time-change X with generator $\mathcal{A} = \beta\tilde{\mathcal{A}}$?

We know how to change drift and diffusion, but what to do with the jumps? $\rightarrow$ add extra component in PPP!

$$X_t = X_0 + \int_0^t \beta(X_s)a(X_s)\,ds + \int_0^t \sqrt{\beta(X_s)}\,\sigma(X_s)\,dB_s$$
$$+ \int_0^t \int_0^{\beta(X_{s-})} \int_U c(X_{s-},u)(\mathcal{N}-\mathcal{N}')(ds,dr,du)$$
$$+ \int_0^t \int_0^{\beta(X_{s-})} \int_V d(X_{s-},u)\mathcal{M}(ds,dr,du),$$

where

- $\mathcal{N}$ PPP on $[0,\infty)\times[0,\infty)\times U$ with $\mathcal{N}'(ds,dr,du) = ds\,dr\,\nu(du)$
- $\mathcal{M}$ PPP on $[0,\infty)\times[0,\infty)\times V$ with $\mathcal{M}'(ds,dr,du) = ds\,dr\,\mu(du)$

Exercise: Calculate generator for X with Itō formula to confirm $\mathcal{A} = \beta\tilde{\mathcal{A}}$.

Jump Diffusions and Time-Change

Suppose solution $\tilde{X}$ of a jump diffusion has generator $\tilde{\mathcal{A}}$. How to produce time-change X with generator $\mathcal{A} = \beta\tilde{\mathcal{A}}$?

We know how to change drift and diffusion, but what to do with the jumps? $\to$ add extra component in PPP!

$$\begin{aligned} X_t = X_0 &+ \int_0^t \beta(X_s)a(X_s)\,ds + \int_0^t \sqrt{\beta(X_s)}\,\sigma(X_s)\,dB_s \\ &+ \int_0^t \int_0^{\beta(X_{s-})} \int_U c(X_{s-},u)(\mathcal{N}-\mathcal{N}')(ds,dr,du) \\ &+ \int_0^t \int_0^{\beta(X_{s-})} \int_V d(X_{s-},u)\mathcal{M}(ds,dr,du), \end{aligned}$$

where

- $\mathcal{N}$ PPP on $[0,\infty)\times[0,\infty)\times U$ with $\mathcal{N}'(ds,dr,du) = ds\,dr\,\nu(du)$
- $\mathcal{M}$ PPP on $[0,\infty)\times[0,\infty)\times V$ with $\mathcal{M}'(ds,dr,du) = ds\,dr\,\mu(du)$

Exercise: Calculate generator for X with Itō formula to confirm $\mathcal{A} = \beta\tilde{\mathcal{A}}$.

Exercise

Please find an SDE representation for pssMps with $\alpha = 1$!

Lamperti SDE

Theorem (Barczy, D.)

Every pssMp can be written as solution to

$$X_t = X_0 + \left(a + \frac{\sigma^2}{2} + \int_{\{|u| \le 1\}} (e^u - 1 - u)\, \Pi(\mathrm{d}u) \right) t + \sigma \int_0^t \sqrt{X_s} \mathrm{d}B_s$$
$$+ \int_0^t \int_0^{1/X_{s-}} \int_{\{|u| \le 1\}} X_{s-}[e^u - 1](\mathcal{N} - \mathcal{N}')(\mathrm{d}s, dr, \mathrm{d}u)$$
$$+ \int_0^t \int_0^{1/X_{s-}} \int_{\{|u| > 1\}} X_{s-}[e^u - 1]\mathcal{N}(\mathrm{d}s, dr, \mathrm{d}u), \quad t \le T_0,$$

where (a, σ^2, Π) is a Lévy triplet and

- B is a BM
- $\mathcal{N}$ is a PPP on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$ with intensity $ds \otimes dr \otimes \Pi(du)$

Lamperti SDE

The equation is not very nice.

But:

- If we assume $E[e^{\xi_1}] < \infty$ we learn something.
- If we assume ξ is spec neg, we can do everything we wish.

Lamperti SDE

If $E[e^{\xi_1}] < \infty$, then

$$\begin{aligned}&\int_0^t \int_0^{1/X_{s-}} \int_{\{|u|>1\}} X_{s-}[e^u - 1]\mathcal{N}'(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u)\\ &= \int_0^t \int_0^{1/X_{s-}} \int_{\{|u|>1\}} X_{s-}[e^u - 1]\, ds\, dr\, \Pi(du)\\ &= t \int_{\{|u|>1\}} [e^u - 1]\Pi(du) < \infty,\end{aligned}$$

hence,

$$\begin{aligned}&\int_0^t \int_0^{1/X_{s-}} \int_{\{|u|>1\}} X_{s-}[e^u - 1]\mathcal{N}(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u)\\ &= \int_0^t \int_0^{1/X_{s-}} \int_{\{|u|>1\}} X_{s-}[e^u - 1](\mathcal{N} - \mathcal{N}')(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u)\\ &+ t \int_{\{|u|>1\}} [e^u - 1]\Pi(du).\end{aligned}$$

Lamperti SDE

Using

$$\log E[e^{\xi_1}] = a + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^u - 1 - u1_{\{|u|\leq 1\}})\,\Pi(du)$$

we can simplify the SDE to

$$X_t = X_0 + \log E[e^{\xi_1}]t + \sigma \int_0^t \sqrt{X_s} \mathrm{d}B_s + \int_0^t \int_0^{1/X_{s-}} \int_{\mathbb{R}} X_{s-}[e^u - 1](\mathcal{N} - \mathcal{N}')(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u).$$

Note: Call both SDEs Lamperti SDE because they are equivalent to Lamperti's representation.

Lamperti SDE

Theorem (Barczy, D.)

- Pathwise uniqueness holds for the Lamperti SDE.
- Precisely for $\log E[e^{\xi_1}] > 0$ there are strong solutions for all $X_0 \geq 0$ to the Lamperti SDE and pathwise uniqueness holds.

Proof: Ugly Yamada/Watanabe type arguments.

Lamperti SDE

Theorem

The self-similar recurrent extensions of Fitzsimmons, Rivero and also the limit laws P^0 of Bertoin, Caballero, Chaumont, Kyprianou, Pardo, Rivero, Savov, ... are solutions to the SDE.

Proof: As above for $Bes^2(\delta)$: Show that (X_t) and $(cX_{tc^{-1}})$ solve same equation, then use uniqueness.

Exercise: Please proof uniqueness also for positive jumps!

Warning: Spec neg case also has different easier proofs.

Lamperti SDE

Theorem

The self-similar recurrent extensions of Fitzsimmons, Rivero and also the limit laws P^0 of Bertoin, Caballero, Chaumont, Kyprianou, Pardo, Rivero, Savov, ... are solutions to the SDE.

Proof: As above for $Bes^2(\delta)$: Show that (X_t) and $(cX_{tc^{-1}})$ solve same equation, then use uniqueness.

Exercise: Please proof uniqueness also for positive jumps!

Warning: Spec neg case also has different easier proofs.

Lamperti SDE

Theorem

The self-similar recurrent extensions of Fitzsimmons, Rivero and also the limit laws P^0 of Bertoin, Caballero, Chaumont, Kyprianou, Pardo, Rivero, Savov, ... are solutions to the SDE.

Proof: As above for $Bes^2(\delta)$: Show that (X_t) and $(cX_{tc^{-1}})$ solve same equation, then use uniqueness.

Exercise: Please proof uniqueness also for positive jumps!

Warning: Spec neg case also has different easier proofs.

Lamperti SDE

Theorem

The self-similar recurrent extensions of Fitzsimmons, Rivero and also the limit laws P^0 of Bertoin, Caballero, Chaumont, Kyprianou, Pardo, Rivero, Savov, ... are solutions to the SDE.

Proof: As above for $Bes^2(\delta)$: Show that (X_t) and $(cX_{tc^{-1}})$ solve same equation, then use uniqueness.

Exercise: Please proof uniqueness also for positive jumps!

Warning: Spec neg case also has different easier proofs.

Lamperti SDE

Why is Lamperti SDE special?

- Lamperti SDE for $t \leq T_0 \quad \Longleftrightarrow \quad$ Lamperti's representation.
- Lamperti's respresentation does not work immediately for $t > T_0$.
- BUT: Lamperti SDE works immediately for $t > T_0$ iff the necessary and sufficient condition is fulfilled.

Lamperti SDE

Why is Lamperti SDE special?

- Lamperti SDE for $t \leq T_0 \quad \Longleftrightarrow \quad$ Lamperti's representation.
- Lamperti's respresentation does not work immediately for $t > T_0$.
- BUT: Lamperti SDE works immediately for $t > T_0$ iff the necessary and sufficient condition is fulfilled.

Summary

We discussed definitions, examples and connections for

- time-change
- generators
- SDEs

In some sense those are equivalent, but approaches have different advantages.

For pssMps we discussed

- time-change representation
- generator representation
- SDE representation

For pssMps the SDE representation has a magic feature: Can be extended after hitting zero.

Summary

We discussed definitions, examples and connections for

- time-change
- generators
- SDEs

In some sense those are equivalent, but approaches have different advantages.

For pssMps we discussed

- time-change representation
- generator representation
- SDE representation

For pssMps the SDE representation has a magic feature: Can be extended after hitting zero.