Self-avoiding walk, loop-erased random walk and self-repelling walk on a fractal

(Preview version: This version includes more than my actual talk.)

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Joint work with Noriaki Ogo and Takafumi Otsuka

Stochastic Processes and their Applications, Mongolia 2015

Outline

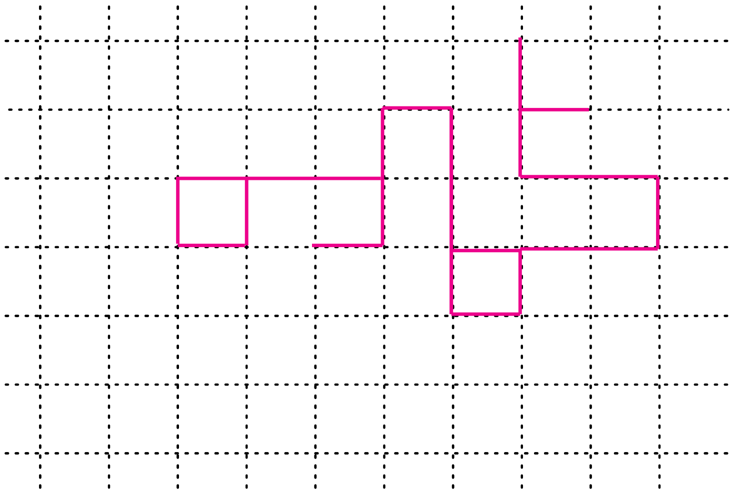
0. Introduction

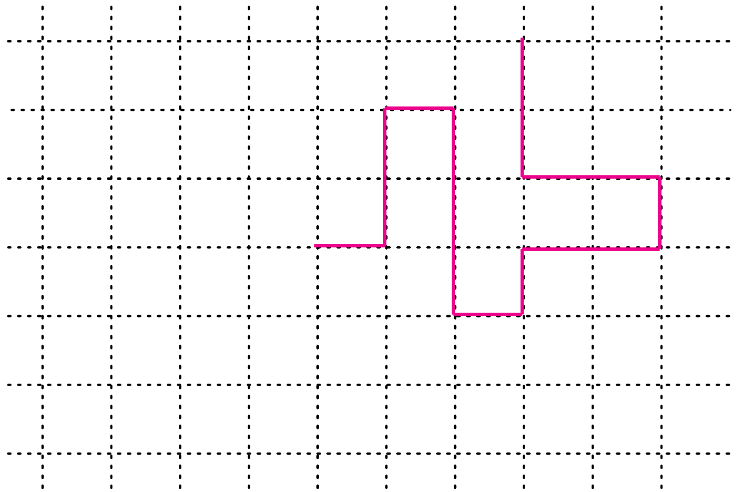
We consider non-Markovian processes/random walks, that is, processes/random walks whose future behavior depends not only on the present state but also their past history.

In general, stochastic processes that lack Markovian property are tough to study, but on some fractal spaces, rigorous and interesting results have been obtained.

Markovian vs. Non-Markovian

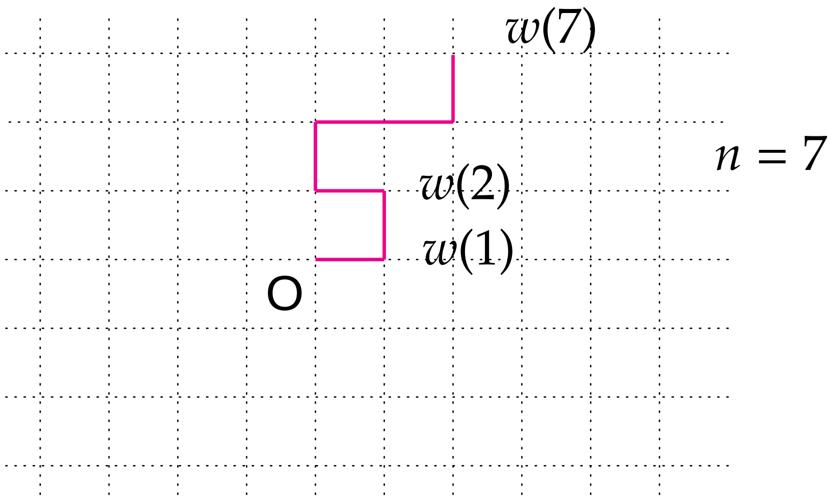
Markovian	non-Markovian
A Simple random walk jumps to one of the nearest sites with equal probability.	A self-avoiding walk cannot visit any sites more than once.





Definition of 'standard' self-avoiding walk (SAW) on $\mathbb{Z}^d$:

For each fixed n, consider the set of all n-step self-avoiding paths on $\mathbb{Z}^d$ starting from O, that is, $(w(0), w(1), \cdots, w(n))$ satisfying $w(0) = O$, $w(i) \in \mathbb{Z}^d$, $|w(i) - w(i-1)| = 1$, $i = 1, 2, \cdots, n$, $w(i) \neq w(j)$, $i \neq j$. Assign equal probability to each n-step path.

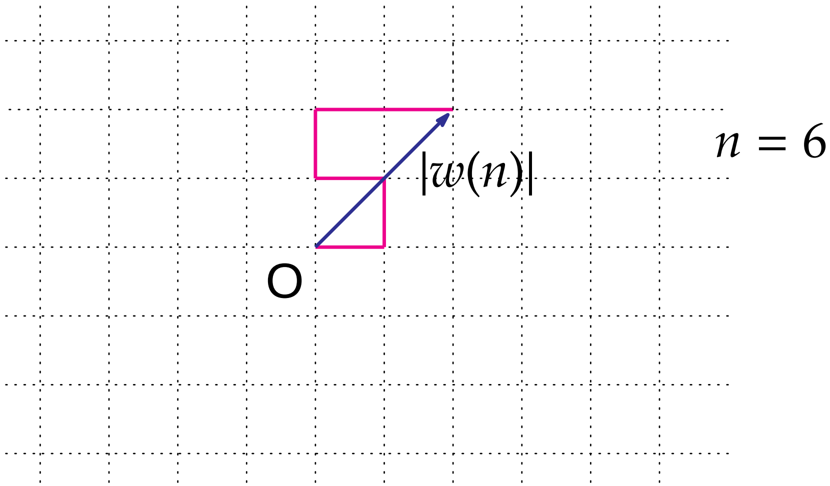


SAW originated in chemistry as a model of long polymers (1950's, 1960's).

Two basic questions

(1) How far does an n-step walk go on average?

$w(n)$: the location after n-steps,
$|\, w(n) \,|$: Euclidian distance from O.



Mean square displacement $E[|w(n)|^2] \sim \ ? \quad n \to \infty$

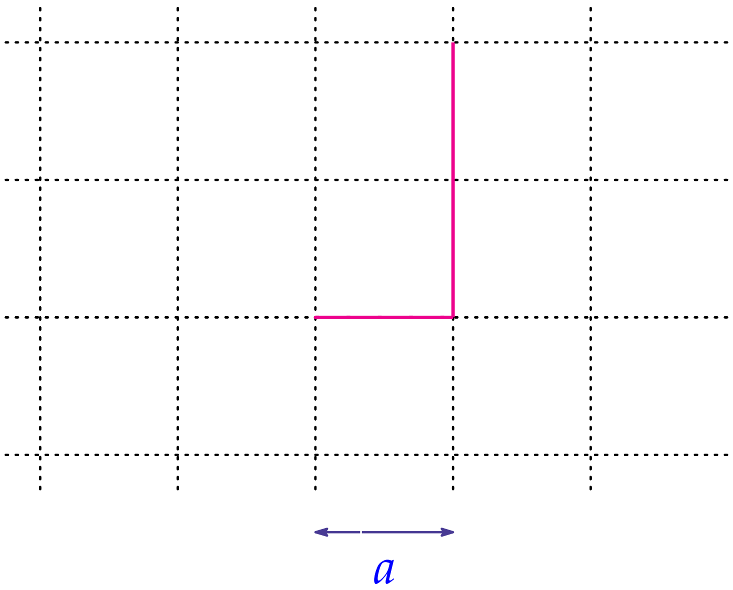
If the mean square displacement shows a power behavior like $E[|w(n)|^2] \sim n^{2\nu}, \quad n \to \infty$,
we call ν the displacement exponent .

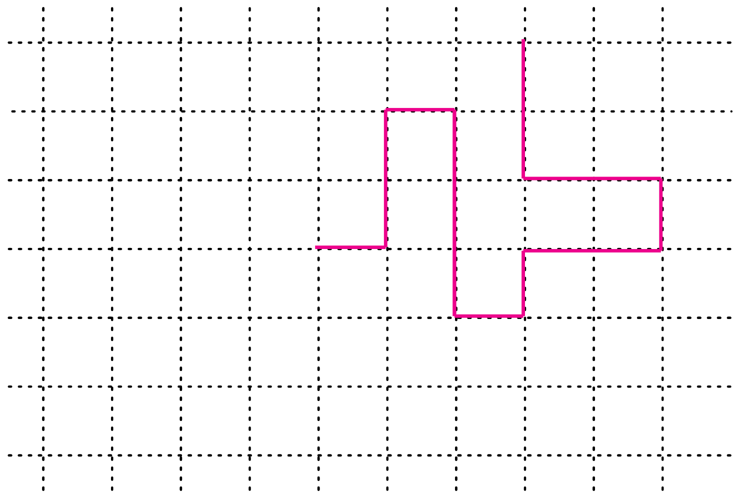
cf. Simple random walk on $\mathbb{Z}^d$

$$E[|w(n)|^2] = n, \ \nu = 1/2.$$

As we will see later, the same exponent ν governs the short time behavior of the scaling limit.

(2) Scaling limit (the limit as the edge length $a \to 0$)
Does the scaling limit exist? (Does the SAW converge to any limit process as $a \to 0$?)
If yes, what is the limit process like?





cf. The simple random walk on $(a\mathbb{Z})^d$ converges to the d-dimensional Brownian motion as $a \to 0$.

SAW on $\mathbb{Z}^d$

displacement exponent (See also Sec. 4)		scaling limit
$d = 1$	$\nu = 1$	trivial
$d = 2$	$\nu = \frac{3}{4}$	$\mathrm{SLE}_{8/3}$
$d = 3$	$\nu = 0.5876 \cdots$	?
$d = 4$	$\nu = \frac{1}{2}$+(log correction)	BM
$d \geq 5$	$\nu = \frac{1}{2}$	BM (Hara, Slade)

blue : conjectures.

Low dimensions are extremely tough!

$\Longrightarrow$ What about SAW on fractals?

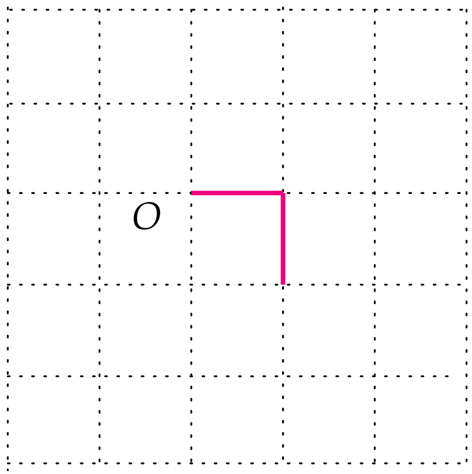
	displacement exponent	scaling limit
$d=1$	$\nu=1$	trivial
Sierpinski gasket		
$d_H=1.58$	$\nu=\log 2/\log\lambda_{SAW}=0.798\cdots$	SA process
	$\lambda_{SAW}=(7-\sqrt{5})/2$	
$d=2$	$\nu=\frac{3}{4}$	$\mathrm{SLE}_{8/3}$
$d=3$	$\nu=0.5876\cdots$	?
$d=4$	$\nu=\frac{1}{2}$+(log correction)	BM
$d\geq 5$	$\nu=\frac{1}{2}$	BM (Hara, Slade)

Though low-dimentional, results on the scaling limit of the SAW on the Sierpinski gasket have been obtained.

Next section $\longrightarrow$ A new kind of non-Markovian random walk introduced in 1980.

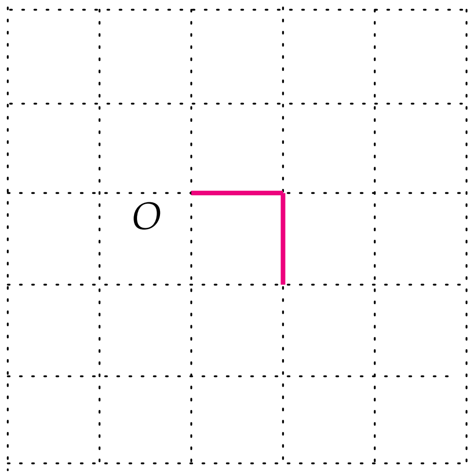
1. Loop-erased random walk

A simple random walk on a graph
jumps to a nearest neighbor site with equal probability.



1. Loop-erased random walk

From a simple random walk on a graph, erase loops chronologically → Loop-erased random walk.



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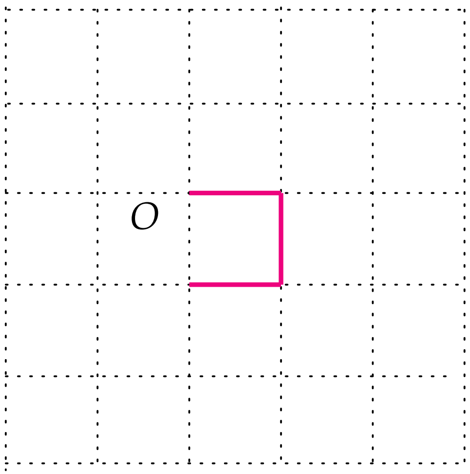

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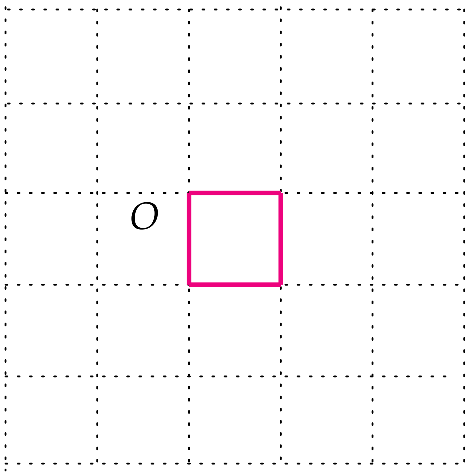
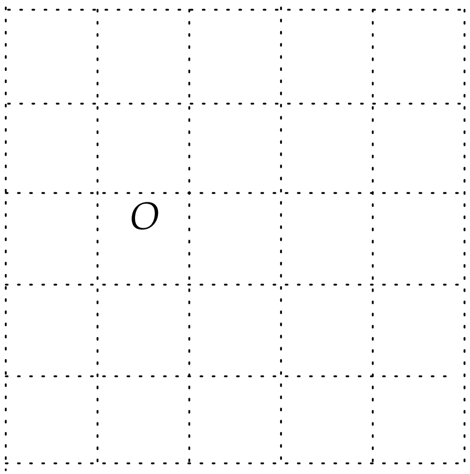


We've got a loop.

1. Loop-erased random walk

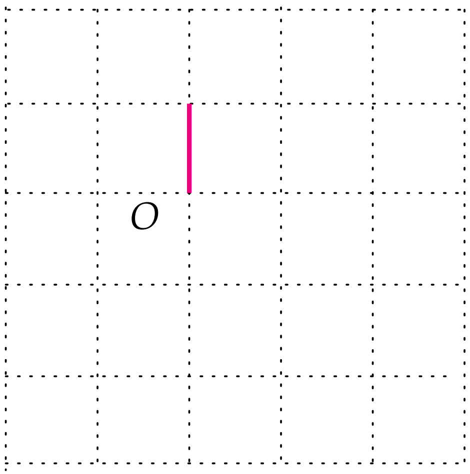
From a simple random walk on a graph, erase loops chronologically $\to$ Loop-erased random walk.



We erased the loop!

1. Loop-erased random walk

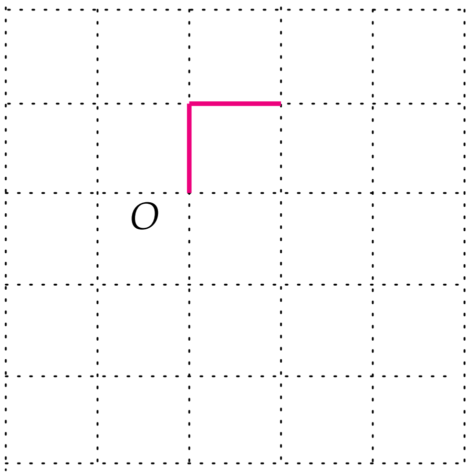
From a simple random walk on a graph, erase loops chronologically $\rightarrow$ Loop-erased random walk.



Start anew.

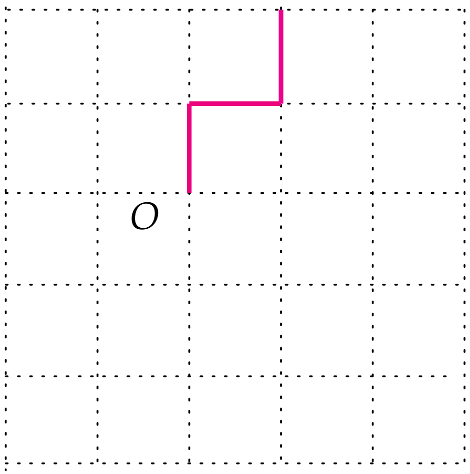
1. Loop-erased random walk

From a simple random walk on a graph, erase loops chronologically → Loop-erased random walk.



1. Loop-erased random walk

From a simple random walk on a graph, erase loops chronologically $\rightarrow$ Loop-erased random walk.



LERW (introduced by G. Lawler, 1980) does not intersect with itself (self-avoiding), but has a different distribution from the standard SAW.

Results concerning LERW on $\mathbb{Z}^d$ (2000∼) by Lawler, Schramm, Werner, Kozma, Kenyon, Masson, Shiraishi, Suzuki, $\cdots$

	growth exponent	scaling limit (edge length $\to 0$)
$d = 2$	$\alpha^{-1} = 4/5$,	SLE_2 curve
$d = 3$	$\exists\alpha^{-1}$, $\alpha^{-1} = 0.617\cdots$,	$\exists$ a scaling limit.
$d \geq 4$	$\alpha^{-1} = 1/2$	Brownian motion

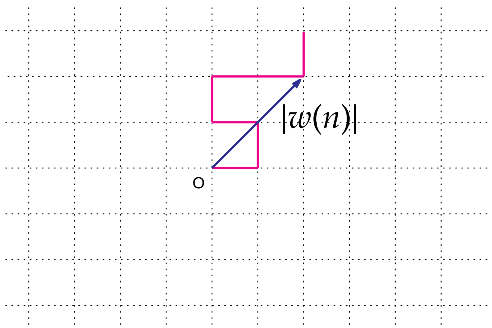
α: growth exponent (believed : $\nu = \alpha^{-1}$)

M_n : number of steps from O to the circle of radius n.

$$M_n \sim n^{\alpha}.$$

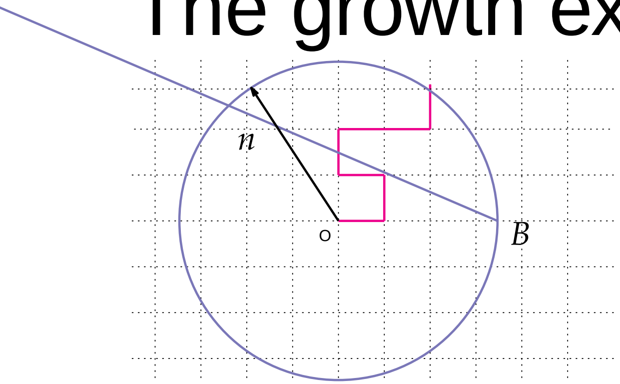
Remark

The displacement exponent



$E[|w(n)|^2] \sim n^{2\nu}$

The growth exponent α



$E[M_n] \sim n^{\alpha}$

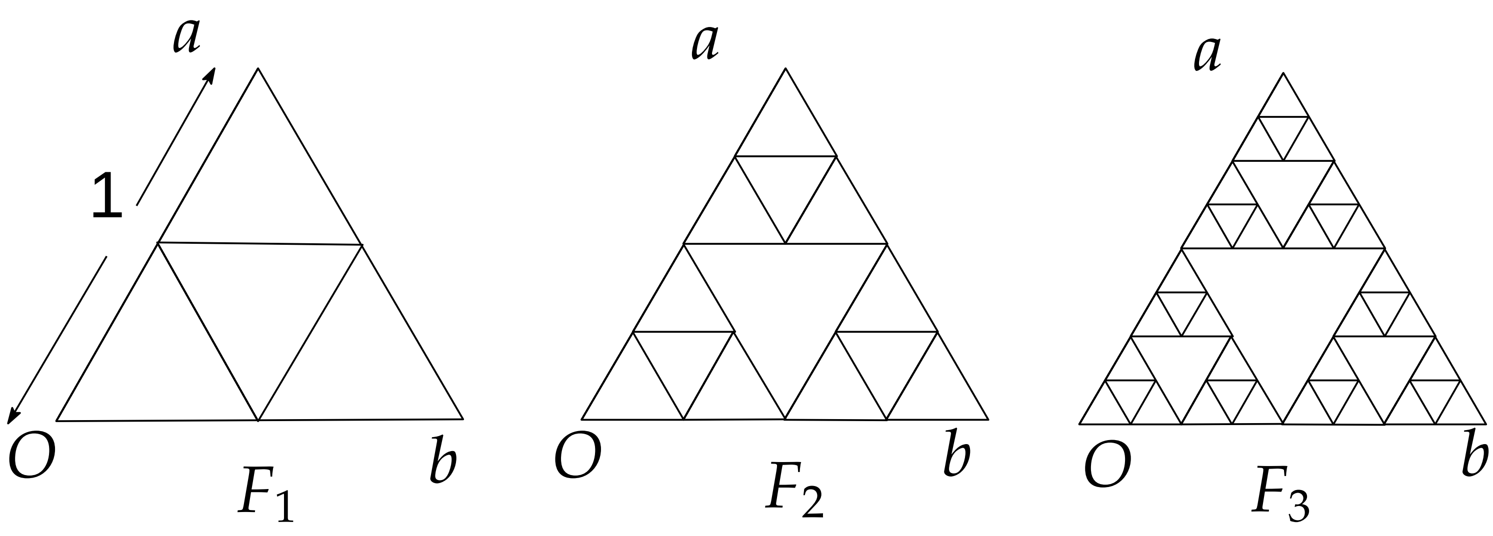
M_n : steps to ∂B

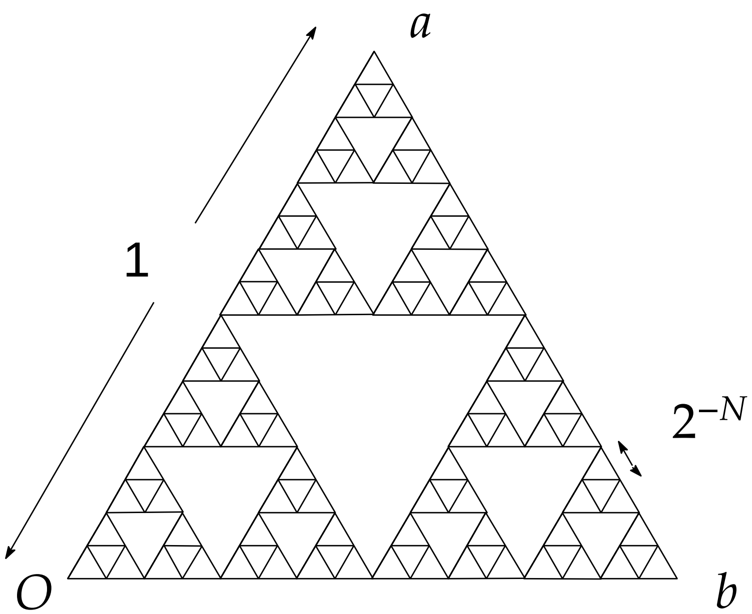
Next section → What about LERW on a fractal graph?

2. Pre-Sierpinski gaskets

$\triangle Oab$: a unit triangle.

pre-Sierpinski gasket F_N : a graph with edge length 2^{-N}.





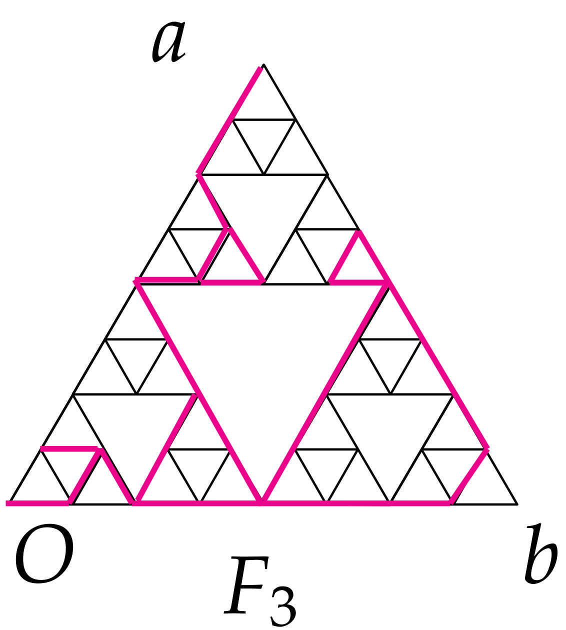
pre-Sierpinski gasket F_N : a graph with edge length 2^{-N}.

Sierpinski gasket $F = \overline{\cup_{N=1}^{\infty} F_N}$ (closure) a fractal

We are interested in the limit of the loop-erased random walks on F_N as the edge length $\delta = 2^{-N} \to 0$ (scaling limit).

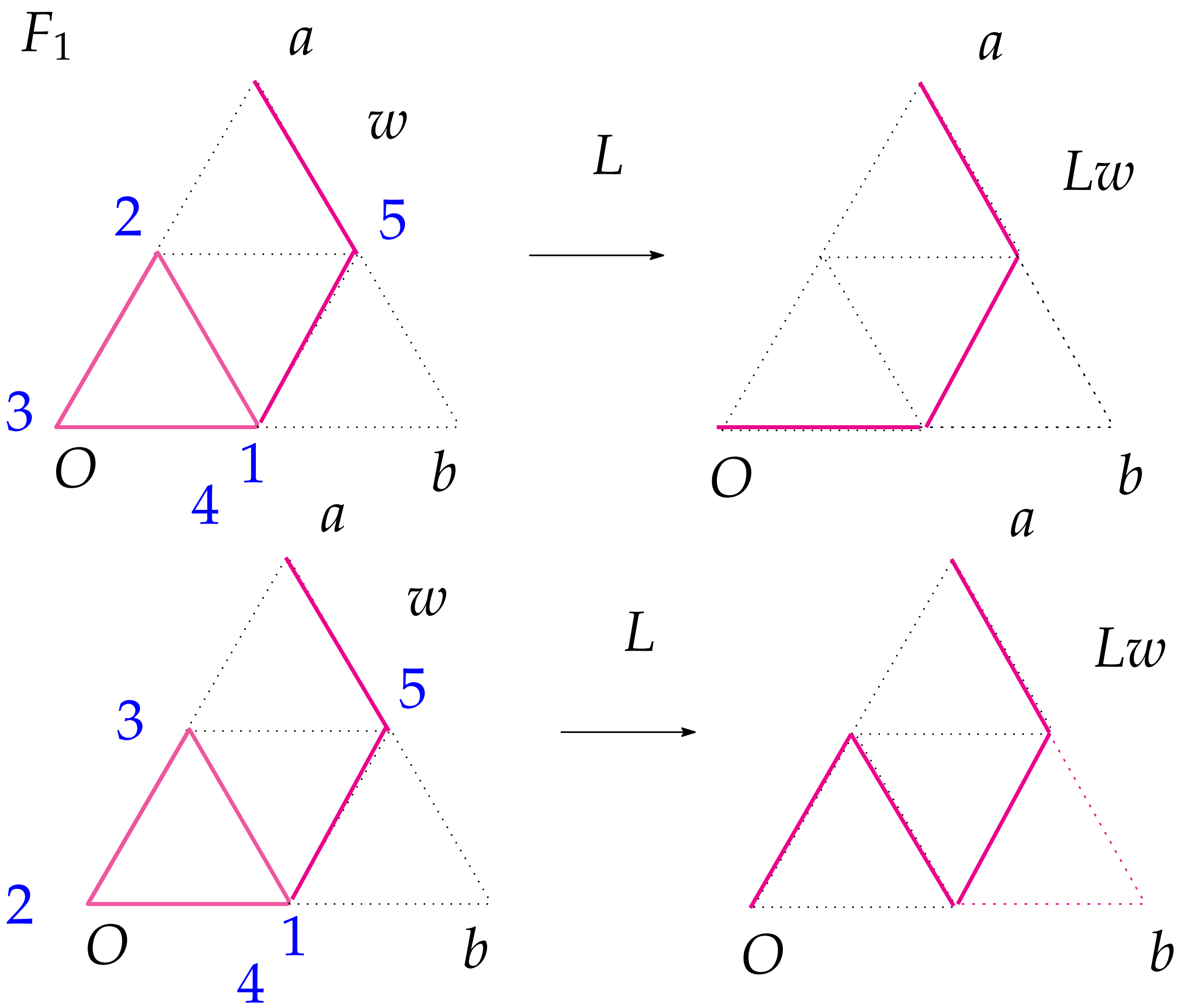
3. Erasing-larger-loops-first rule (ELLF)

Z_N : a simple random walk on F_N, starting at O and stopped at a. We condition that it has no loops of diameter 1 (ex. no return trip $O \to b \to O$ etc.).



Erase loops from its path.

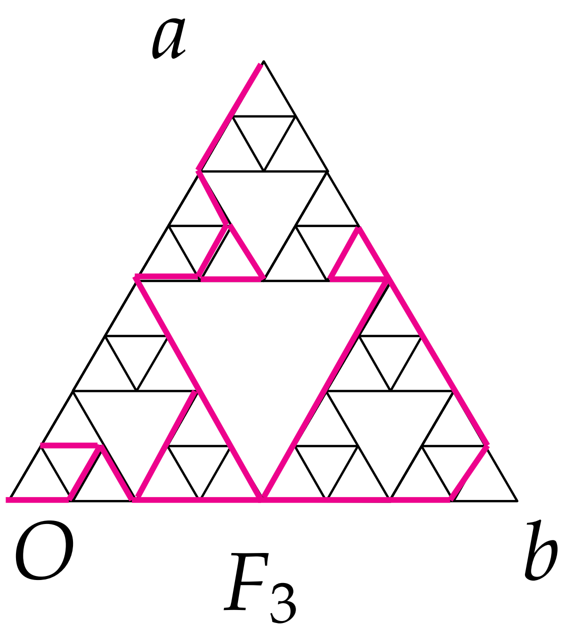
On F_1 we erase loops chronologically.



L: the loop-erasing operator.

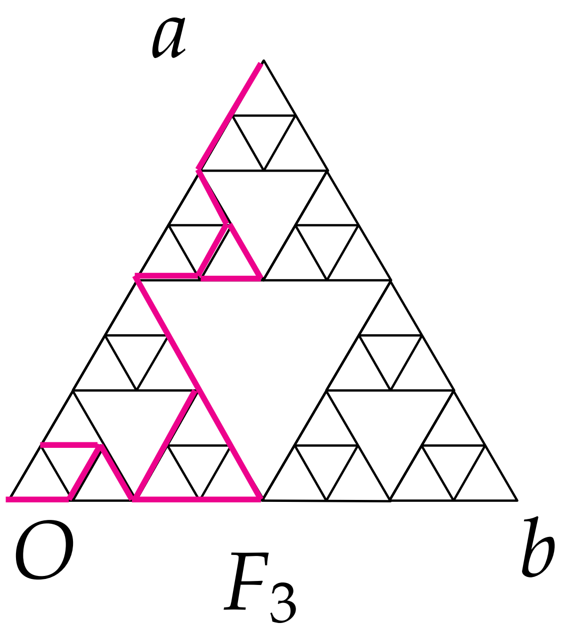
On F_N, $N \geq 2 \to$ Erasing-larger-loops-first rule

(ELLF) not chronological



Erase loops with diameter in $[1/2, 1) \Longrightarrow$ Erase loops with diameter in $[1/4, 1/2) \Longrightarrow$ Erase loops with diameter in $[1/8, 1/4) \Longrightarrow \cdots \to$ Recursions

On F_N, $N \geq 2 \rightarrow$ Erasing-larger-loops-first rule (ELLF)



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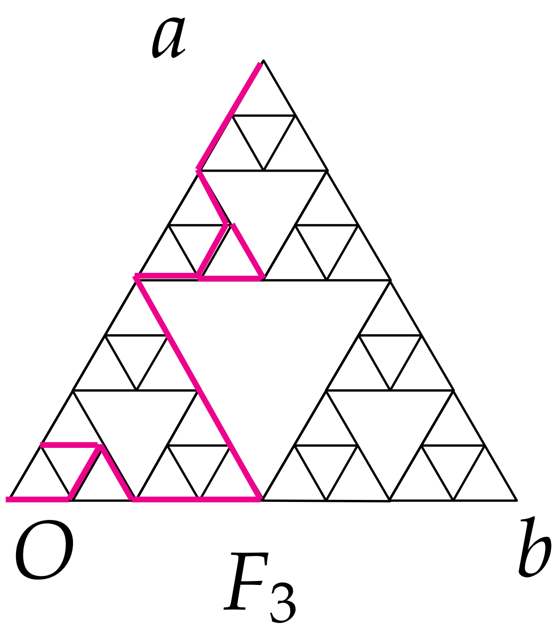

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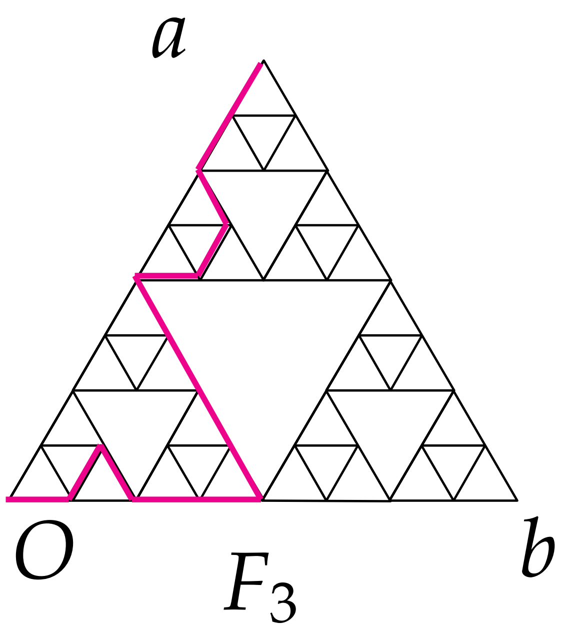


Erase loops with diameter in $[1/2, 1) \Longrightarrow$ Erase loops with diameter in $[1/4, 1/2) \Longrightarrow$ Erase loops with diameter in $[1/8, 1/4) \Longrightarrow \cdots \to$ Recursions for the generating functions of the number of steps. (See Appendix.)

Thm. 1. (Shinoda-Teufl-Wagner, H-Mizuno 2014)

Υ_N : LERW on F_N. $\lambda_{LERW} = (20 + \sqrt{205})/15 = 2.2878\ldots$

$\Upsilon_N(\lambda_{LERW}^N\, t\,) \to \Upsilon(t)$ uniformly in t a.s. as $N \to \infty$. (The scaling limit exists.) (steps interpreted as ‘time’)

Υ is almost surely self-avoiding. (Not obvious)

ELLF LERW $\overset{\mathrm{d}}{=}$ ‘standard’ LERW. (Not obvious)

HM (ELLF) STW (chronological erasing)

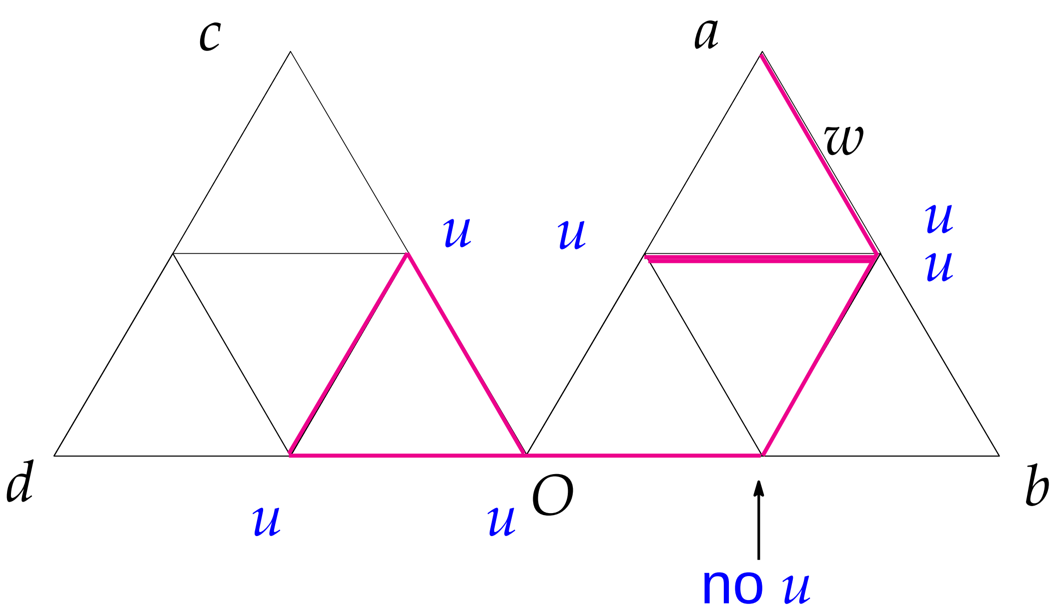
use of uniform spanning tree

Next section $\to$ELLF is applicable to other kinds of walks.

4. Erasing loops from self-repelling walks

To construct a self-repelling walk on $F_1 \times 2$, consider paths $O \to a$, not visiting b, c, d on the way.

Penalty u for sharp turns and returns to O $(0 \le u \le 1)$.



$N(w) = 6, \ L(w) = 8$

$L(w) =$ no. of steps

$P_1[w] = u^{N(w)} x_u^{L(w)-1}, \quad x_u > 0; \sum P_1[w] = 1$.

A natural generalization, for $u = 1$ gives the simple RW :

$P[w] = (\frac{1}{4})^{L(w)-1}$, $x_1 = 1/4$.

We can define a one-parameter family of self-repelling walks on $F_N \times 2$ recursively.

$u = 1 \rightarrow$ the simple random walk.
$u = 0 \rightarrow$ a self-avoiding walk with the same displacement exponent ν as that for the standard SAW.

This model of self-repelling walks interpolates the simple random walk and SAW continuously.

Thm. 2 (Hambly, T. Hattori, K.H. 2002)
The scaling limit $Z(t)$ exists. (BM for $u = 1$)
(a special feature) $\forall s > 0$,
$\exists C_1 = C_1(u,s), C_2 = C_2(u,s) > 0$

$$C_1 \leq \liminf_{t\to 0} \frac{E[|Z(t)|^s]}{t^{\nu s}} \leq \limsup_{t\to 0} \frac{E[|Z(t)|^s]}{t^{\nu s}} \leq C_2.$$

$\nu = f(u)$ is a continuous function in u,
$f(0) = \nu_{SAW} = \log 2/\log \lambda_{SAW}$, $f(1) = \log 2/\log 5$.
ν is equal to the displacement exponent for the self-repelling walk (edge length=1).
cf. ν is either 1 or $1/2$ for some models of self-repelling walks on $\mathbb{Z}$.

5. Main Theorems

Self-repelling walks + erasing-larger-loops-first rule
$\rightarrow$ a new one-parameter family of 'self-avoiding' walks.

Thm. 3 (H-Ogo-Otsuka 2015)   scaling limit
X_N^u : Loop-erased self-repelling walk on $F_N \times 2$ (edge length 2^{-N}).
$2 < \exists \lambda_u < 3$ (a continuous function of u) ;
$X_N^u(\lambda_u^N t) \rightarrow X^u(t)$ unif. in t a.s as $N \rightarrow \infty$.
(The scaling limit exists.)

Thm. 4 (H-Ogo-Otsuka 2015) path properties

1) X^u is almost surely self-avoiding.

2) Hausdorff dimension of the sample path :

$d(u) = \log \lambda_u / \log 2 > 1$ a.s..

3) $\forall s > 0$, $\exists C_3 = C_3(u, s), C_4 = C_4(u, s) > 0$

$$C_3 \leq \liminf_{t \to 0} \frac{E[|X(t))|^s]}{t^{\nu s}} \leq \limsup_{t \to 0} \frac{E[|X(t))|^s]}{t^{\nu s}} \leq C_4,$$

where $\nu = 1/d(u)$, a continuous function of u.

(λ_u : the time-scaling factor $\quad X_N^u(\lambda_u^N t) \to X^u(t)$)

Summary

We applied ELLF to a family of self-repelling walks to obtain a new family of self-avoiding walks interpolating LERW and the ‘standard’ SAW, and studied their scaling limits.

References

Self-repelling walk on the SG

- B.M. Hambly, K. Hattori, T. Hattori, *Self-repelling walk on the Sierpinski gasket*, PTRF, 124 (2002) 1-25

LERW on the SG

- M. Shinoda, E.Teufl, S. Wagner, *Uniform spanning trees on Sierpinski graphs*, Lat. Am. J. of Prob. Math. Stat. 11 (2014) 737-780
- K. Hattori, M. Mizuno, *Loop-erased random walk on the Sierpinski gasket*, SPA, 124 (2014) 566-585

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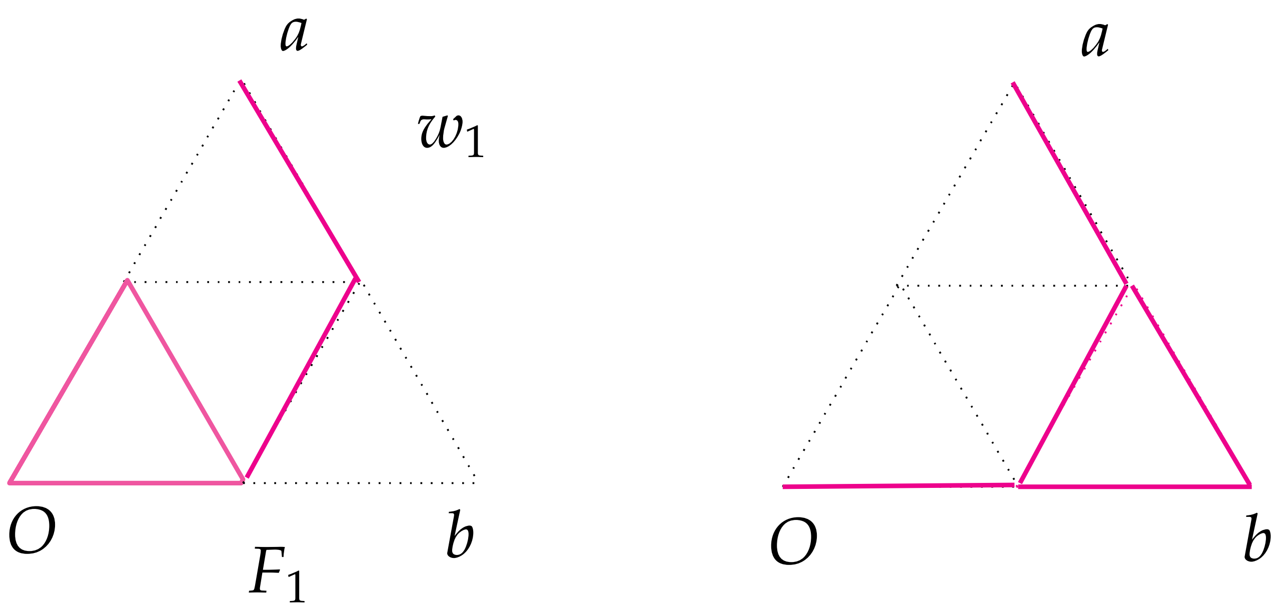
Appendix



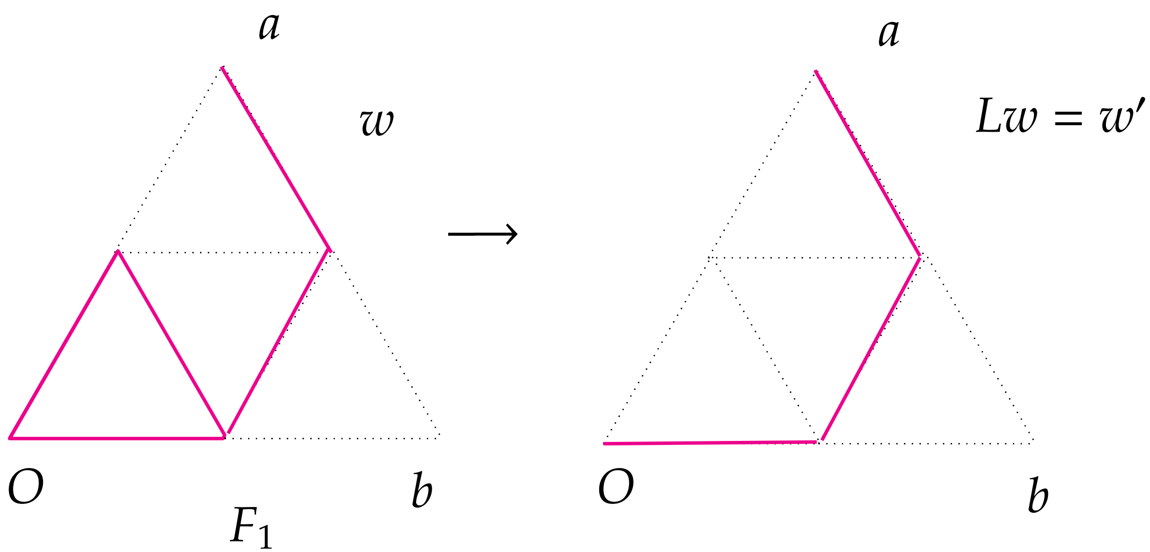
Two conditioned simple random walks on F_N from O to a.

P_N : the path measure of SRW not via b.

P'_N : the path measure of SRW via b.

For example, $P_1[w_1] = (\frac{1}{2})^2(\frac{1}{4})^4/(\frac{1}{2})$. $\leftarrow$ Conditioned

Loop erasure from the simple random walk on F_1 (chronological).



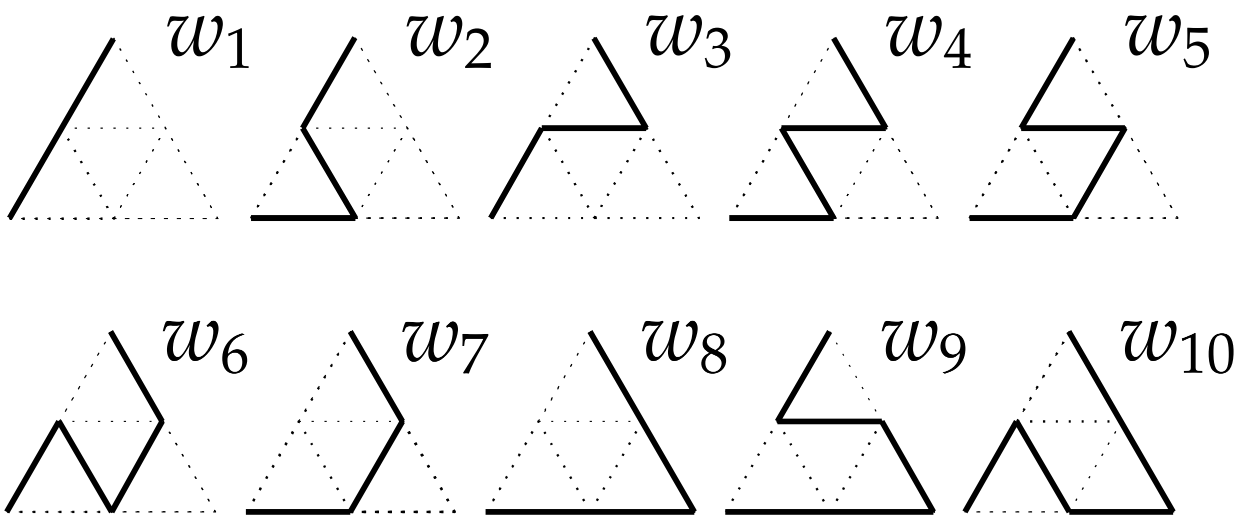
L : Loop-erasing operator.

$\hat{P}_1 = P_1 \circ L^{-1}$, $\hat{P}'_1 = P'_1 \circ L^{-1}$: LERW measures

($\hat{P}_1[w']$ is the probability to get a path w' as a result of loop-erasure.) Infinitely many paths result in a same path by L.

These probabilities can be calculated directly by hand.

$\hat{P}_1 = P_1 \circ L_1^{-1}$: LERW measure (SRW not via b)

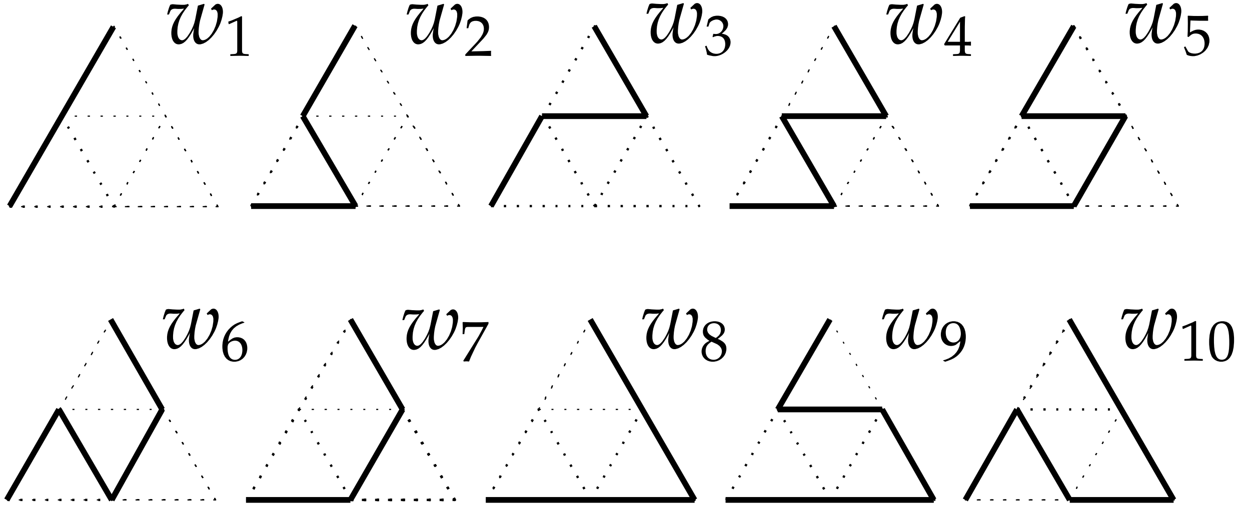


$$\hat{P}_1[w_1] = \frac{1}{2},\ \hat{P}_1[w_2] = \hat{P}_1[w_3] = \frac{2}{15},$$

$$\hat{P}_1[w_4] = \hat{P}_1[w_5] = \hat{P}_1[w_6] = \frac{1}{30},\ \hat{P}_1[w_7] = \frac{2}{15},$$

$$\hat{P}_1[w_i] = 0,\ i = 8, 9, 10.$$

$\hat{P}'_1 = P'_1 \circ L_1^{-1}$: LERW measure (SRW via b)



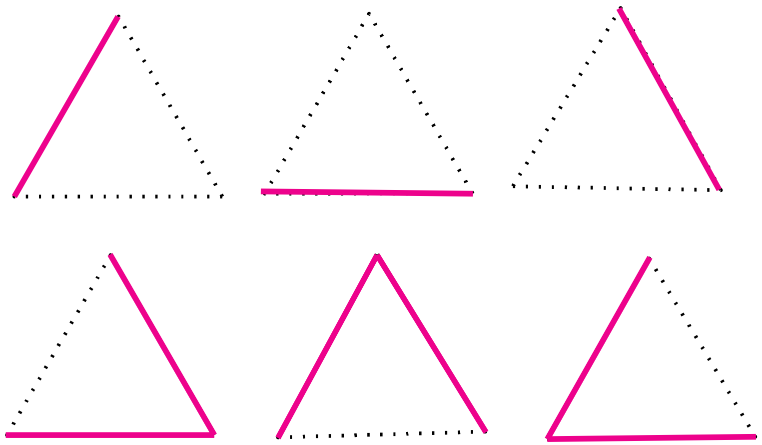
$$\hat{P}'_1[w_1] = \frac{1}{9},\ \hat{P}'_1[w_2] = \hat{P}'_1[w_3] = \frac{11}{90},$$

$$\hat{P}'_1[w_4] = \hat{P}'_1[w_5] = \hat{P}'_1[w_6] = \frac{2}{45},$$ (b can be erased)

$$\hat{P}'_1[w_7] = \frac{8}{45},\ \hat{P}'_1[w_8] = \frac{2}{9},\ \hat{P}'_1[w_9] = \hat{P}'_1[w_{10}] = \frac{1}{18}.$$

Generating functions

$\hat{W}_N$: The set of loopless paths on F_N from O to a. For a path $w \in \hat{W}_N$, count the numbers of 2^{-N} - sized triangles w passes through:



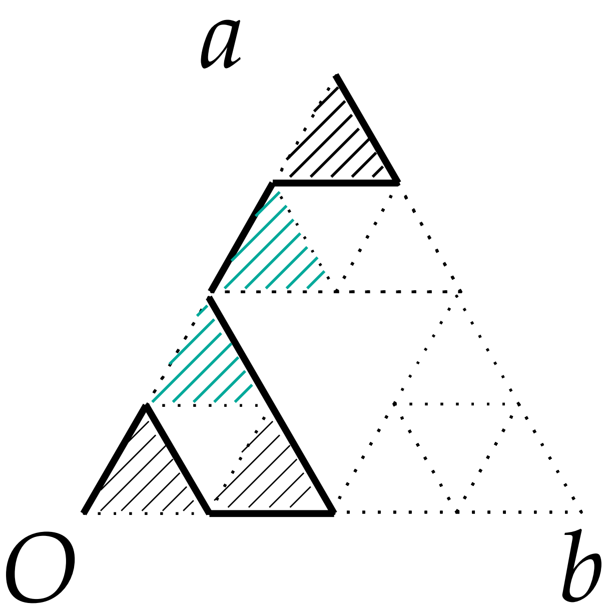
Type 1

Type 2

$s_1(w) = \sharp\{$triangles of Type 1$\}$, $s_2(w) = \sharp\{$triangles of Type 2$\}$

Random variables

$s_1(w) = \sharp\{\text{Type 1}\}$, $s_2(w) = \sharp\{\text{Type 2}\}$



$$s_1(w) = 2,\ s_2(w) = 3$$

Number of steps : $L(w) = s_1(w) + 2s_2(w)$

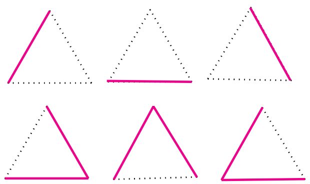
(In other words, the ‘time’ it takes to go $O \to a$ if jumps occur at integer times.)

$\hat{P}_N = P_N \circ L^{-1}$, $\hat{P}'_N = P'_N \circ L^{-1}$: LERW path measures

Define generating functions by

$$\Phi_N(x,y) = \sum_{w \in \hat{W}_N} \hat{P}_N(w)\, x^{s_1(w)}\, y^{s_2(w)},$$

$$\Theta_N(x,y) = \sum_{w \in \hat{W}_N} \hat{P}'_N(w)\, x^{s_1(w)}\, y^{s_2(w)}, \quad x, y \geq 0.$$



Type 1

Type 2

$s_1(w) = \sharp\{2^{-N}$-triangles of Type 1$\}$, $s_2(w) = \sharp\{$Type 2$\}$.

The ELLF rule leads to the following recursions.

Recursions

$$\Phi_{N+1}(x,y) = \Phi_1(\Phi_N(x,y), \Theta_N(x,y)).$$

$$\Theta_{N+1}(x,y) = \Theta_1(\Phi_N(x,y), \Theta_N(x,y)), \quad N \in \mathbb{N}.$$

$$\Phi_1(x,y) = \frac{1}{30}(15x^2 + 8xy + y^2 + 2x^2y + 4x^3).$$

$$\Theta_1(x,y) = \frac{1}{45}(5x^2 + 11xy + 2y^2 + 14x^2y + 8x^3 + 5xy^2).$$

Mean matrix of the number of triangles

$$\mathbf{M} = \begin{bmatrix} \frac{\partial}{\partial x}\Phi_1(1,1) & \frac{\partial}{\partial y}\Phi_1(1,1) \\ \frac{\partial}{\partial x}\Theta_1(1,1) & \frac{\partial}{\partial y}\Theta_1(1,1) \end{bmatrix} = \begin{bmatrix} \frac{9}{5} & \frac{2}{5} \\ \frac{26}{15} & \frac{13}{15} \end{bmatrix}$$

The larger eigenvalue

$$\lambda_{LERW} = \frac{1}{15}(20 + \sqrt{205}) = 2.2878\ldots.$$

This number determines the time-scaling factor, the growth exponent, the exponent for short time behavior and the path Hausdorff dimension.

Kolmogorov extension theorem

\+ the limit theorem of branching processes

\+ recursions

$\Longrightarrow$ the scaling limit and ν.