# Optimal transport, heat flow and coupling of Brownian motions

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Stochastic Processes and Applications
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Outline of the talk

**1. Basics on optimal transport**

**2. Lower Ricci curvature bound**

**3. Coupling(s) of Brownian motions**

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3. Coupling(s) of Brownian motions

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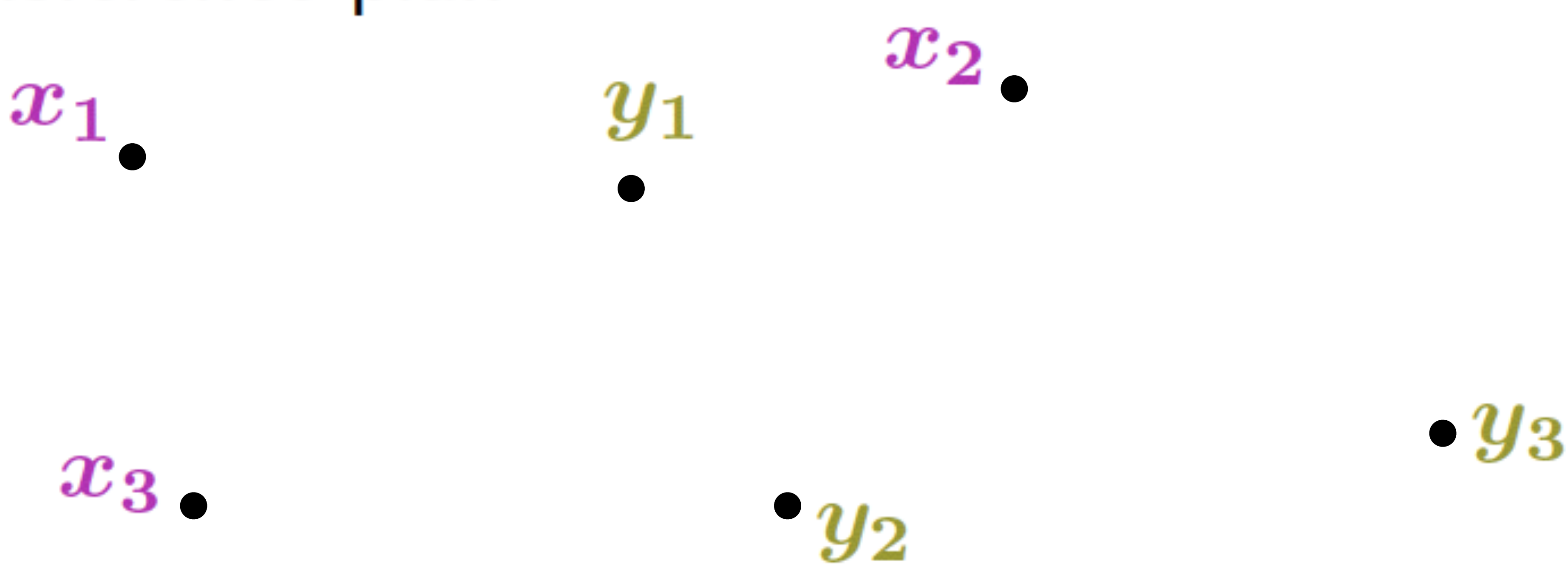

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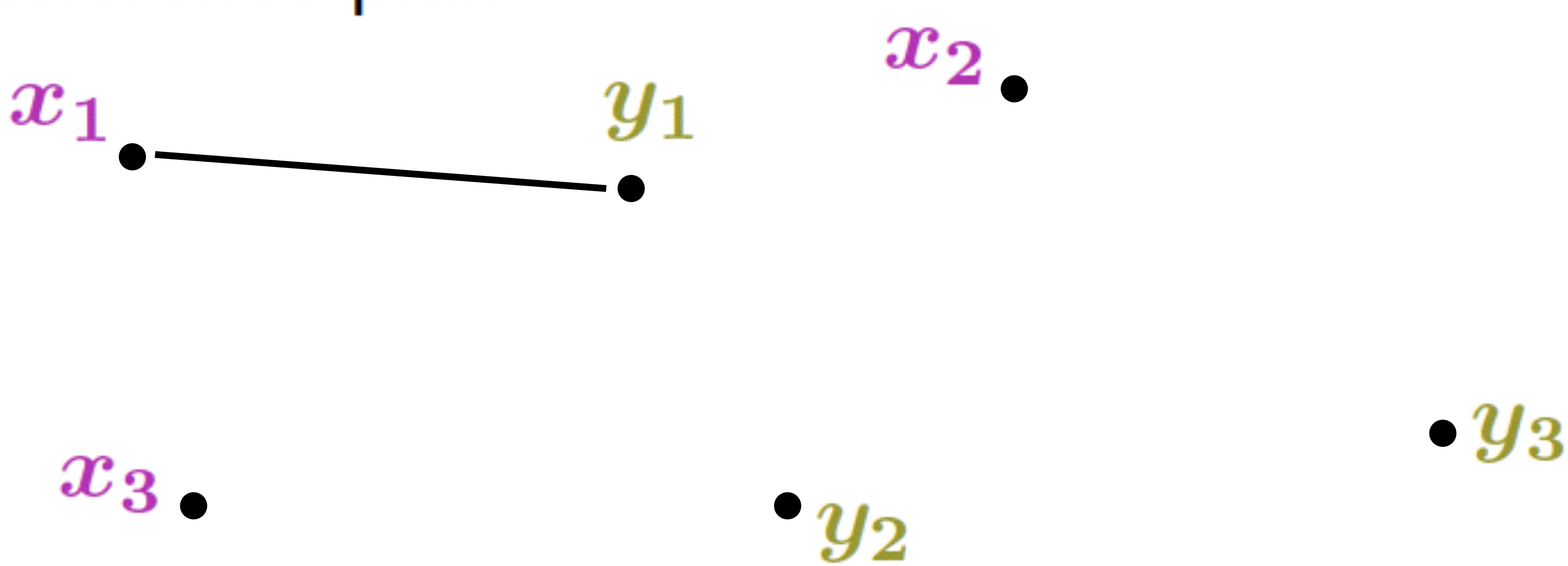

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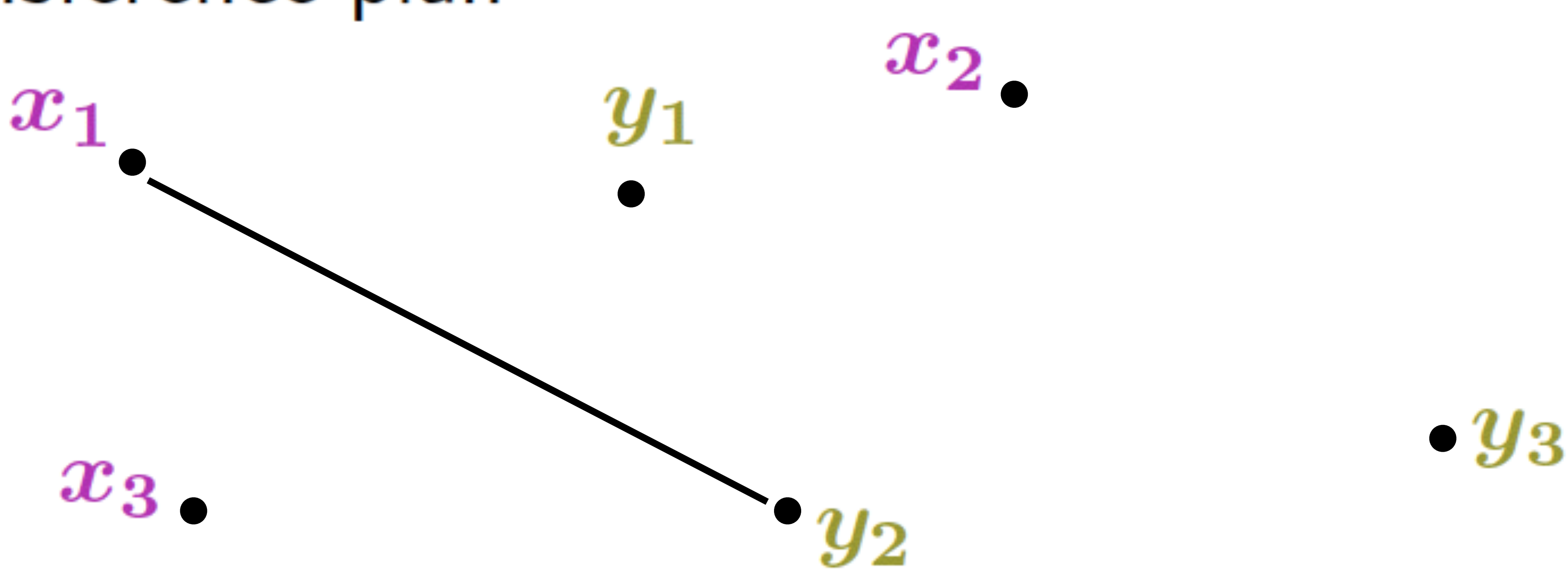

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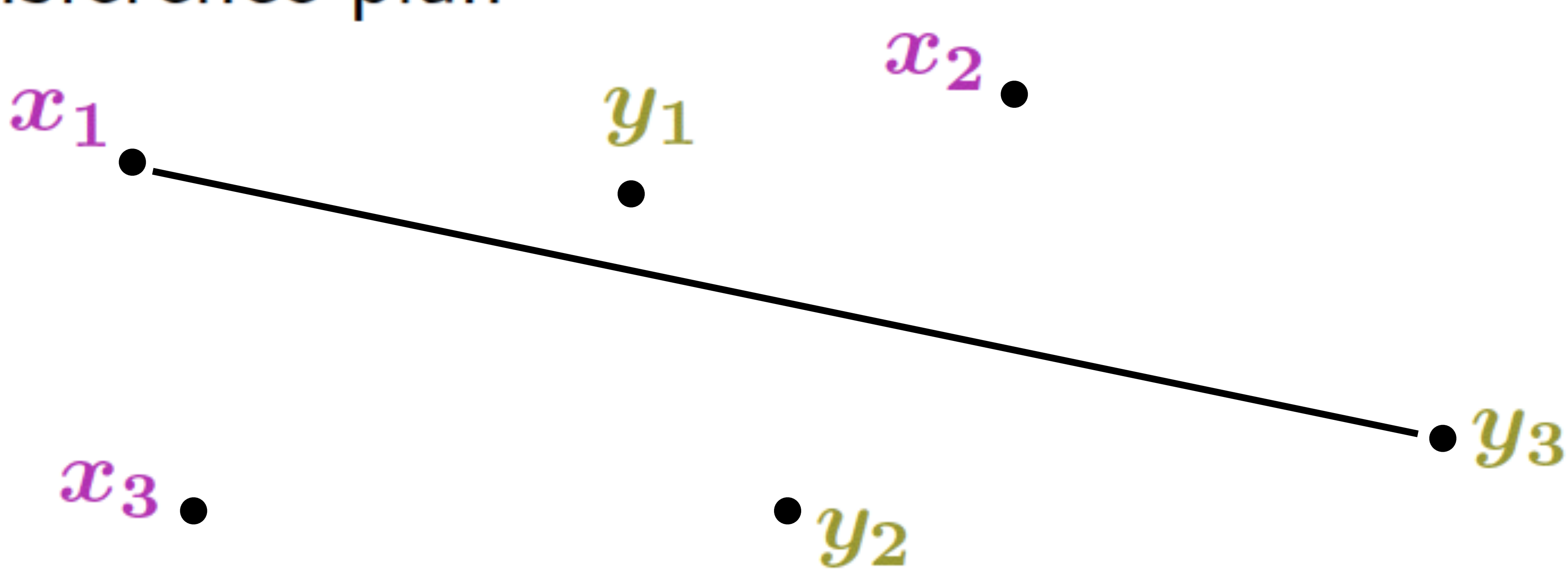

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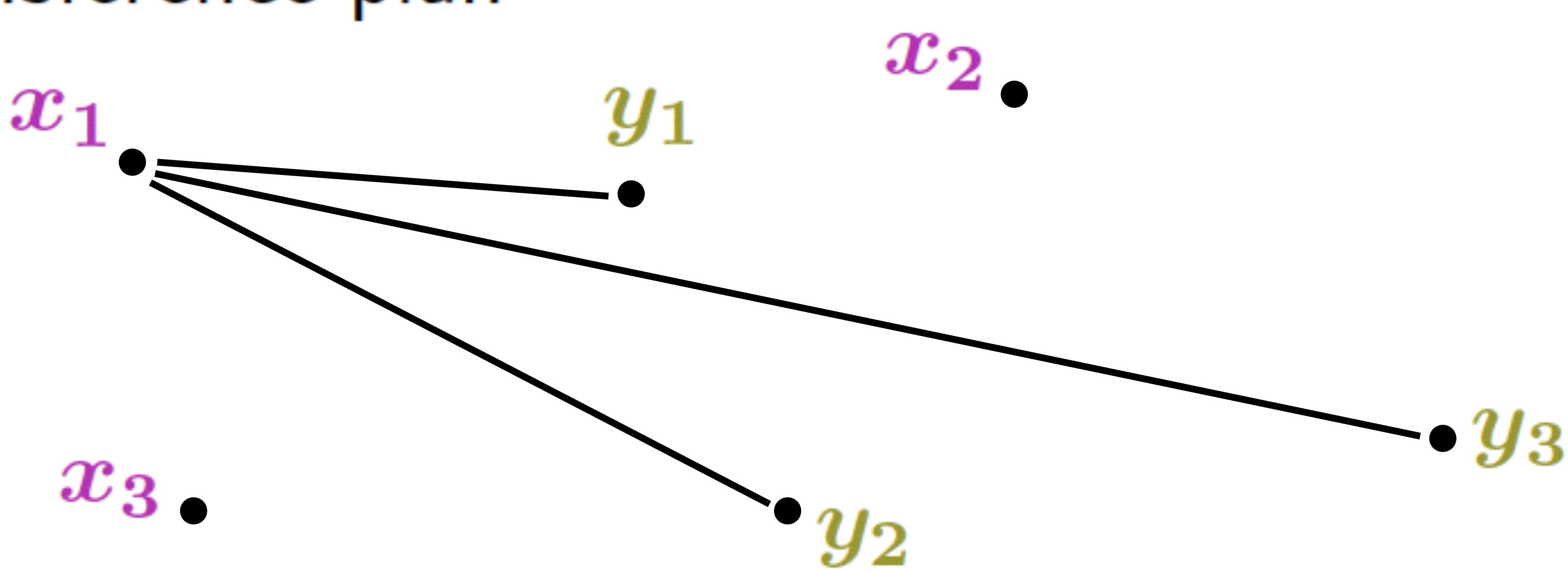

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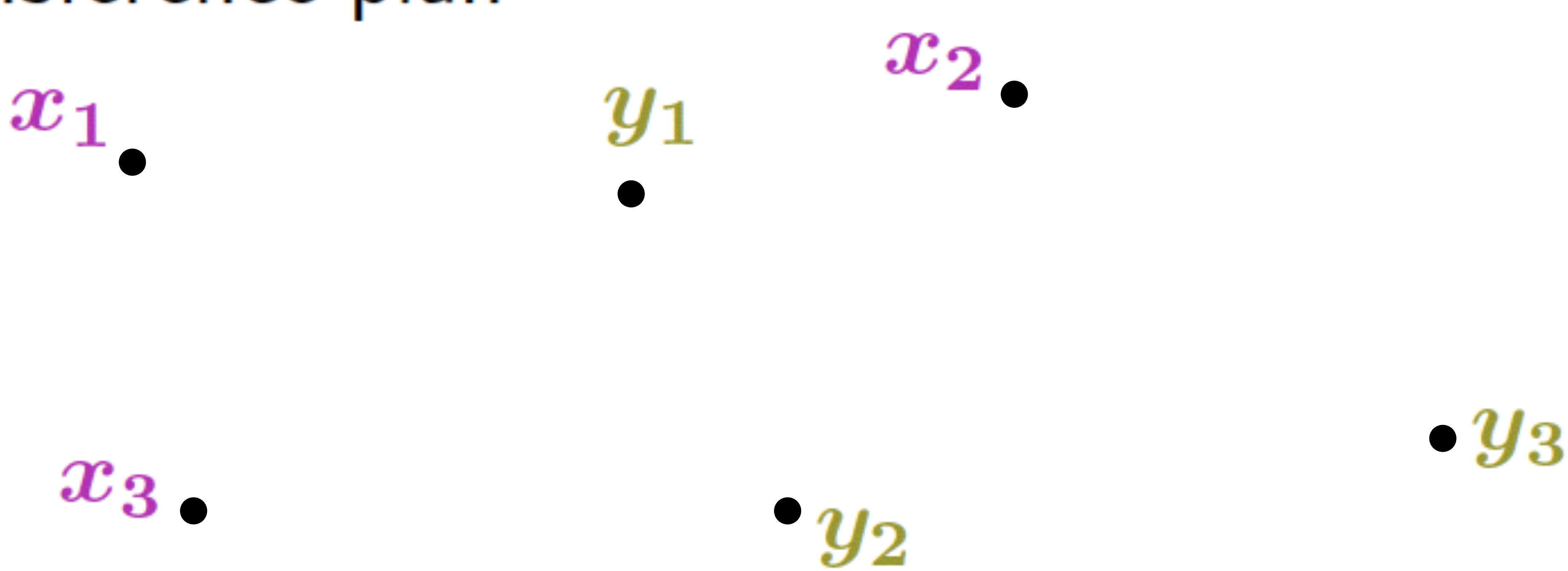

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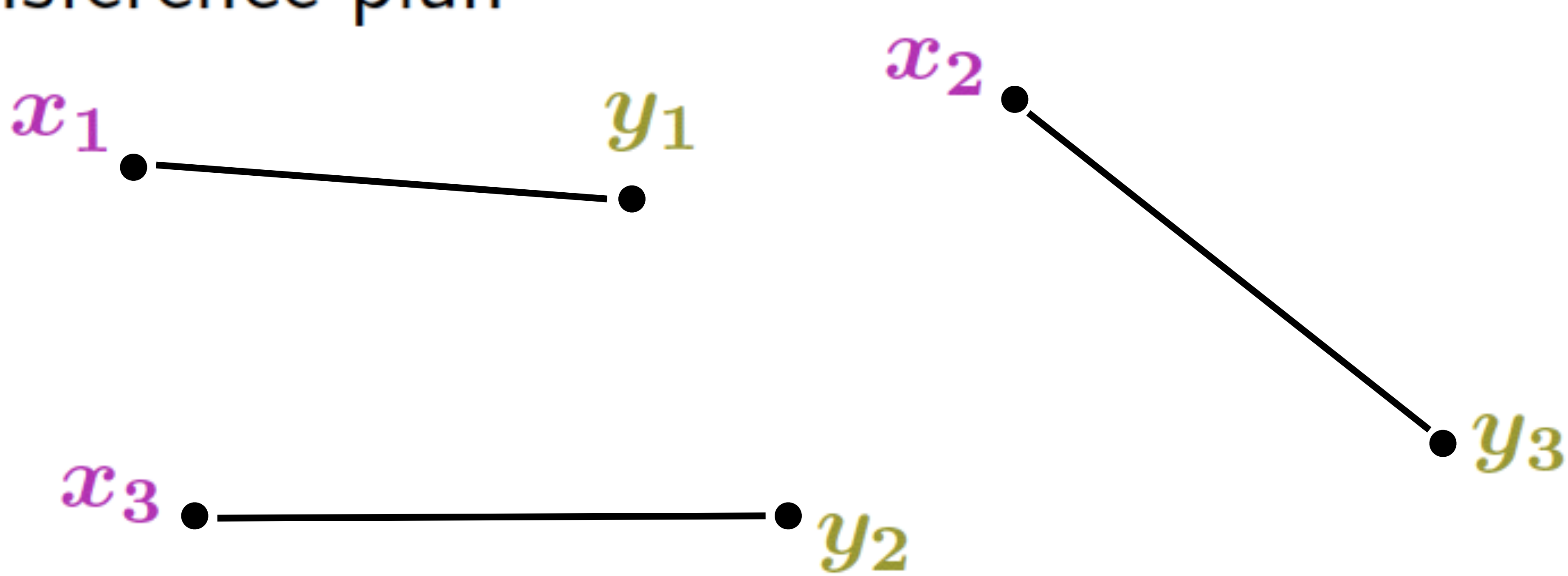

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$$\pi \in \mathcal{P}(X \times X), \begin{cases} \pi(A \times X) = \mu(A), \\ \pi(X \times A) = \nu(A) \end{cases} ?$$

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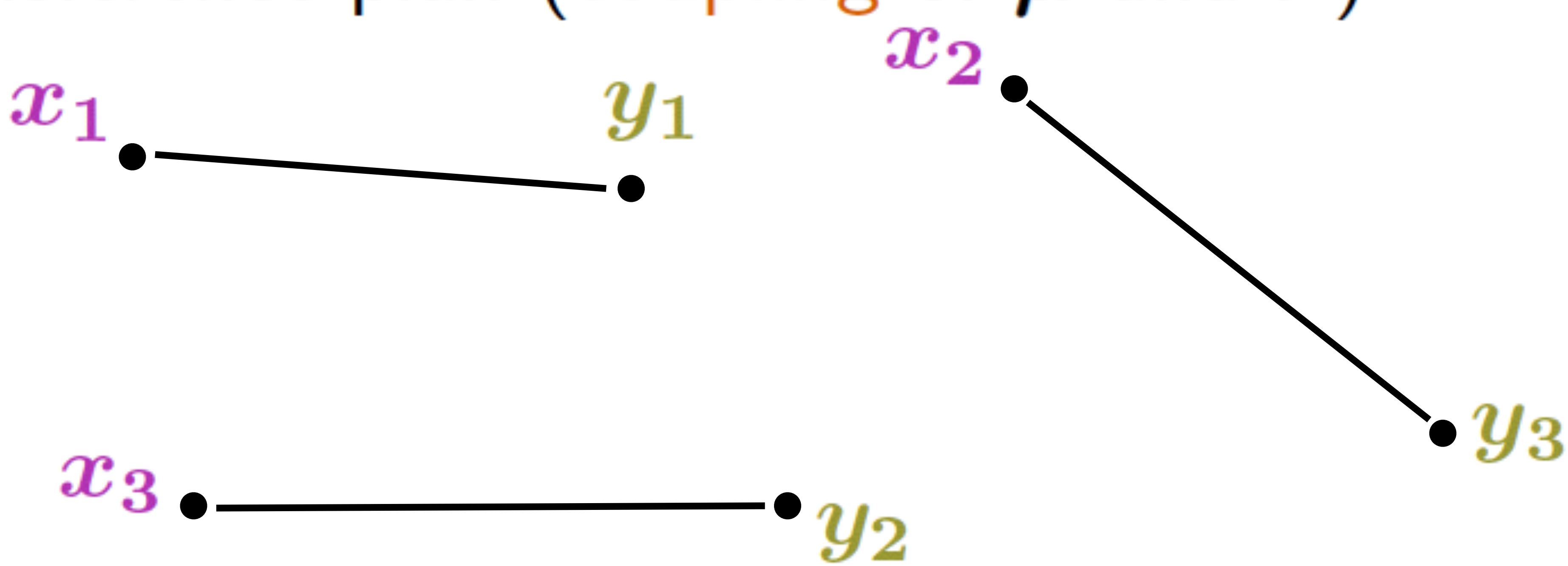

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Given $\mu, \nu \in \mathcal{P}(X)$ & $c : X \times X \to \mathbb{R}$,

$$\mathcal{T}_c(\mu, \nu) := \inf \left\{ \int_{X \times X} c \, d\pi \;\middle|\; \pi\text{: coupling of } \mu \text{ \& } \nu \right\}$$

Basic problems in Opt. trans.

- Characterization of minimizer(s) of $\mathcal{T}_c(\mu, \nu)$
- Properties of $\mathcal{T}_c(\mu, \nu)$ as a function of $\mu$ & $\nu$
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$(X, d)$: metric space

<u>$L^p$-Wasserstein distance</u> $(p \in [1, \infty])$

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★ $W_p(\mu, \nu)^p = \mathcal{T}_{d^p}(\mu, \nu) \qquad (p \in [1, \infty))$

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- $\lim_n W_p(\mu_n, \mu) = 0$

$$\Leftrightarrow \begin{cases} \mu_n \to \mu \text{ (weakly) \&} \\ \int d(x_0, x)^p \mu_n(dx) \to \int d(x_0, x)^p \mu(dx) \end{cases}$$

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$\rightsquigarrow$ Applications to rate of conv. of prob. meas.'s

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- $^{\exists}$A "good" coupling of $\mu$ & $\nu$

  $\Rightarrow$ Upper bound of $W_p(\mu, \nu)$
- $W_p$ is "stable" under perturbations of $(X, d)$
- Geometry of $(\mathcal{P}_p(X), W_p) \leftrightsquigarrow$ Geom. of $(X, d)$

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$$d\text{: Geod.} \overset{\text{def}}{\Leftrightarrow} \begin{cases} {}^{\forall}x_0, x_1 \in X, {}^{\exists}\gamma : [0,1] \to X \text{ s.t.} \\ \gamma_0 = x_0,\ \gamma_1 = x_1, \\ d(\gamma_s, \gamma_t) = |s - t| d(x, y) \end{cases}$$

($\gamma$: minimal geodesic)

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Superposition principle [Lisini '07]

$(\mu_s)_{s\in[0,1]}$: $W_p$-geodesic ($p < \infty$)

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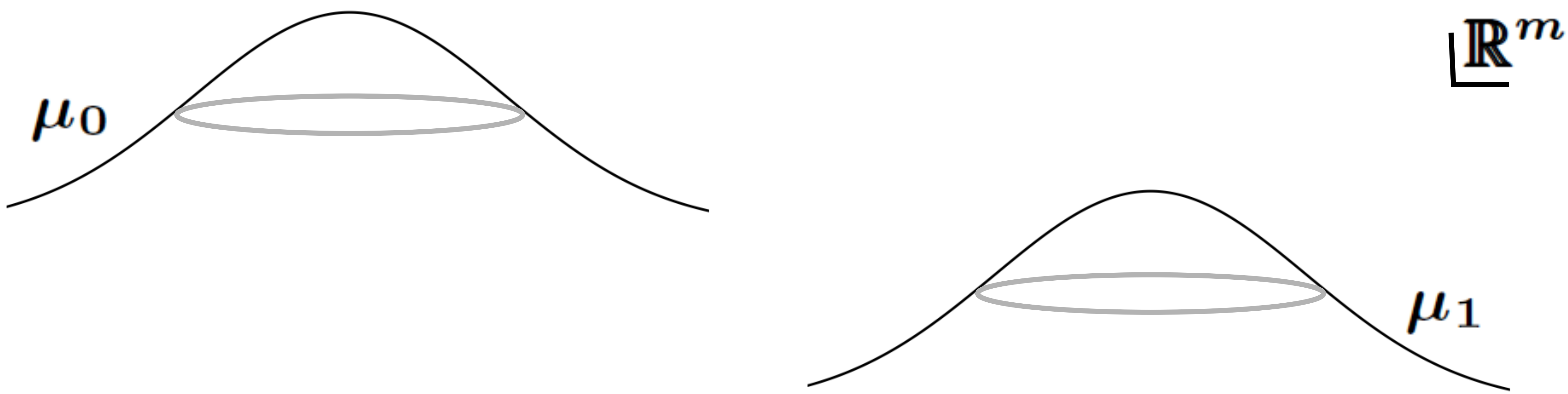

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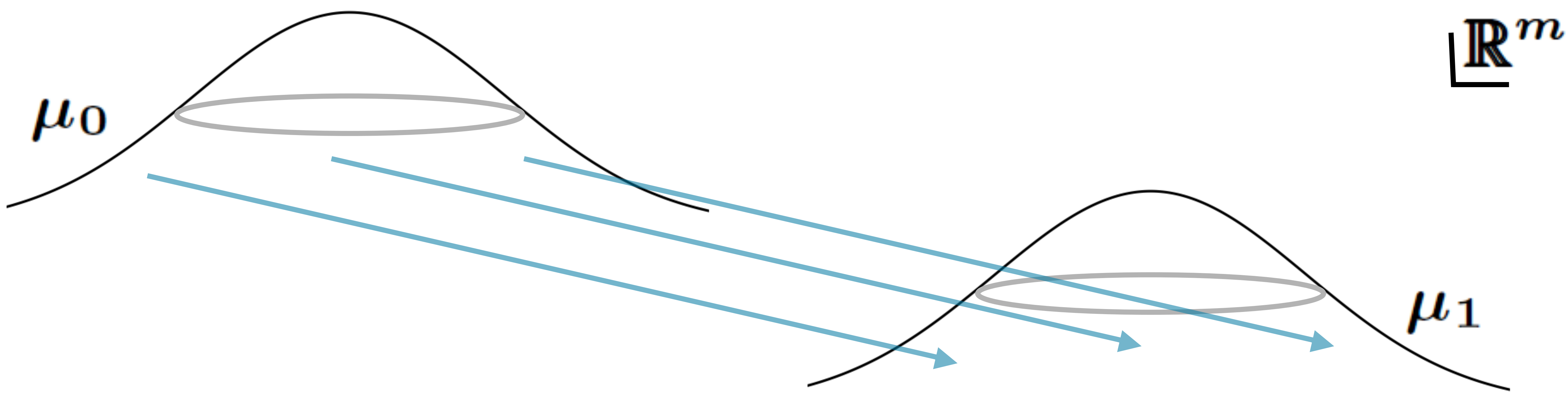

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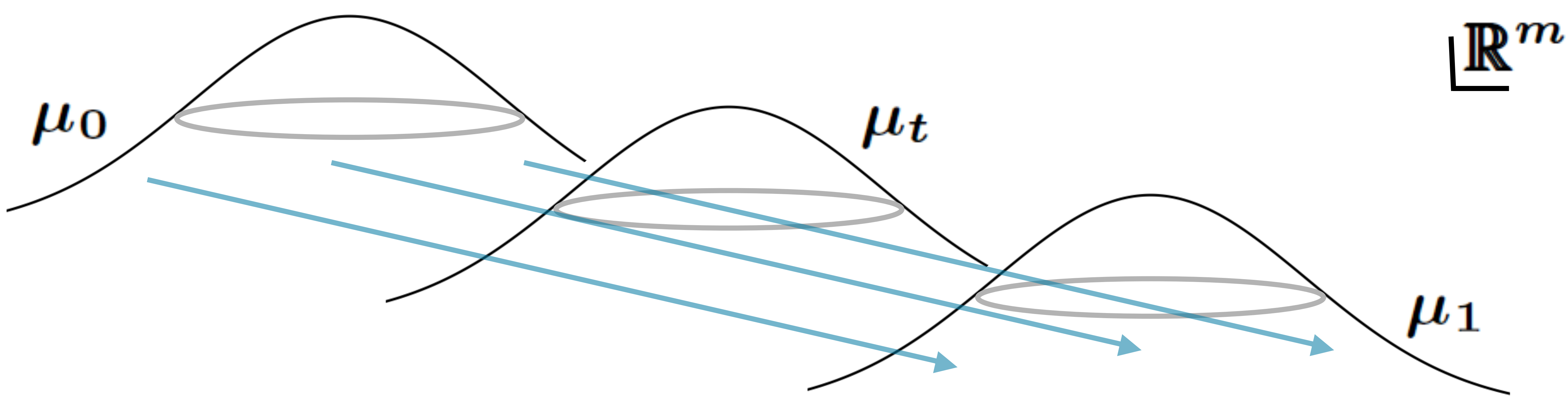

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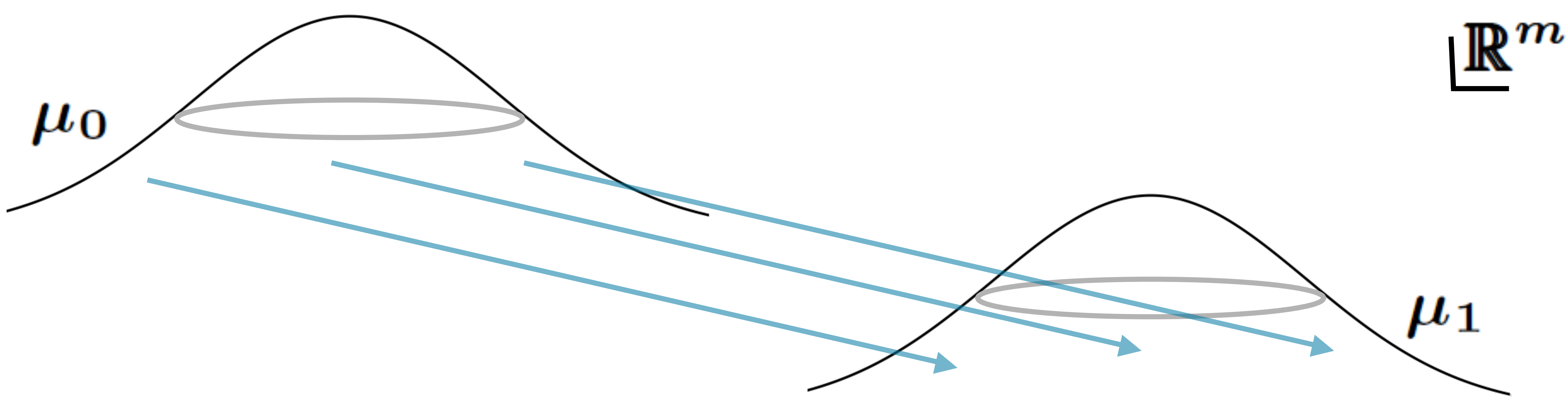

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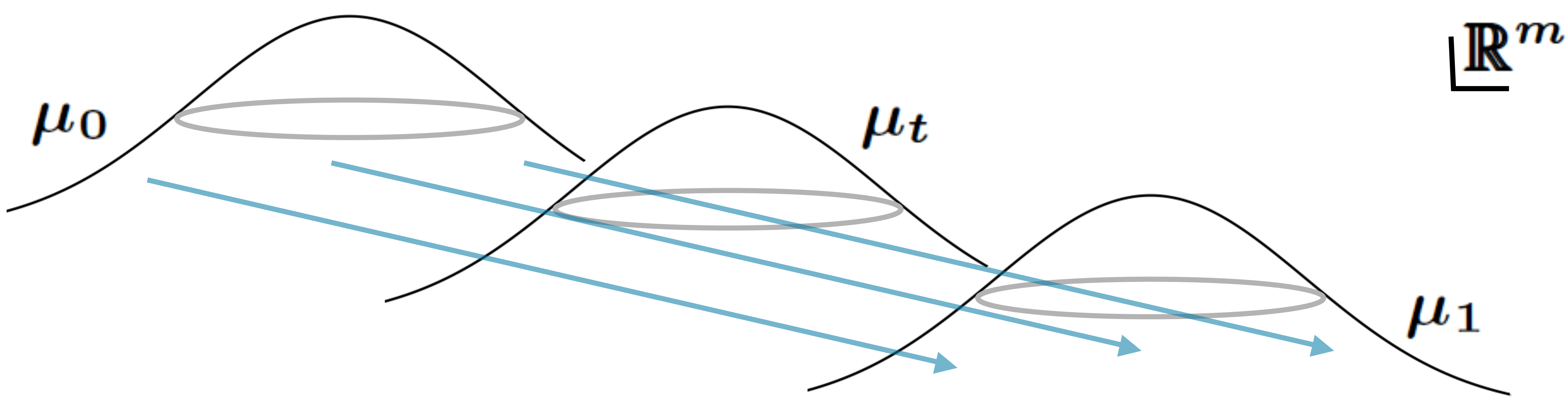

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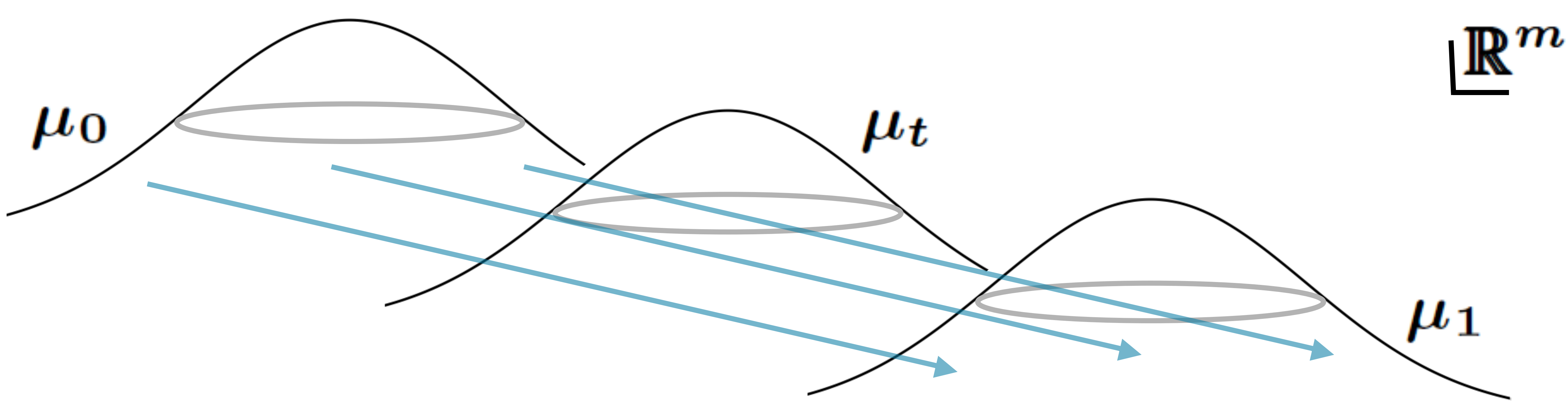

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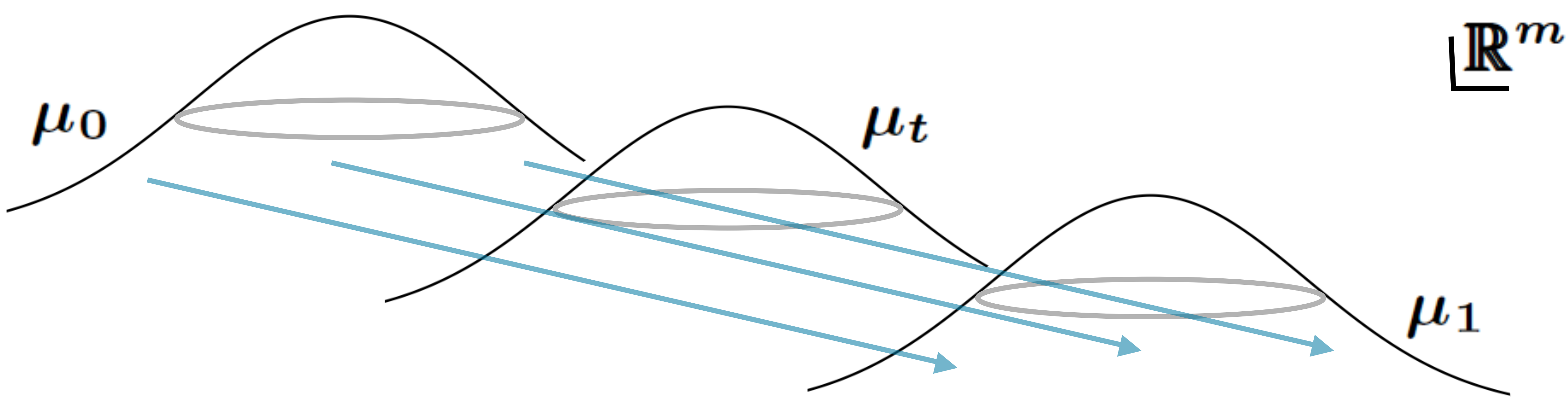

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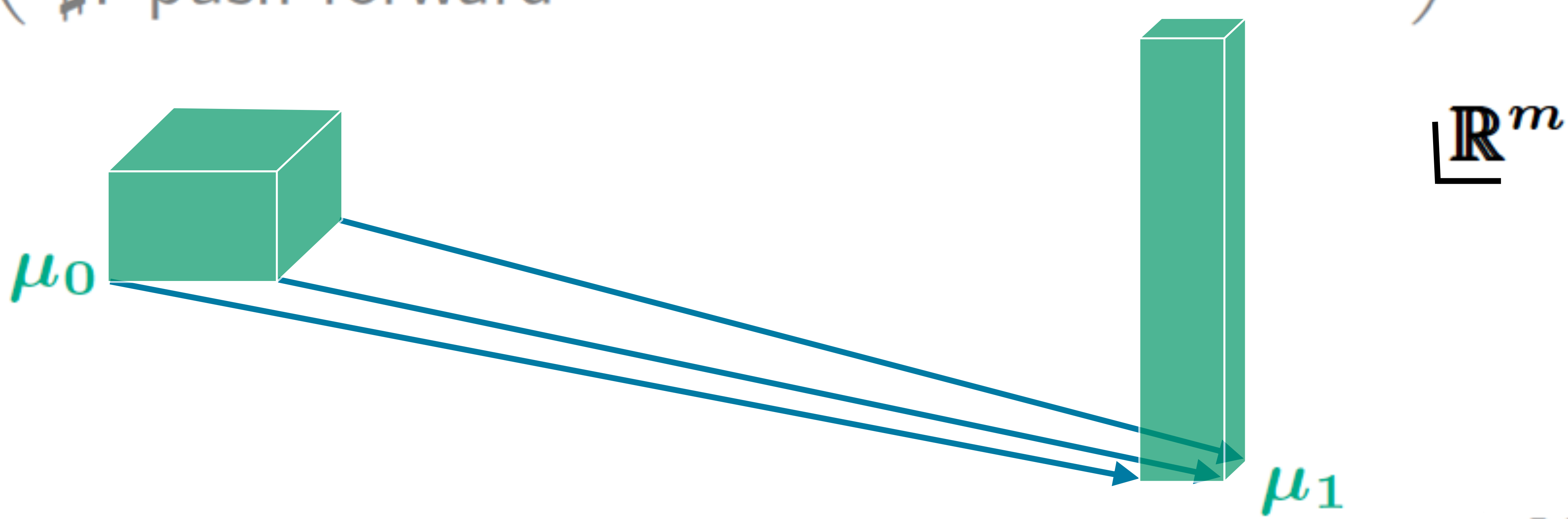

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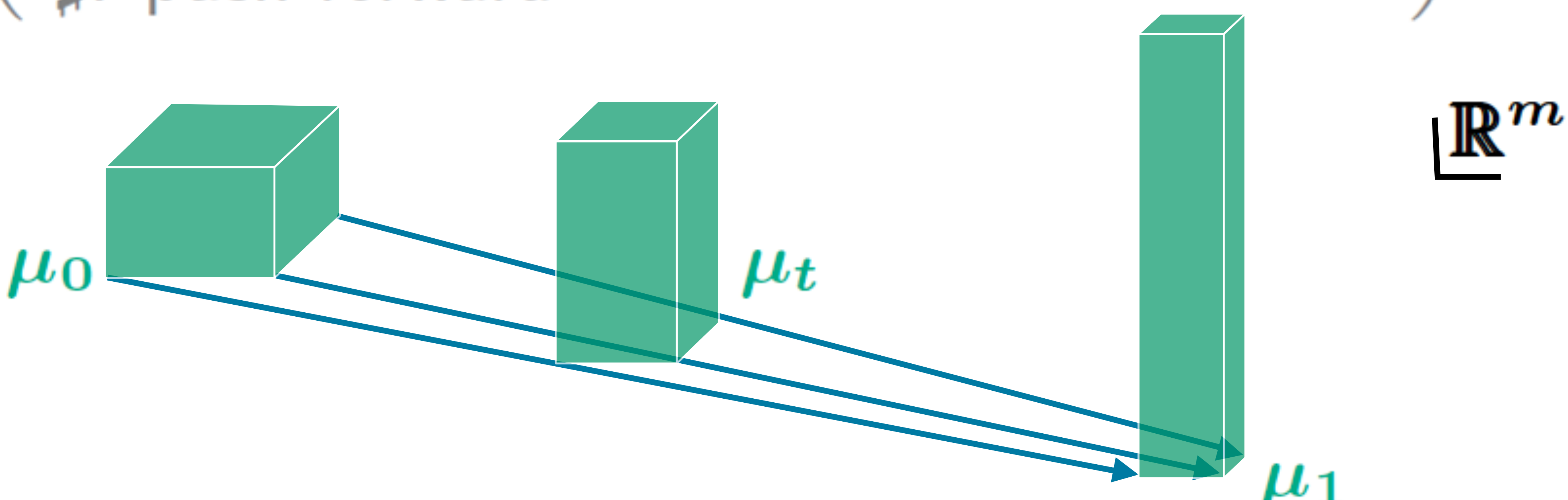
$\sharp$: push-forward )



# Kantorovich duality

$c : X \times X \to [0, \infty]$: lower semi-conti. (for simplicity)

$$\mathcal{T}_c(\mu, \nu) = \sup_{g,f} \left[ \int g \, d\mu + \int f \, d\nu \right] = \sup_{f} \left[ \int \hat{f} \, d\mu + \int f \, d\nu \right],$$

where $f, g \in C_b(M)$,

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⇝ Extension to $L^p/L^q$-duality [K.'10 / K.'13, ...]

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$^{\exists}$A formal Riemannian structure on $\mathcal{P}_2(X)$ canonically associated with $W_2$ [Otto '01 / Otto & Villani '00]

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- Heat flow = a grad. flow of $\mathrm{Ent}_{\mathfrak{m}}$ ($\mathfrak{m}$: ref. meas.)

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \begin{cases} \displaystyle\int \rho \log \rho \, d\mathfrak{m} & (\mu = \rho \mathfrak{m}) \\ \infty & (\text{otherwise}) \end{cases}$$

1. Basics on optimal transport

2. **Lower Ricci curvature bound**

3. Coupling(s) of Brownian motions

# Weighted Riemannian manifold

$(X, g)$: Riem. mfd.,

$d$: Riem. dist., $\mathfrak{m} = \mathrm{e}^{-V} \mathrm{vol}$: weighted volume meas.

$\mathcal{L} := \Delta - \nabla V \cdot \nabla$, $\quad P_t := \mathrm{e}^{t\mathcal{L}}$

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- $V \equiv 0 \Rightarrow \mathrm{Ric}_V = \mathrm{Ric}$: Ricci curvature
- $X = \mathbb{R}^m$, $g$: canonical metric

  $\Rightarrow d$: Eucl. dist., $\mathrm{vol}$: Lebesgue meas.,

  $\mathrm{Ric}_V = \mathrm{Hess}\, V$

# Characterization of $\mathrm{Ric} \geq K$

TFAE for $K \in \mathbb{R}$ ([von Renesse & Sturm '05] etc.)

**(i)** $\mathrm{Ric}_V \geq K$

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(ii) $\operatorname{Hess}\operatorname{Ent}_{\mathfrak{m}} \geq K$ on $(\mathcal{P}_2(X), W_2)$
(iii) $W_2(\mu_0 P_t, \mu_1 P_t) \leq \mathrm{e}^{-Kt} W_2(\mu_0, \mu_1)$

Recall: $\mu_t^{(i)} := \mu^{(i)} P_t$ solves $\dot{\mu}_t^{(i)} = -\nabla \operatorname{Ent}_{\mathfrak{m}}(\mu_t^{(i)})$

$\sigma : [0,1] \to \mathcal{P}_2(X)$ $W_2$-min. geod. from $\mu_t^{(0)}$ to $\mu_t^{(1)}$

$$\frac{1}{2}\frac{d}{dt} W_2(\mu_t^{(0)}, \mu_t^{(1)})^2 = \langle \dot{\mu}_t^{(1)}, \dot{\sigma}_1 \rangle - \langle \dot{\mu}_t^{(0)}, \dot{\sigma}_0 \rangle$$
$$= -\langle \nabla \operatorname{Ent}_{\mathfrak{m}}(\sigma_1), \dot{\sigma}_1 \rangle + \langle \nabla \operatorname{Ent}_{\mathfrak{m}}(\sigma_0), \dot{\sigma}_0 \rangle$$
$$= -\int_0^1 \operatorname{Hess}\operatorname{Ent}_{\mathfrak{m}}(\dot{\sigma}_r, \dot{\sigma}_r)\, dr$$
$$\leq -K W_2(\mu_t^{(0)}, \mu_t^{(1)})^2$$

# "(ii) $\Rightarrow$ (iii)" via Otto calc.

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**(ii)** $\leq -K W_2(\mu_t^{(0)}, \mu_t^{(1)})^2 \quad \Rightarrow$ **(iii)**

# Applications on $\mathsf{RCD}^*(K,N)$ sp's

Basics on analysis ($N = \infty$ is sufficient)

- $L^\infty$-Lipschitz regularization of $P_t$ ($\Rightarrow$ str. Feller)
- $L^p$-Sobolev spaces, theory of BV functions
- Approaches from theory of Dirichlet form
- $^\exists$ "Brownian motion" on $X$

Potential theoretic properties

- Two-sided Gaussian heat kernel estimates
- Li-Yau inequality
- Cheng's gradient estimate
- Regularity of harmonic fn.'s
- Harm. fn.'s of polynomial growth

# Applications on $\mathrm{RCD}^*(K, N)$ sp's

## Functional inequalities

- ($N$-)HWI ineq.
- (log-)Sobolev ineq., Poincaré ineq.
- Talagrand ineq., Gaussian concentration ineq.
- F.-Y. Wang's (log-)Harnack ineq.

## Differential geometric properties

- Bishop-Gromov ineq., Brunn-Minkowski ineq.
- Cheeger-Gromoll isometric splitting thm
- Bonnet-Myers thm & maximal diameter thm
- Lichnerowicz-Obata thm (sharp & rigid spec. gap)
- Isoperimetric inequalities

1. Basics on optimal transport

2. Lower Ricci curvature bound

**3. Coupling(s) of Brownian motions**

# Coupling by para. trans. on RCD sp.'s

$$W_2(\mu P_t, \nu P_t) \le \mathrm{e}^{-Kt} W_2(\mu, \nu)$$

$$\Downarrow$$

$$|\nabla P_t f| \le \mathrm{e}^{-Kt} P_t(|\nabla f|^2)^{1/2}$$

$$\Downarrow$$

$$\frac{1}{2}\mathcal{L}|\nabla f|^2 - \langle \nabla f, \nabla \mathcal{L} f \rangle \ge K|\nabla f|^2$$

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$$\Downarrow$$

$$\frac{1}{2}\mathcal{L}|\nabla f|^2 - \langle \nabla f, \nabla \mathcal{L} f \rangle \ge K |\nabla f|^2 + \frac{|\nabla |\nabla f|^2|^2}{4|\nabla f|^2}$$

(Bakry–Émery's self-improvement property [Savaré '14])

# Coupling by para. trans. on RCD sp.'s

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$$|\nabla P_t f| \leq \mathrm{e}^{-Kt} P_t(|\nabla f|)$$

$$\Downarrow$$

$$W_\infty(\mu P_t, \nu P_t) \leq \mathrm{e}^{-Kt} W_\infty(\mu, \nu)$$

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$$W_\infty(\mu P_t, \nu P_t) \leq \mathrm{e}^{-Kt} W_\infty(\mu, \nu)$$

$$\Downarrow$$

${}^\forall \pi \in \mathcal{P}(X \times X)$, ${}^\exists (B_t^{(0)}, B_t^{(1)})_{t \geq 0}$: a cplg of BM's,

s.t. $\begin{cases} (B_0^{(0)}, B_0^{(1)}) \overset{d}{=} \pi, \\ d(B_t^{(0)}, B_t^{(1)}) \leq \mathrm{e}^{-K(t-s)} d(B_s^{(0)}, B_s^{(1)}) \end{cases}$

[Sturm '14]

# Coupling by reflection on Riem. mfd

[Kendall '86 / Cranston '91 / ...] $\mathrm{Ric}_V \geq 0$

$\Rightarrow {}^\forall x_0, x_1 \in X$, ${}^\exists(B_t^{(0)}, B_t^{(1)})$: coupling of BM's starting at $(x_0, x_1)$ & a 1-dim (std.) BM $W_t$ s.t.

$$\boxed{d(B_t^{(0)}, B_t^{(1)}) \leq d(x_0, x_1) + 2\sqrt{2}W_t}$$

if $t < \tau := \inf\{s \geq 0 \mid B_{s'}^{(0)} = B_{s'}^{(1)}$ for $s' \geq s\}$

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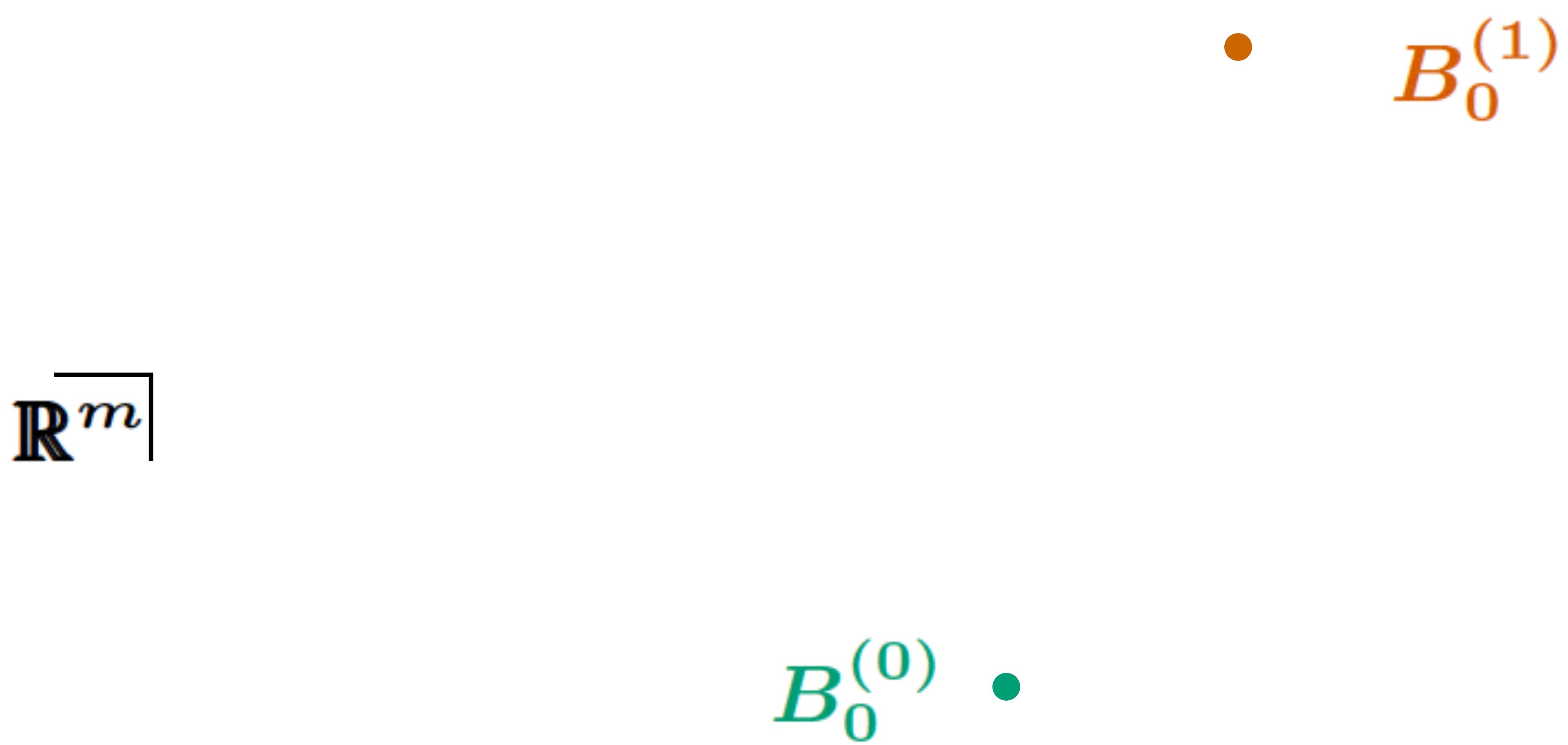

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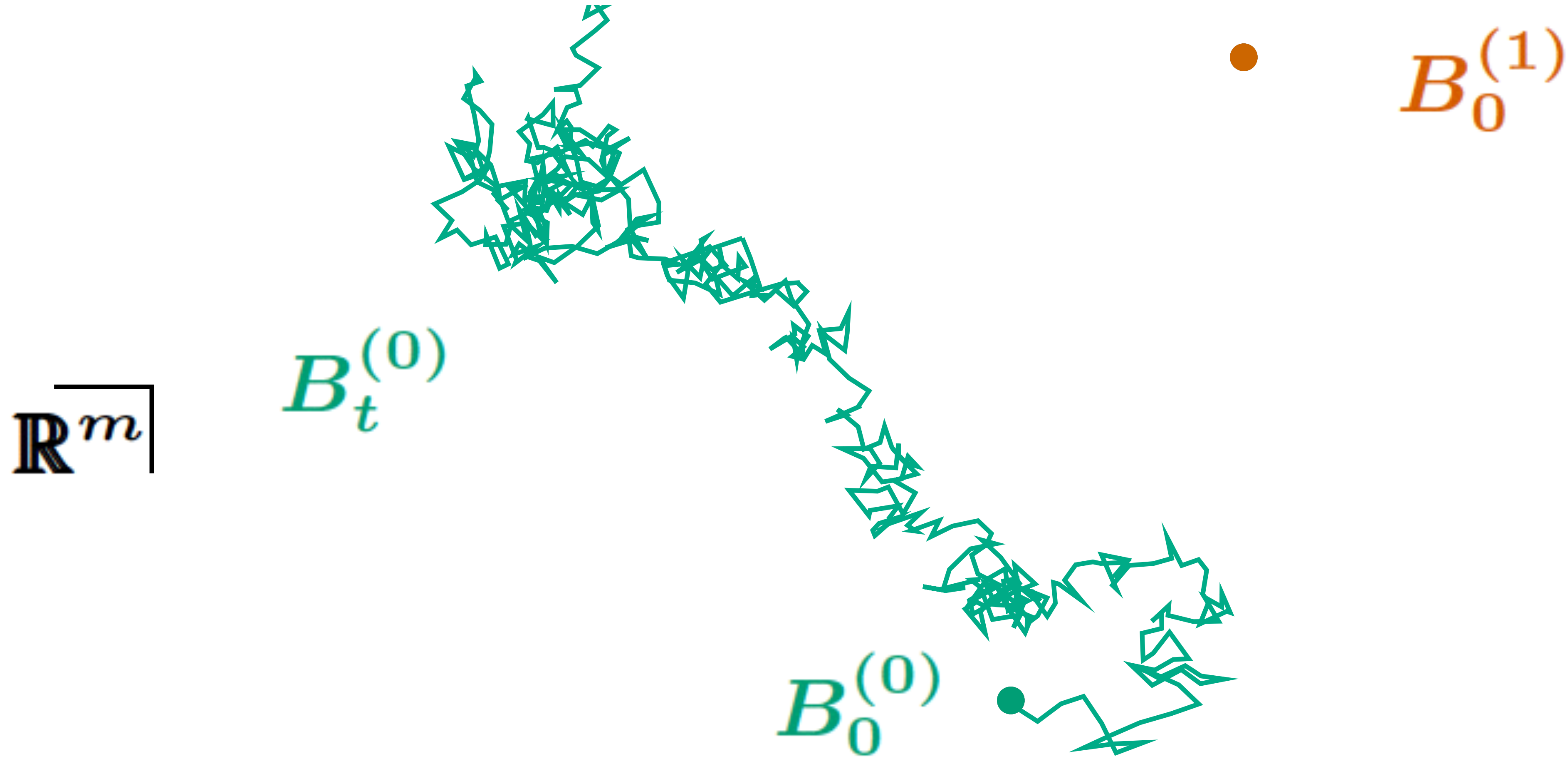

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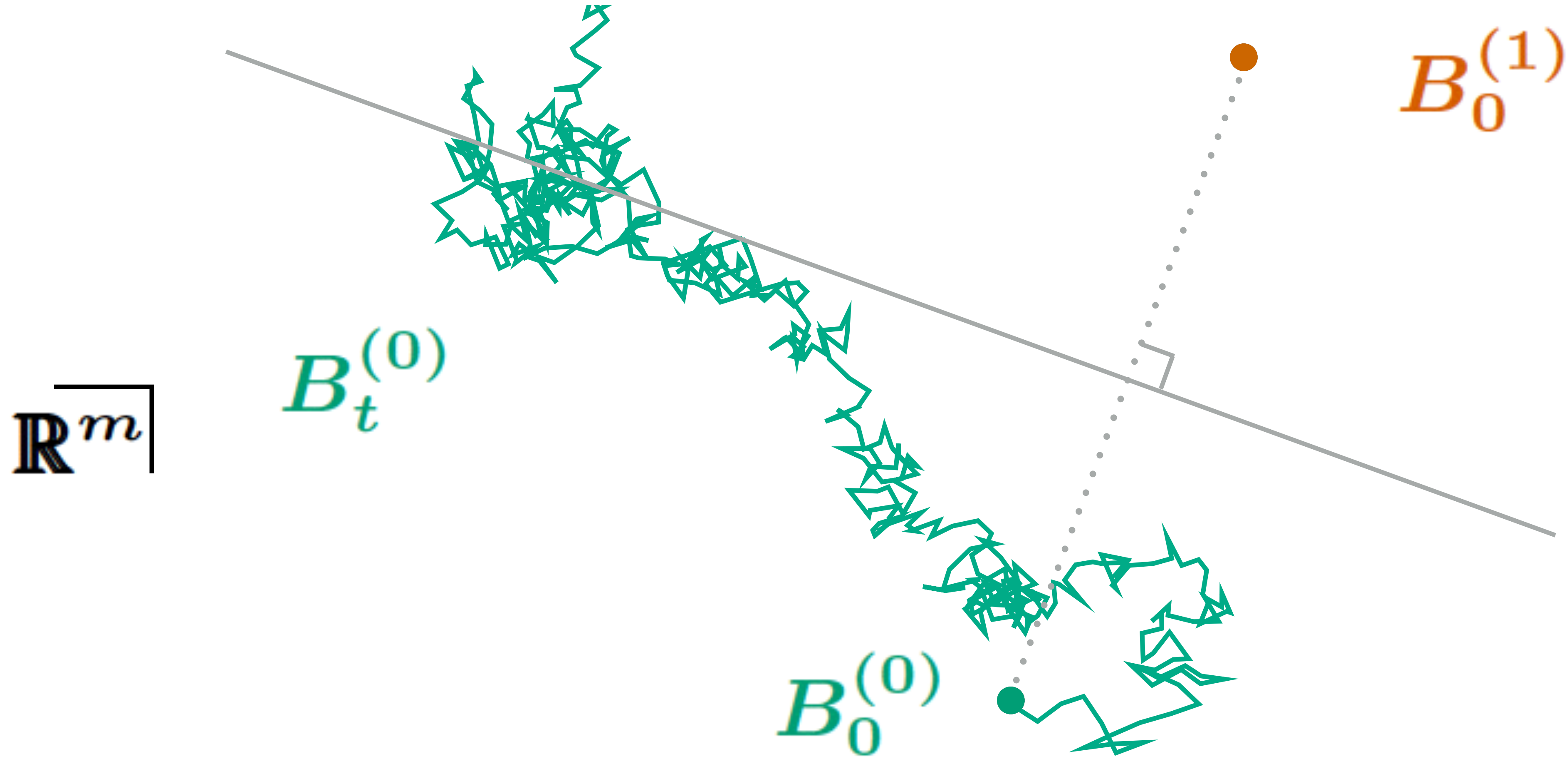

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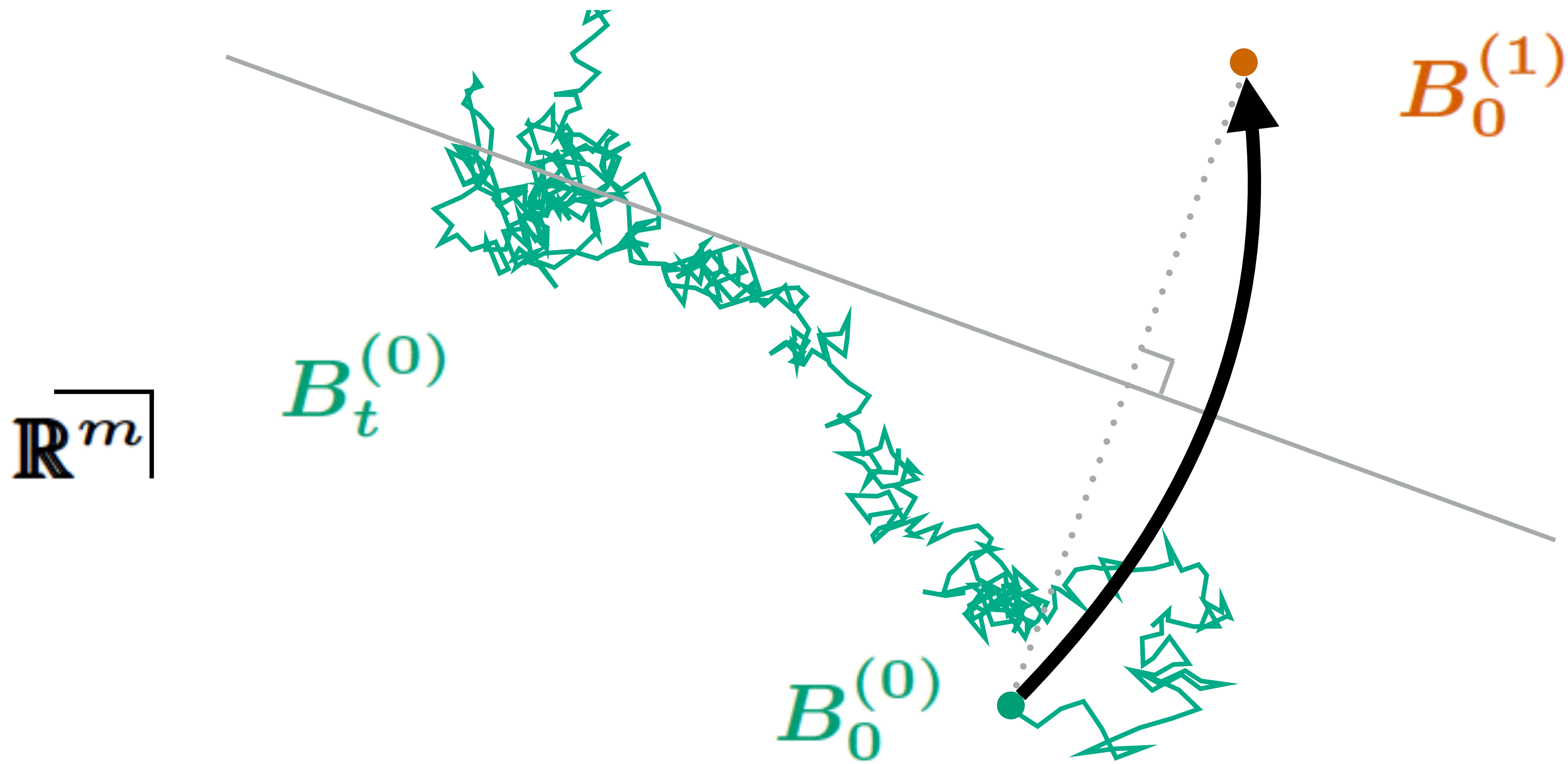

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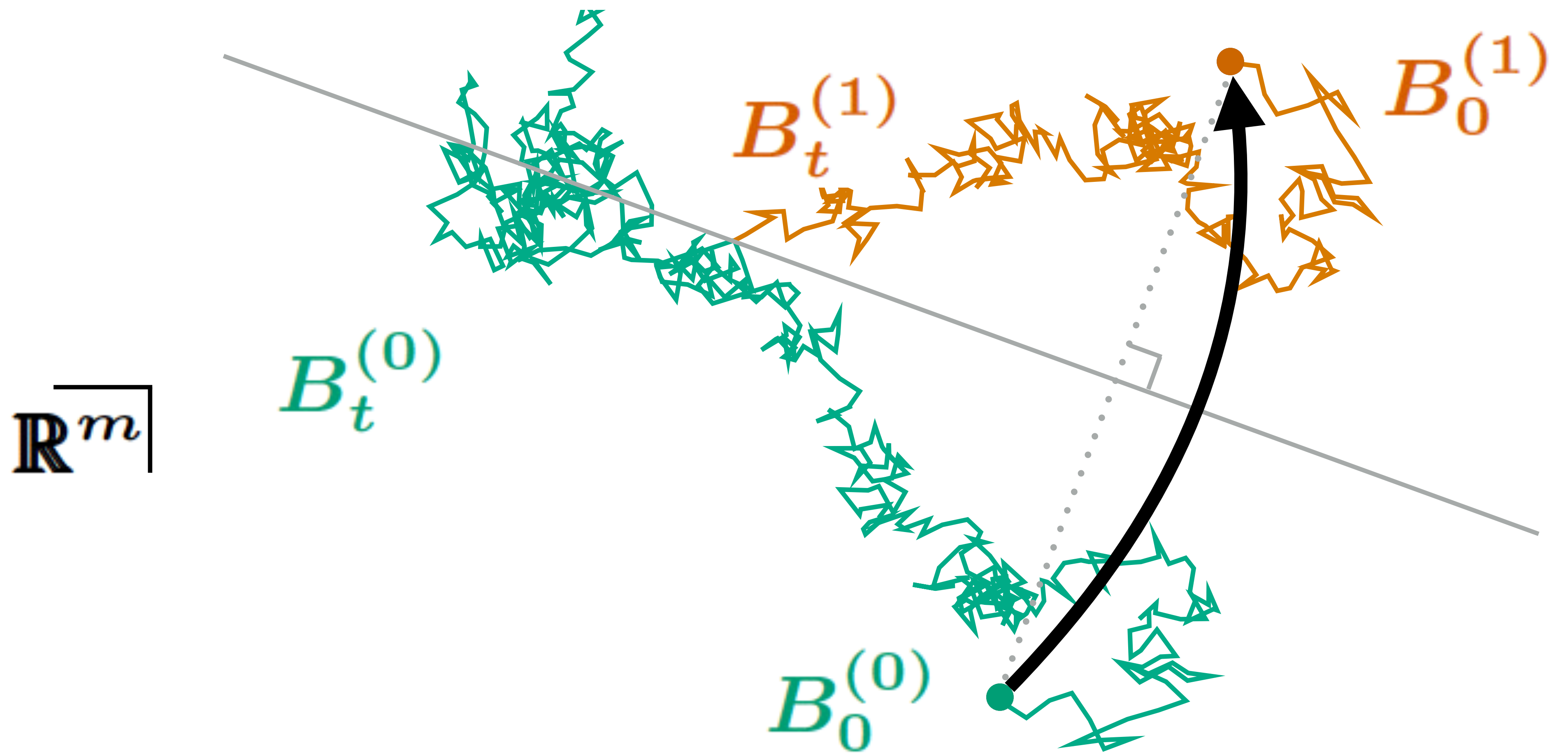

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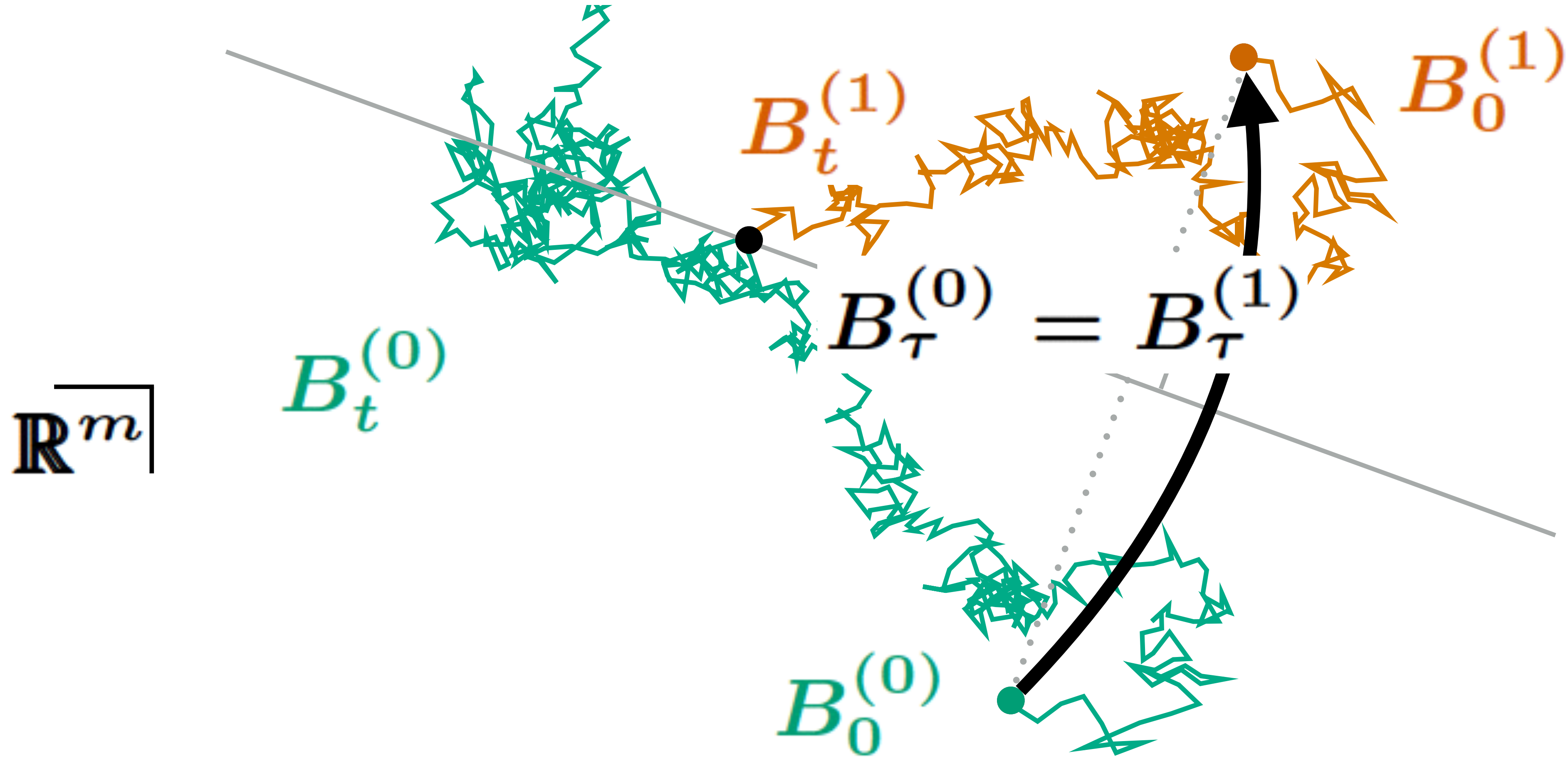

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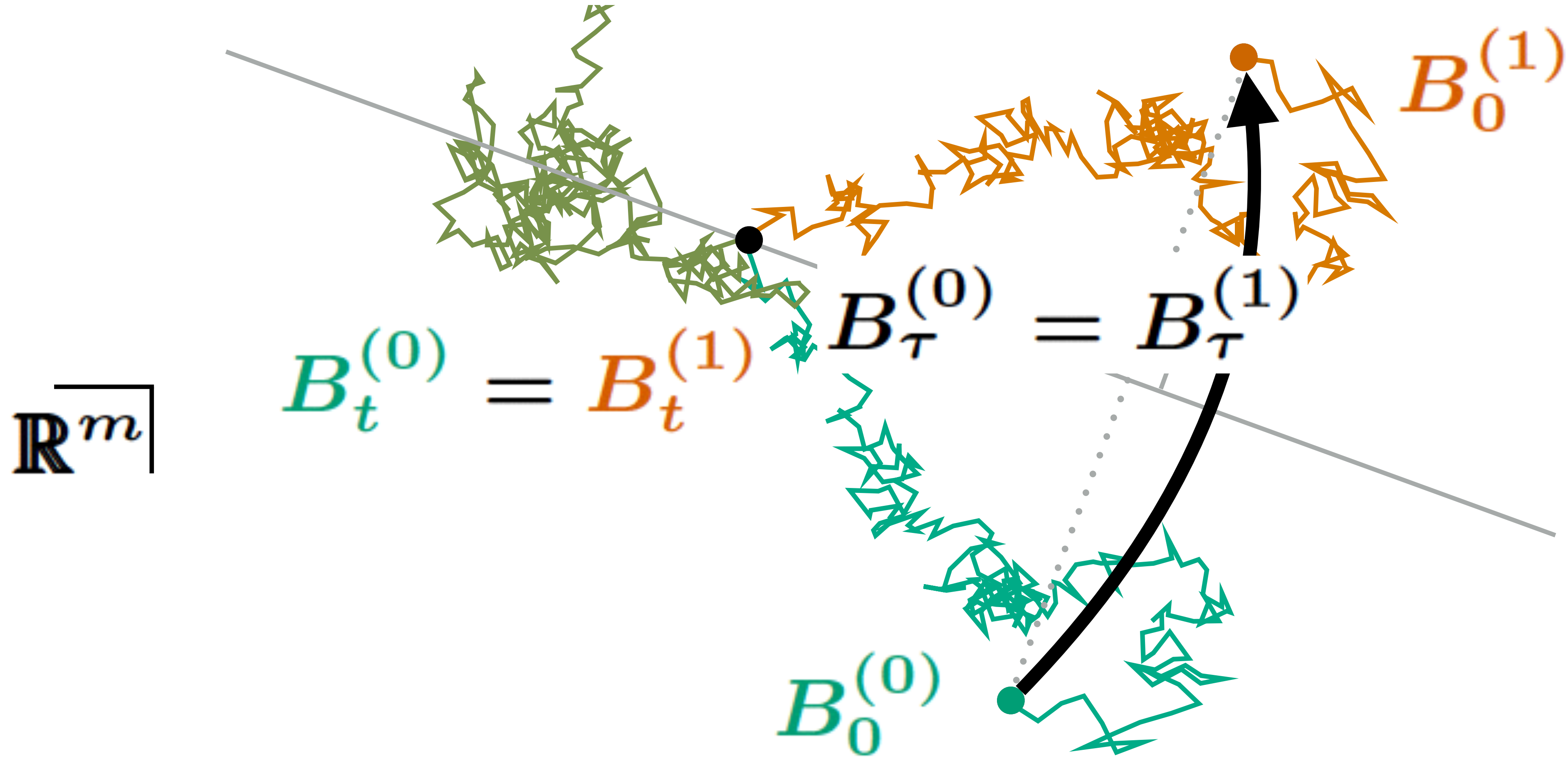

# Coupling by refl. and opt. trans.

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$$\varphi_t(r) := \mathbb{E}\left[\inf_{s \leq t} \left(r + 2\sqrt{2} W_s\right) > 0\right]$$

# Coupling by refl. and opt. trans.

$$d(B_t^{(0)}, B_t^{(1)}) \leq d(x_0, x_1) + 2\sqrt{2} W_t$$

$$(\Rightarrow \mathbb{P}[\tau > t] \leq \varphi_t(d(x_0, x_1)))$$

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$\Rightarrow {}^\forall s, t > 0$,

$$\mathbb{E}[\varphi_s(d(B_t^{(0)}, B_t^{(1)}))] \le \varphi_{s+t}(d(x_0, x_1)),$$

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$$\Downarrow$$

${}^\forall T > 0$, $\mathcal{T}_{\varphi_{T-t}(d)}(\delta_{x_0} P_t, \delta_{x_1} P_t) \searrow$ [K. & Sturm '13]

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$$\left(\Rightarrow \frac{1}{2} \|\delta_{x_0} P_t - \delta_{x_1} P_t\|_{\mathrm{var}} \le \varphi_t(d(x_0, x_1))\right)$$

# F'nal ineq. $\Rightarrow$ coupling by refl.

**Theorem 1 ([K.])**

*On* $\mathsf{RCD}^*(0,\infty)$ *sp's,* $\mathcal{T}_{\varphi_{T-t}(d)}(\mu P_t, \nu P_t) \searrow$ *in* $t$

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**Theorem 2 ([K.])**

*On* **RCD**$^*(0,\infty)$ *sp's,* $^\forall x_0, x_1 \in X$,

$^\exists(B_t^{(0)}, B_t^{(1)})_{t\ge 0}$: *a coupling of BMs s.t.*

- $(B_0^{(0)}, B_0^{(1)}) = (x_0, x_1)$,
- $^\forall t > 0$, $\mathbb{P}[\tau > t] \le \varphi_t(d(x_0, x_1))$

*In particular,* $\mathbb{P}[\tau = \infty] = 0$

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*In particular,* $\mathbb{P}[\tau = \infty] = 0$

★ Extension to $K \neq 0$: OK.

# Idea of the pf. of Thm 1

- Kantorovich duality
- Reverse Gaussian f'nal isoperimetry for $\boldsymbol{P_t}$

$$\frac{\mathrm{e}^{2Kt}-1}{K}|\nabla P_t f|^2 \le I(P_t f)^2 - P_t(I(f))^2,$$

$$I := \Phi' \circ \Phi^{-1}, \ \ \Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/2} dy$$

- $\boldsymbol{W_\infty(\mu P_t, \nu P_t) \le \mathrm{e}^{-Kt} W_\infty(\mu, \nu)}$

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$$\Rightarrow \;\; \mathcal{T}_{\varphi_0(d)}(\delta_{x_0}P_t, \delta_{x_1}P_t) \le \varphi_t(d(x_0, x_1))$$

- $W_\infty(\mu P_t, \nu P_t) \le e^{-Kt} W_\infty(\mu, \nu)$

$$\Rightarrow \;\; \boxed{\mathcal{T}_{\varphi_{T-t}(d)}(\delta_{x_0}P_t, \delta_{x_1}P_t) \le \varphi_T(d(x_0, x_1))}$$