An Introduction to Stochastic Control, with Applications to Mathematical Finance

Bernt Øksendal
Department of Mathematics, University of Oslo, Norway
and
Norwegian School of Economics (NHH),Bergen, Norway

Stochastic Processes and Applications,
Ulan Bator, Mongolia, 29-31 July 2015

These lectures are partially based on joint works with
Agnès Sulem, INRIA, Paris, France

On the wall of the entrance hall of the main building of the Humboldt University, Berlin (freely translated):

"For centuries scientists and philosophers have tried to understand the world. Now is the time to control it!"

Karl Marx

Abstract

We give a short introduction to the stochastic calculus for Itô-Lévy processes and review briefly the two main methods of optimal control of systems described by such processes:

(i) Dynamic programming and the Hamilton-Jacobi-Bellman (HJB) equation

(ii) The stochastic maximum principle and its associated backward stochastic differential equation (BSDE).

The two methods are illustrated by application to the classical portfolio optimization problem in finance. A second application is the problem of risk minimization in a financial market. Using a dual representation of risk, we arrive at a stochastic differential game, which is solved by using the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation, which is an extension of the HJB equation to stochastic differential games.

Introduction

The purpose of the course is to give a quick introduction to stochastic control of jump diffusions, with applications to mathematical finance, with emphasis on portfolio optimization and risk minimization. The content of these lectures is the following:

In Section 2 we review some basic concepts and results from the stochastic calculus of Itô-Lévy processes.

In Section 3 we present a *portfolio optimization* problem in an Itô-Lévy type financial market. We recognize this as a special case of a stochastic control problem and we present the first general method for solving such problems: *Dynamic programming* and the *HJB* equation. We show that if the system is Markovian, we can use this method to solve the problem.

In Section 4 we study a *risk minimization* problem in the same market. By a general representation of convex risk measures, this problem may be regarded as a *stochastic differential game*, which also can be solved by dynamic programming (HJBI equation) if the system is Markovian.

Finally, in Section 5 we study the portfolio optimization problem by means of the second main stochastic control method: *The maximum principle*. The advantage with this method is that it also applies to non-Markovian systems.

Stochastic calculus for Itô-Lévy processes

In this section we give a brief survey of stochastic calculus for Itô-Lévy processes. For more details we refer to Chapter 1 in [5]. We begin with a definition of a Lévy process:

Definition

A Lévy process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a process, $\eta(t) \equiv \eta(t, \omega)$ with the following properties

(i) $\eta(0) = 0$.

(ii) η has stationary, independent increments.

(iii) η is stochastically continuous.

The jump of η at time t is $\Delta\eta(t) = \eta(t) - \eta(t-)$.

Remark. One can prove that η always has a càdlàg (i.e. right continuous with left sided limits) version. We will use this version from now on.

The *jump measure* $N([0,t],U)$ gives the number of jumps of η up to time t with jump size in the set $U \subset \mathbb{R}_0 :\equiv \mathbb{R} \setminus \{0\}$. If we assume that $\overline{U} \subset \mathbb{R}_0$, then it can be shown that U contains only finitely many jumps in any finite time interval.

The *Lévy measure* $\nu(\cdot)$ of η is defined by

$$\nu(U) = \mathbb{E}[N([0,1],U)]. \tag{2.1}$$

and $N(\mathrm{d}t, \mathrm{d}\zeta)$ is the differential notation of the random measure $N([0,t],U)$. Intuitively, ζ can be regarded as generic jump size.

Let $\tilde{N}(\cdot)$ denote the *compensated jump measure* of η, defined by

$$\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta) \equiv N(\mathrm{d}t, \mathrm{d}\zeta) - \nu(\mathrm{d}\zeta)\mathrm{d}t. \tag{2.2}$$

For convenience we shall from now on impose the following additional integrability condition on $\nu(\cdot)$:

$$\int_{\mathbb{R}} \zeta^2 \nu(\mathrm{d}\zeta) < \infty, \tag{2.3}$$

which is equivalent to the assumption that for all $t \geq 0$

$$\mathbb{E}[\eta^2(t)] < \infty. \tag{2.4}$$

This condition still allows for many interesting kinds of Lévy processes. In particular, it allows for the possibility that a Lévy process has the following property:

$$\int_{\mathbb{R}} (1 \wedge |\zeta|) \nu(\mathrm{d}\zeta) = \infty. \tag{2.5}$$

This implies that there are infinitely many small jumps.

Under the assumption (2.3) above the *Itô-Lévy decomposition theorem* states that any Lévy process has the form

$$\eta(t) = at + bB(t) + \int_0^t \int_{\mathbb{R}} \zeta \tilde{N}(\mathrm{d}s, \mathrm{d}\zeta), \tag{2.6}$$

where $B(t)$ is a Brownian motion, and a, b are constants.

More generally, we study the *Itô-Lévy processes*, which are the processes of the form

$$(2.7) \qquad X(t) = x + \int_0^t \alpha(s,\omega)\mathrm{d}s + \int_0^t \beta(s,\omega)\mathrm{d}B(s) + \int_0^t \int_{\mathbb{R}} \gamma(s,\zeta,\omega)\tilde{N}(\mathrm{d}s,\mathrm{d}\zeta),$$

where $\int_0^t |\alpha(s)|\mathrm{d}s + \int_0^t \beta^2(s)\mathrm{d}s + \int_0^t \int_{\mathbb{R}} \gamma^2(s,\zeta)\nu(\mathrm{d}\zeta)\mathrm{d}s < \infty$ a.s., and $\alpha(t)$, $\beta(t)$, and $\gamma(t,\zeta)$ are predictable processes (predictable w.r.t. the filtration $\mathcal{F}_t$ generated by $\eta(s)$, for $s \leq t$).

In differential form we have

$$(2.8)\, \mathrm{d}X(t) = \alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t) + \int_{\mathbb{R}} \gamma(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta).$$

We now proceed to the *Itô formula* for Itô-Lévy processes: Let $X(t)$ be an Itô-Lévy process defined as above. Let $f : [0, T] \times \mathbb{R}$ be a $\mathcal{C}^{1,2}$ function and put $Y(t) = f(t, X(t))$.
Then $Y(t)$ is also an Itô-Lévy process, with representation:

$$
\begin{aligned}
&\mathrm{d}Y(t)\\
&= \frac{\partial f}{\partial t}(t, X(t))\mathrm{d}t + \frac{\partial f}{\partial x}(t, X(t))(\alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t))\\
&+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X(t))\beta^2(t)\mathrm{d}t\\
&+ \int_{\mathbb{R}} \{f(t, X(t^-) + \gamma(t,\zeta)) - f(t, X(t^-))\}\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta)\\
&+ \int_{\mathbb{R}} \{f(t, X(t) + \gamma(t,\zeta)) - f(t, X(t)) - \frac{\partial f}{\partial x}(t, X(t))\gamma(x,\zeta)\}\nu(\mathrm{d}\zeta)\mathrm{d}t,
\end{aligned}
$$

where the last term can be interpreted as the quadratic variation of jumps. To simplify the notation we will in the following always assume that the predictable version version $(X(t^-))$ is chosen when $X(t)$ appears in the $\tilde{N}(dt, d\zeta)$-integrals.

The *Itô isometries* state the following:

$$\mathbb{E}\left[\left(\int_0^T \beta(s)\mathrm{d}B(s)\right)^2\right] = \mathbb{E}\left[\int_0^T \beta^2(s)\mathrm{d}s\right] \tag{2.9}$$

$$\mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}} \gamma(s,\zeta)\tilde{N}(\mathrm{d}s,\mathrm{d}\zeta)\right)^2\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \gamma^2(s,\zeta)\nu(\mathrm{d}\zeta)\mathrm{d}s\right] \tag{2.10}$$

Martingale properties: If the quantities of (2.10) are finite, then

$$M(t) = \int_0^t \int_{\mathbb{R}} \gamma(s,\zeta)\tilde{N}(\mathrm{d}s,\mathrm{d}\zeta) \tag{2.11}$$

is a martingale for $t \leq T$.

The *Itô representation theorem* states that any $F \in L^2(\mathcal{F}_T, \mathbb{P})$ has the representation

$$F = \mathbb{E}[F] + \int_0^T \varphi(s)\mathrm{d}B(s) + \int_0^T \int_{\mathbb{R}} \psi(s,\zeta)\tilde{N}(\mathrm{d}s, \mathrm{d}\zeta)$$

for suitable predictable (unique) L^2-processes $\varphi(\cdot)$ and $\psi(\cdot)$.
Remark: Using *Malliavin calculus* (see [1]), we get the representation

$$\varphi(s) = \mathbb{E}[D_s F | \mathcal{F}_t]$$

and

$$\psi(s,\zeta) = \mathbb{E}[D_{s,\zeta} F | \mathcal{F}_s],$$

where D_s and $D_{s,\zeta}$ are the Malliavin derivatives at s and (s,ζ) w.r.t. $B(\cdot)$ and $\tilde{N}(\cdot,\cdot)$, respectively (the Clark-Ocone Theorem).

Example

Suppose $\eta(t) = \eta_0(t) = \int_0^t \int_{\mathbb{R}} \zeta \tilde{N}(\mathrm{d}s, \mathrm{d}\zeta)$, i.e. $\eta(t)$ is a pure-jump martingale. We want to find the representation of the random variable $F := \eta_0^2(T)$. By the Itô formula we get

$$
\begin{aligned}
\mathrm{d}(\eta_0^2(t)) &= \int_{\mathbb{R}} \{(\eta_0(t) + \zeta)^2 - (\eta_0(t))^2\} \tilde{N}(\mathrm{d}t, \mathrm{d}\zeta) \\
&\quad + \int_{\mathbb{R}} \{(\eta_0(t) + \zeta)^2 - (\eta_0(t))^2 - 2\eta_0(t)\zeta\} \nu(\mathrm{d}\zeta)\mathrm{d}t \\
&= \int_{\mathbb{R}} 2\eta_0(t)\zeta \tilde{N}(\mathrm{d}t, \mathrm{d}\zeta) + \int_{\mathbb{R}} \zeta^2 \tilde{N}(\mathrm{d}t, \mathrm{d}\zeta) + \int_{\mathbb{R}} \zeta^2 \nu(\mathrm{d}\zeta)\mathrm{d}t \\
&= 2\eta_0(t)\mathrm{d}\eta_0(t) + \int_{\mathbb{R}} \zeta^2 \tilde{N}(\mathrm{d}t, \mathrm{d}\zeta) + \int_{\mathbb{R}} \zeta^2 \nu(\mathrm{d}\zeta)\mathrm{d}t. \qquad (2.12)
\end{aligned}
$$

This implies that

$$\eta_0^2(T) = T\int_{\mathbb{R}} \zeta^2 \nu(\mathrm{d}\zeta) + \int_0^T 2\eta_0(t)\mathrm{d}\eta_0(t) + \int_0^T \int_{\mathbb{R}} \zeta^2 \tilde{N}(\mathrm{d}t, \mathrm{d}\zeta). \tag{2.13}$$

Note that it is not possible to write $F \equiv \eta_0^2(T)$ as a constant + an integral w.r.t. $\mathrm{d}\eta_0(t)$.
This has an interpretation in finance: It implies that in a normalized market with $\eta_0(t)$ as the risky asset price, the claim $\eta_0^2(T)$ is *not replicable*. This illustrates that markets based on Lévy processes are typically not complete.

Consider the following stochastic differential equation (SDE):

$$\begin{aligned} \mathrm{d}X(t) &= b(t, X(t))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}B(t) \\ &\quad + \int_{\mathbb{R}} \gamma(t, X(t^-), \zeta)\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta); \quad X(0) = x. \end{aligned}$$

(2.14)

Here $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$; $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$; and $\gamma : [0, T] \times \mathbb{R}^n \times \mathbb{R}_0^\ell \to \mathbb{R}^{n \times \ell}$ are given functions. If these functions are Lipschitz continuous with respect to x and with at most linear growth in x, uniformly in t, then a unique L^2 - solution to the above SDE exists.

Example

The (generalized) geometric Itô-Lévy process X is defined by:

$$
\begin{aligned}
(2.15)\quad \mathrm{d}X(t) &= X(t^-)\left[\alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t)\right. \\
&\quad \left. + \int_{\mathbb{R}} \gamma(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta)\right]; \quad X(0) = x > 0
\end{aligned}
$$

If $\gamma > -1$ then $X(t)$ can never jump to 0 or a negative value. Then we see by the Itô formula that the solution is

$$
\begin{aligned}
X(t) &= x \exp\left[\int_0^t \beta(s)\mathrm{d}B(s) + \int_0^t (\alpha(s) - \frac{1}{2}\beta^2(s))\mathrm{d}s\right. \\
&\quad + \int_0^t \int_{\mathbb{R}} \{\ln(1+\gamma(s,\zeta)) - \gamma(s,\zeta)\}\nu(\mathrm{d}\zeta)\mathrm{d}s \\
(2.16)\qquad &\quad \left. + \int_0^t \int_{\mathbb{R}} \ln(1+\gamma(s,\zeta))\tilde{N}(\mathrm{d}s,\mathrm{d}\zeta)\right]
\end{aligned}
$$

If $b(t,x) = b(x)$, $\sigma(t,x) = \sigma(x)$, and $\gamma(t,x,\zeta) = \gamma(x,\zeta)$, i.e. $b(\cdot)$, $\sigma(\cdot)$, and $\gamma(\cdot,\cdot)$ do not depend on t, the corresponding SDE takes the form

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}B(t) + \int_{\mathbb{R}} \gamma(X(t),\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta). \tag{2.17}$$

Then $X(t)$ is called an Itô-Lévy diffusion or simply a *jump-diffusion*.

The *generator* A of a jump-diffusion $X(t)$ is defined by

$$(Af)(x) = \lim_{t\to 0} \frac{\mathbb{E}^x[f(X(t))] - f(x)}{t}, \tag{2.18}$$

if the limit exists. The form of the generator A of the process $X(\cdot)$ is given explicitly in the following lemma:

Lemma

If $X(\cdot)$ is a jump-diffusion and $f \in \mathcal{C}_0^2(\mathbb{R})$ (the twice continuously differentiable functions with compact support on $\mathbb{R}$), then $(Af)(x)$ exists for all x and

$$(Af)(x) = \sum_{i=1}^{n} b_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma\sigma^T)_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

$$+ \sum_{k=1}^{\ell} \int_{\mathbb{R}} \{f(x + \gamma^{(k)}(x,\zeta)) - f(x) - \nabla f(x) \cdot \gamma^{(k)}(x,\zeta)\} \nu_k(\mathrm{d}\zeta)$$

where $\gamma^{(k)}$ is column number k of the $n \times \ell$ matrix γ.

The Dynkin formula

The generator gives a crucial link between jump diffusions and (deterministic) partial differential equations. We will exploit this when we come to the dynamic programming approach to stochastic control problems in the next section. One of the most useful expressions of this link is the following result, which may be regarded as a giant generalization of the classical mean-value theorem in classical analysis:

The Dynkin formula

Let X be a jump-diffusion process and let τ be a *stopping time*. Let $h \in \mathcal{C}^2(\mathbb{R})$ and assume that $\mathbb{E}^x\left[\int_0^\tau |Ah(X(t))|\mathrm{d}t\right] < \infty$ and $\{h(X(t))\}_{t \leq \tau}$ is uniformly integrable. Then

$$\text{(2.19)} \quad \mathbb{E}^x[h(X(\tau))] \;=\; h(x) + \mathbb{E}^x\left[\int_0^\tau Ah(X(t))\mathrm{d}t\right].$$

Application to Stochastic Control

Recall the two main methods of optimal control of systems described by Itô - Lévy processes:

- *Dynamic programming and the Hamilton-Jacobi-Bellman (HJB) equation*
  R. Bellman, 1950's.
- *The stochastic maximum principle*
  Pontryagin et al (1950's, deterministic case),
  Bismut (Brownian motion driven SDE's, 1970),
  Bensoussan, Peng, Pardoux ... (Brownian motion driven SDE's, 1970 -1990),
  Tang & Li, Framstad, Sulem & Ø. (jump diffusions, 1990 -).

Dynamic programming is efficient when applicable, but it requires that the system is Markovian. The maximum principle has the advantage that it also applies to non-Markovian SDE's, but the drawback is the corresponding complicated BSDE for the adjoint processes.

Stochastic Control (1): Dynamic Programming

We start by a motivating example:

Example

(Optimal portfolio problem). Suppose we have a financial market with two investment possibilities:

(i) A risk-free asset with unit price $S_0(t) = 1$.

(ii) A risky asset with unit price $S(t)$ at time t given by

$$\begin{aligned} \mathrm{d}S(t) &= S(t^-)\left[\alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t) \right. \\ &\left. + \int_{\mathbb{R}} \gamma(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta)\right], \quad \gamma > -1, \quad S(0) > 0. \end{aligned} \tag{3.1}$$

Let $\pi(t)$ denote a portfolio representing the fraction of the total wealth invested in the risky asset at time t. If we assume that $\pi(t)$ is *self-financing*, the corresponding wealth $X(t) = X_\pi(t)$ satisfies the state equation

$$\begin{cases} \mathrm{d}X(t) = X(t^-)\pi(t)\left[\alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t) + \int_{\mathbb{R}} \gamma(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta)\right]. \\ X(0) = x. \end{cases}$$

The problem is to maximize $\mathbb{E}[U(X_\pi(T))]$ over all $\pi \in \mathcal{A}$, where $\mathcal{A}$ denotes the set of all admissible portfolios and U is a given *utility function*.

This is a special case of the following *general stochastic control problem*:

The *state equation* is given by:

$$\begin{aligned} \mathrm{d}Y(t) = \mathrm{d}Y_u(t) &= b(Y(t), u(t))\mathrm{d}t + \sigma(Y(t), u(t))\mathrm{d}B(t) \\ &+ \int_{\mathbb{R}} \gamma(Y(t), u(t), \zeta)\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta), \quad Y(0) = y \in \mathbb{R}^k. \end{aligned} \tag{3.2}$$

The *performance functional* is assumed to have the form:

$$J_u(y) = \mathbb{E}^y \left[\int_0^{\tau_{\mathcal{S}}} \underbrace{f(Y(s), u(s))}_{\text{profit rate}} \mathrm{d}s + \underbrace{g(Y(\tau_{\mathcal{S}}))}_{\text{bequest function}} 1_{\{\tau_{\mathcal{S}} < \infty\}} \right], \tag{3.3}$$

where $\tau_{\mathcal{S}} = \inf\{t \geq 0 : Y(t) \notin \mathcal{S}\}$ (*bankruptcy time*), and $\mathcal{S}$ is a given *solvency region*.

Problem: Find $u^* \in \mathcal{A}$ and $\Phi(y)$ such that

$$\Phi(y) = \sup_{u \in \mathcal{A}} J_u(y) = J_{u^*}(y).$$

Theorem

(Hamilton-Jacobi-Bellman (HJB) equation)

(a) *Suppose we can find a function $\varphi \in \mathcal{C}^2(\mathbb{R}^n)$ such that*

(i) *$A_v \varphi(y) + f(y, v) \leq 0$, for all $v \in \mathcal{V}$, where $\mathcal{V}$ is the set of possible control values, and $A_v \varphi(y)$ is given by*

$$A_v \varphi(y) =$$

$$\sum_{i=1}^{k} b_i(y, v) \frac{\partial \varphi}{\partial y_i}(y) + \frac{1}{2} \sum_{i,j=1}^{k} (\sigma \sigma^T)_{ij}(y, v) \frac{\partial^2 \varphi}{\partial y_i \partial y_j}$$

$$+ \sum_{k=1}^{\ell} \int_{\mathbb{R}} \{\varphi(y + \gamma^{(k)}(y, v, \zeta)) - \varphi(y) - \nabla \varphi(y) \gamma^{(k)}(y, v, \zeta)\} \nu_k(\mathrm{d}\zeta)$$

(3.4)

(ii) $\lim_{t \to \tau_{\mathcal{S}}} \varphi(Y(t)) = g(Y(\tau_{\mathcal{S}}))1_{\{\tau_{\mathcal{S}} < \infty\}}$

(iii) "growth conditions:"

$$E^y \Big[|\varphi(Y(\tau))| + \int_0^{\tau_{\mathcal{S}}} \{|A\varphi(Y(t))| + |\sigma^T(Y(t))\nabla\varphi(Y(t))|^2$$

$$+ \sum_{j=1}^{\ell} \int_{\mathbb{R}} |\varphi(Y(t) + \gamma^{(j)}(Y(t), u(t), \zeta_j)) -$$

$$\varphi(Y(t))|^2 \nu_j(d\zeta_j)\} dt \Big] < \infty,$$ for all $u \in \mathcal{A}$ and all stopping times τ.

(iv) $\{\varphi^-(Y(\tau))\}_{\tau \le \tau_{\mathcal{S}}}$ is uniformly integrable for all $u \in \mathcal{A}$ and $y \in \mathcal{S}$,

where, in general, $x^- := max\{-x, 0\}$ for $x \in \mathbb{R}$.

Then

$$\varphi(y) \ge \Phi(y).$$

(b) Suppose we for all $y \in \mathcal{S}$ can find $v = \widehat{u}(y)$ such that

$$A_{\widehat{u}(y)}\varphi(y) + f(y, \widehat{u}(y)) = 0$$

and $\widehat{u}(y)$ is an admissible feedback control (Markov control), i.e. $\widehat{u}(y)$ means $\widehat{u}(Y(t))$. Then $\widehat{u}(y)$ is an optimal control and

$$\varphi(y) = \Phi(y).$$

Remark. This is a useful result because it, in some sense, basically reduces the original highly complicated stochastic control problem to a classical problem of maximizing a function of (possibly several) real variable(s), namely the function $v \mapsto A_v\varphi(y) + f(y, v); v \in \mathcal{V}$. We will illustrate this by examples below.

Sketch of proof: Using the "growth conditions" (*iii*), one can prove by an approximation argument that the Dynkin formula holds with $h = \varphi$ and $\tau = \tau_{\mathcal{S}}$, for any given $u \in \mathcal{A}$.
This gives (if $\tau_{\mathcal{S}} < \infty$)

$$\mathbb{E}^y[\varphi(Y(\tau_{\mathcal{S}}))] = \varphi(y) + \mathbb{E}^y\left[\int_0^{\tau_{\mathcal{S}}} A\varphi(Y(t))\mathrm{d}t\right]$$

$$\leq_{(A\varphi + f \leq 0)} \varphi(y) - \mathbb{E}^y\left[\int_0^{\tau_{\mathcal{S}}} f(Y(t), u(t))\mathrm{d}t\right]. \tag{3.5}$$

This implies

$$\varphi(y) \geq \mathbb{E}^y\left[\int_0^{\tau_{\mathcal{S}}} f(Y(t), u(t))\mathrm{d}t + g(Y(\tau_{\mathcal{S}}))\right] \tag{3.6}$$

$$= J_u(y), \quad \text{for all} \quad u \in \mathcal{A}, \tag{3.7}$$

which means that

$$\varphi(y) \geq \sup_{u \in \mathcal{A}} J_u(y) = \Phi(y). \tag{3.8}$$

This proves (*a*).

To prove (b), observe that if we have a control $\hat{u}$ with *equality* above, i.e. $A\varphi + f = 0$, then by the argument in (a) we get

$$\varphi(y) = J_{\hat{u}}(y).$$

Hence

$$\Phi(y) \leq \varphi(y) = J_{\hat{u}}(y) \leq \Phi(y).$$

It follows that $\hat{u}$ is optimal. $\square$

To illustrate this result, let us return to the optimal portfolio problem (Example) :
Suppose we have *logarithmic utility*, i.e. $U(x) = \ln(x)$. Then the problem is to maximize $\mathbb{E}[\ln X_\pi(T)]$. Put

$$
\begin{aligned}
\mathrm{d}Y(t) &= \begin{bmatrix} \mathrm{d}t \\ \mathrm{d}X(t) \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ X(t)\pi(t)\alpha(t) \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0 \\ X(t)\pi(t)\beta(t) \end{bmatrix} \mathrm{d}B(t) \qquad (3.9) \\
&\quad + \begin{bmatrix} 0 \\ X(t)\pi(t) \end{bmatrix} \int_{\mathbb{R}} \gamma(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta) \qquad (3.10)
\end{aligned}
$$

and

$$
\begin{aligned}
&A_\pi\varphi(t,x) \\
&= \frac{\partial \varphi}{\partial t}(t,x) + x\pi\alpha(t)\frac{\partial \varphi}{\partial x}(t,x) + \frac{1}{2}x^2\pi^2\beta^2(t)\frac{\partial^2 \varphi}{\partial x^2}(t,x) \\
&+ \int_{\mathbb{R}} \{\varphi(t, x + x\pi\gamma(t,\zeta)) - \varphi(t,x) - \frac{\partial \varphi}{\partial x}(t,x)x\pi\gamma(t,\zeta)\}\nu(\mathrm{d}\zeta)
\end{aligned}
$$

Here $f = 0$ and $g(t,x) = \ln x$.

We guess that the value function is of the form

$$\varphi(t, x) = \ln x + \kappa(t),$$

where $\kappa(t)$ is some deterministic function (to be determined). Then if we maximize $A_\pi \varphi$ over all π we find, if we assume that $\alpha(t)$, $\beta(t)$, and $\gamma(t, \zeta)$ are deterministic (this ensures that the system is Markovian; see Remark below), that our candidate $\hat{\pi}$ for the optimal portfolio is the solution of the equation

$$\hat{\pi}(t)\beta^2(t) + \hat{\pi}(t) \int_{\mathbb{R}} \frac{\gamma^2(t, \zeta)\nu(\mathrm{d}\zeta)}{1 + \hat{\pi}(t)\gamma(t, \zeta)} = \alpha(t). \tag{3.11}$$

In particular, if $\nu = 0$ and $\beta^2(t) \neq 0$, then

$$\hat{\pi}(t) = \frac{\alpha(t)}{\beta^2(t)}.$$

We can now proceed to find $\kappa(t)$, and then verify that with this choice of φ and $\hat{\pi}$ all the conditions of the HJB equation. Thus we can conclude that

$$\pi^*(t) := \hat{\pi}(t)$$

is indeed the optimal portfolio.

Remark. The assumption that $\alpha(t)$, $\beta(t)$, and $\gamma(t, \zeta)$ are deterministic functions is used when applying the dynamic programming techniques in solving this type of stochastic control problems. More generally, for the dynamic programming/HJB method to work it is necessary that the system is *Markovian*, i.e. that the coefficients are deterministic functions of t and $X(t)$. This is a limitation of the dynamic programming approach to solving stochastic control problems.
In Section 5 we shall see that there is an alternative approach to stochastic control, called *the maximum principle*, which does not require that the system is Markovian.

Risk minimization

Let $p \in [1, \infty]$. A *convex risk measure* is a map $\rho : L^p(\mathcal{F}_T) \to \mathbb{R}$ with the following properties:

(i) (Convexity): $\rho(\lambda F + (1-\lambda)G) \leq \lambda\rho(F) + (1-\lambda)\rho(G)$; for all $F, G \in L^p(\mathcal{F}_T)$,
i.e. diversification reduces the risk.

(ii) (Monotonicity): $F \leq G \Rightarrow \rho(F) \geq \rho(G)$; for all $F, G \in L^p(\mathcal{F}_T)$,
i.e. smaller wealth has bigger risk.

(iii) (Translation invariance): $\rho(F + \alpha) = \rho(F) - \alpha \quad$ if $a \in \mathbb{R}$; for all $F \in L^p(\mathcal{F}_T)$,
i.e. adding a constant to F *reduces* the risk accordingly.

Remark. We may regard $\rho(F)$ as the amount we need to add to the position F in order to make it "acceptable", i.e. $\rho(F + \rho(F)) = 0$. (F is acceptable if $\rho(F) \leq 0$).

One can prove that basically any risk convex measure ρ can be represented as follows:

$$\rho(F) = \sup_{Q \in \wp} \{\mathbb{E}_Q(-F) - \zeta(Q)\}$$

for some family $\wp$ of measures $Q \ll \mathbb{P}$ and for some convex *penalty function* $\zeta : \wp \to \mathbb{R}$. We refer to [3] for more information about risk measures.

Returning to the financial market above, suppose we want to *minimize the risk* of the terminal wealth, rather than maximize the expected utility. Then the problem is to minimize $\rho(X_\pi(T))$ over all possible admissible portfolios $\pi \in \mathcal{A}$.

Hence we want to solve the problem

$$\inf_{\pi \in \mathcal{A}} \left(\sup_{Q \in \wp} \{ \mathbb{E}_Q[-X_\pi(T)] - \zeta(Q) \} \right). \tag{4.1}$$

This is an example of a *stochastic differential game* (of zero-sum type). Heuristically, this can be interpreted as the problem to find the best possible π under the worst possible scenario Q.

The game above is a special case of the following general *zero-sum stochastic differential game*:

We have 2 players and 2 types of controls, u_1 and u_2, and we put $u = (u_1, u_2)$. We assume that player number i controls u_i, for $i = 1, 2$. Suppose the state $Y(t) = Y_u(t)$ has the form

$$\begin{aligned} \mathrm{d}Y(t) &= b(Y(t), u(t))\mathrm{d}t + \sigma(Y(t), u(t))\mathrm{d}B(t) \\ &\quad + \int_{\mathbb{R}} \gamma(Y(t), u(t), \zeta)\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta)\,;\; Y(0) = y. \end{aligned} \tag{4.2}$$

We define the *performance functional* as follows:

(4.3)

$$J_{u_1,u_2}(y) = \mathbb{E}^y\Big[\int_0^{\tau_{\mathcal{S}}} f(Y(t), u_1(t), u_2(t))\mathrm{d}t + g(Y(\tau_{\mathcal{S}}))\mathbf{1}_{\tau_{\mathcal{S}}<\infty}\Big].$$

Problem: Find $\Phi(y)$ and $u_1^* \in \mathcal{A}_1$, $u_2^* \in \mathcal{A}_2$ such that

(4.4)

$$\Phi(y) := \inf_{u_2\in\mathcal{A}_2}\big(\sup_{u_1\in\mathcal{A}_1} J_{u_1,u_2}(y)\big) = \sup_{u_1\in\mathcal{A}_1}\big(\inf_{u_2\in\mathcal{A}_2} J_{u_1,u_2}(y)\big) = J_{u_1^*,u_2^*}(y).$$

The HJBI equation for stochastic differential games

This type of problem is not solvable by the classical Hamilton-Jacobi-Bellman (HJB) equation. We need a new tool, namely the *Hamilton-Jacobi-Bellman-Isaacs (HJBI)* equation, which in this setting goes as follows:

Theorem

(The HJBI equation for zero-sum games ([4]))
Suppose we can find a function $\varphi \in \mathcal{C}^2(\mathcal{S}) \bigcap \mathcal{C}(\bar{\mathcal{S}})$ *(continuous up to the boundary of* $\mathcal{S}$*) and a Markov control pair* $(\hat{u}_1(y), \hat{u}_2(y))$ *such that*

(i) $A_{u_1,\hat{u}_2(y)}\varphi(y) + f(y, u_1, \hat{u}_2(y)) \leq 0$; $\forall u_1 \in \mathcal{A}_1$ *and* $\forall y \in \mathcal{S}$

(ii) $A_{\hat{u}_1(y),u_2}\varphi(y) + f(y, \hat{u}_1(y), u_2) \geq 0$; $\forall u_2 \in \mathcal{A}_2$ *and* $\forall y \in \mathcal{S}$

(iii) $A_{\hat{u}_1(y),\hat{u}_2(y)}\varphi(y) + f(y, \hat{u}_1(y), \hat{u}_2(y)) = 0$; $\forall y \in \mathcal{S}$

(iv) $\lim\limits_{t\to\tau_{\mathcal{S}}} \varphi(Y_u(t)) = g(Y_u(\tau_{\mathcal{S}}))\mathbf{1}_{\tau_{\mathcal{S}}<\infty}$ *for all* u

(v) *"growth conditions".*
Then

$$\varphi(y) = \Phi(y) = \inf_{u_2}(\sup_{u_1} J_{u_1,u_2}(y)) = \sup_{u_1}(\inf_{u_2} J_{u_1,u_2}(y))$$
$$= \inf_{u_2} J_{\hat{u}_1,u_2}(y) = \sup_{u_1} J_{u_1,\hat{u}_2}(y)$$
$$= J_{\hat{u}_1,\hat{u}_2}(y).$$

Proof.

The proof is similar to the proof of the HJB equation. □

Remark. For the sake of the simplicity of the presentation, in (v) above and also in (iv) of Theorem 10 we choose not to specify the rather technical "growth conditions"; we just mention that they are analogous to the growth conditions in earlier theorems. We refer to [4] for details. For a specification of the growth conditions in Theorem 11 we refer to Theorem 2.1 in [9].

To apply this risk minimization problem to our zero-sum game results, we parametrize the family $\wp$ of measures $Q \ll P$ as follows: For given predictable processes $\theta_0(t), \theta_1(t,\zeta)$ we put $\theta := (\theta_1, \theta_2)$ and define the process $Z_\theta(t)$ as follows:

$$\mathrm{d}Z_\theta(t) = Z_\theta(t^-)[\theta_0(t)\mathrm{d}B(t) + \int_{\mathbb{R}} \theta_1(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta)]; Z_\theta(0) > 0, \theta_1 > -1$$

i.e.

$$\begin{aligned} Z_\theta(t) = Z_\theta(0)\exp[&\int_0^t \theta_0(s)\mathrm{d}B(s) - \frac{1}{2}\int_0^t \theta_0^2(s)ds \\ &+ \int_0^t \int_{\mathbb{R}} \ln(1+\theta_1(s,\zeta))\tilde{N}(\mathrm{d}s,\mathrm{d}\zeta) \\ &+ \int_0^t \int_{\mathbb{R}} \{\ln(1+\theta_1(s,\zeta)) - \theta_1(s,\zeta)\}\nu(\mathrm{d}\zeta)\mathrm{d}s]. \end{aligned} \tag{4.5}$$

Define a probability measure $Q_\theta \ll \mathbb{P}$ on $\mathcal{F}_T$ by putting $\frac{\mathrm{d}Q_\theta}{\mathrm{d}P} = Z_\theta(T)$. Then $Z_\theta(t) = \frac{\mathrm{d}(Q_\theta|\mathcal{F}_t)}{\mathrm{d}(P|\mathcal{F}_t)}$ and $Z_\theta(t) = \mathbb{E}[Z_\theta(T)|\mathcal{F}_t]$ for all $t \leq T$. If we restrict ourselves to this family $\wp$ of measures $Q = Q_\theta$ for $\theta \in \Theta$, the risk minimization problem gets the form:

$$\inf_{\pi\in\Pi}(\sup_{\theta\in\Theta}\{\mathbb{E}_{Q_\theta}[-X_\pi(T)]-\zeta(Q_0)\}) = \inf_{\pi\in\Pi}(\sup_{\theta\in\Theta}\{\mathbb{E}[-Z_\theta(T)X_\pi(T)]-\zeta(Q_0)\})$$

For example, if $\zeta(Q_\theta) = \int_0^{\tau_s} \lambda\,(Y(s), \theta(s))\,\mathrm{d}s$, then this problem is a special case of the zero-sum stochastic differential game.

Example: Entropic risk minimization by the HJBI equation

Suppose the financial market is as before, i.e.

(4.6)

$$\begin{cases} S_0(t) = 1 \text{ for all } t \\ dS_1(t) = S_1(t^-)\left[\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t,\zeta)\tilde{N}(dt,d\zeta)\right]; t \geq 0 \\ S_1(0) > 0. \end{cases}$$

If $\beta(t)$ is a self-financing portfolio representing the number of units of the risky asset (with unit price $S_1(t)$) held at time t, the corresponding wealth process $X(t) = X_\beta(t)$ will be given by

$$\begin{aligned} dX(t) &= \beta(t)dS_1(t) \\ &= w(t)\left[\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t,\zeta)\tilde{N}(dt,d\zeta)\right] \\ &= dX^w(t), \end{aligned} \tag{4.7}$$

where $w(t) := \beta(t)S_1(t^-)$ is the *amount* held in the risky asset at time t.

If we choose $\zeta(Q)$ to be the *entropy*, i.e.

$$\zeta(Q) = \zeta_e(Q) = \frac{dQ}{dP} \ln(\frac{dQ}{dP}), \tag{4.8}$$

then the corresponding $\rho = \rho_e$ is called the *entropic risk*.

If we use the above representation of the risk measure $\rho = \rho_e$ corresponding to the family $\mathcal{P}_\Theta$ of measures Q given above and the entropic penalty ζ_e, the entropic risk minimizing portfolio problem becomes

$$\text{(4.9)} \qquad \inf_{w \in \mathcal{W}} \left(\sup_{\theta \in \Theta} \left\{ E \left[-\frac{dQ}{dP} X^w(T) - \frac{dQ}{dP} log \left(\frac{dQ}{dP} \right) \right] \right\} \right)$$

where $\mathcal{W}$ is the family of admissible portfolios w.

To put this problem into the setting of the HJBI equation, we represent Q by $\frac{dQ}{dP} = M^{\theta}(T)$ and we put

$$dY(t) = dY^{\theta,w}(t) = \begin{bmatrix} dt \\ dX^{w}(t) \\ dM^{\theta}(t) \end{bmatrix} = \begin{bmatrix} 1 \\ w(t)\mu(t) \\ 0 \end{bmatrix} dt$$

(4.10)

$$+ \begin{bmatrix} 0 \\ w(t)\sigma(t) \\ M^{\theta}(t)\theta_0(t) \end{bmatrix} dB(t) + \int_{\mathbb{R}} \begin{bmatrix} 0 \\ w(t)\gamma(t,\zeta) \\ M^{\theta}(t^{-})\theta_1(t_1,\zeta) \end{bmatrix} \tilde{N}(dt, d\zeta),$$

with initial value

(4.11) $$Y^{\theta,w}(0) = y = \begin{bmatrix} s \\ x \\ m \end{bmatrix} \; ; \; s \in [0, T], x > 0, m > 0.$$

In this case the solvency region is $\mathcal{S} = [0, T] \times \mathbb{R}_{+} \times \mathbb{R}_{+}$ and the performance functional is

(4.12)

$$J^{\theta,w}(s, x, m) = E^{s,x,m}[-M^{\theta}(T)X^{w}(T) - M^{\theta}(T)\log M^{\theta}(T)].$$

Assume from now on that

(4.13) $\mu(t), \sigma(t)$ and $\gamma(t,\zeta)$ are deterministic.

Then $Y^{\theta,w}(t)$ becomes a controlled jump diffusion, and the risk minimization problem is the following special case of the zero-sum game:

Problem (Entropic risk minimization)

Find $w^* \in \mathcal{W}$, $\theta^* \in \Theta$ and $\Phi(y)$ such that

$$\Phi(y) = \inf_{w \in \mathcal{W}} \left(\sup_{\theta \in \Theta} J^{\theta,w}(y) \right) = J^{\theta^*,w^*}(y) \; ; \; y \in \mathcal{S}. \tag{4.14}$$

We see that the generator $A^{\theta,w}$ is given by

$$\begin{aligned} A^{\theta,w}\varphi(s,x,m) &= \frac{\partial\varphi}{\partial s}(s,x,m) + w\mu(s)\frac{\partial\varphi}{\partial x}(s,x,m) \\ &+ \frac{1}{2}w^2\sigma^2(s)\frac{\partial^2\varphi}{\partial x^2}(s,x,m) + \frac{1}{2}m^2\theta_0^2\frac{\partial^2\varphi}{\partial m^2}(s,x,m) \\ &+ w\theta_0 m\sigma(s)\frac{\partial^2\varphi}{\partial x\partial m}(s,x,m) \\ &+ \int_{\mathbb{R}}\{\varphi(s,x+w\gamma(s,\zeta),m+m\theta_1(\zeta)) - \varphi(s,x,m) \\ &-\frac{\partial\varphi}{\partial x}(s,x,m)w\gamma(s,\zeta) - \frac{\partial\varphi}{\partial m}(s,x,m)m\theta_1(\zeta)\Big\}\nu(d\zeta). \end{aligned} \tag{4.15}$$

Comparing with the general formulation in Section 42, we see that in this case

$$f = 0 \text{ and } g(y) = g(x,m) = -mx - m\log(m).$$

Therefore, according to the HJBI equation, we should try to find a function $\varphi(s, x, m)$ and control values $\theta = \hat{\theta}(y)$, $w = \hat{w}(y)$ such that

$$\inf_{w \in \mathbb{R}} \left(\sup_{\theta \in \mathbb{R}^2} A^{\theta, w} \varphi(y) \right) = A^{\hat{\theta}, \hat{w}} \varphi(y) = 0 \ ; \ y \in \mathcal{S} \tag{4.16}$$

and

$$\lim_{t \to T^-} \varphi(s, x, m) = -xm - m \log(m). \tag{4.17}$$

Let us try a function of the form

$$\varphi(s,x,m) = -xm - m\log(m) + \kappa(s)m \tag{4.18}$$

where κ is a deterministic function, $\kappa(T)=0$. Then by (4.15)

$$\begin{aligned}
&A^{\theta,w}\varphi(s,x,m) = \\
&\kappa'(s)m + m\mu(s)(-m) + \frac{1}{2}m^2\theta_0^2\left(-\frac{1}{m}\right) + w\theta_0 m\sigma(s)(-1) \\
&+ \int_{\mathbb{R}} \{-(x + w\gamma(s,\zeta))(m + m\theta_1(\zeta)) + xm \\
&- (m + m\theta_1(\zeta))\log(m + m\theta_1(\zeta)) \\
&\quad + m\log m + \kappa(s)(m + m\theta_1(\zeta)) - \kappa(s)m + mw\gamma(s,\zeta) \\
&\quad - m\theta_1(\zeta)(-x - 1 - \log m + \kappa(s))\}\nu(d\zeta) \\
&= m[\kappa'(s) - w\mu(s) - \frac{1}{2}\theta_0^2 - w\theta_0\sigma(s) \\
&+ \int_{\mathbb{R}} \theta_1(\zeta)\{1 - \log(1+\theta_1(\zeta)) - w\gamma(s,\zeta)\}\nu(d\zeta)]
\end{aligned} \tag{4.19}$$

Maximizing $A^{\theta,w}\varphi(y)$ with respect to $\theta = (\theta_0, \theta_1)$ and minimizing with respect to w gives the following first order equations

$$\hat{\theta}_0(s) + \hat{w}(s)\sigma(s) = 0 \tag{4.20}$$

$$1 - \log(1 + \hat{\theta}_1(s,\zeta)) - \hat{w}(s)\gamma(s,\zeta) = 0 \tag{4.21}$$

$$\mu(s) + \hat{\theta}_0(s)\sigma(s) - \int_{\mathbb{R}} \hat{\theta}_1(s,\zeta)\gamma(s,\zeta)\nu(d\zeta) = 0. \tag{4.22}$$

These are 3 equations in the 3 unknown candidates $\hat{\theta}_0, \hat{\theta}_1$ and $\hat{w}$ for the optimal control for the entropic risk minimization problem.

To get an explicit solution, let us now assume that

$$N = 0 \text{ and } \gamma = \theta_1 = 0. \tag{4.23}$$

Then (4.20)-(4.22) gives

$$\hat{\theta}_0(s) = -\frac{\mu(s)}{\sigma(s)}, \hat{w}(s) = \frac{\mu(s)}{\sigma^2(s)}. \tag{4.24}$$

Substituted into (4.19) we get by the HJBI equation

$$A^{\hat{\theta},\hat{w}}\varphi(s,x,m) = m\left[\kappa'(s) - \frac{1}{2}\left(\frac{\mu(s)}{\sigma(s)}\right)^2\right] = 0.$$

Combining this with the boundary value for κ we obtain

$$\kappa(s) = -\int_s^T \frac{1}{2}\left(\frac{\mu(t)}{\sigma(t)}\right)^2 dt. \tag{4.25}$$

Now all the conditions of the HJBI equation are satisfied, and we get:

Theorem (Entropic risk minimization)

Assume that (4.13) *and* (4.23) *hold. Then the solution of Problem 8 is*

$$\Phi(s,x,m) = -xm - m\log m - \int_s^T \frac{1}{2}\left(\frac{\mu(t)}{\sigma(t)}\right)^2 dt \tag{4.26}$$

and the optimal controls are

$$\hat{\theta}_0(s) = -\frac{\mu(s)}{\sigma(s)} \text{ and } \hat{w}(s) = \frac{\mu(s)}{\sigma^2(s)}\ ;\ s \in [0,T]. \tag{4.27}$$

In particular, choosing the initial values $s = 0$ *and* $m = 1$ *we get*

$$\Phi(0,x,1) = -x - \int_0^T \frac{1}{2}\left(\frac{\mu(t)}{\sigma(t)}\right)^2 dt. \tag{4.28}$$

Extension of HJBI to non-zero sum games.

In the general case of not necessarily zero-sum games we have two performance functionals, one for each player:

(4.29)

$$J^{(i)}_{u_1,u_2}(y) = \mathbb{E}^y\left[\int_0^{\tau_s} f_i(Y(t), u_1(t), u_2(t))\mathrm{d}t + g_i(Y(\tau_s))\mathbf{1}_{\tau_s<\infty}\right]; i = 1, 2$$

(In the zero-sum game we have $J^{(2)} = -J^{(1)}$). The pair $(\hat{u}_1, \hat{u}_2)$ is called a *Nash equilibrium* if

(i) $J^{(1)}_{u_1,\hat{u}_2}(y) \leq J^{(1)}_{\hat{u}_1,\hat{u}_2}(y)$ for all u_1

(ii) $J^{(2)}_{\hat{u}_1,u_2}(y) \leq J^{(2)}_{\hat{u}_1,\hat{u}_2}(y)$ for all u_2

Remark. This is related to the Nash prisoner dilemma, and it is not a very strong equilibrium: One can sometimes obtain a better result for both players at points which are not Nash equilibria.

The next result is an extension of the HJBI equation to the non-zero sum games:

Theorem

(The HJBI equation for non-zero stochastic differential games [4])

Suppose there exist functions $\varphi_i \in \mathcal{C}^2(\mathcal{S})$; $i = 1, 2$, *and a Markovian control* $(\hat{\theta}, \hat{\pi})$ such that*:*

(i) $A_{u_1, \hat{u}_2(y)}\varphi_1(y) + f_1(y, u_1, \hat{u}_2(y)) \leq A_{\hat{u}_1(y), \hat{u}_2(y)}\varphi_1(y) + f_1(y, \hat{u}_1(y), \hat{u}_2(y)) = 0$; *for all* u_1

(ii) $A_{\hat{u}_1(y), u_2}\varphi_2(y) + f_2(y, \hat{u}_1(y), u_2) \leq A_{\hat{u}_1(y), \hat{u}_2(y)}\varphi_2(y) + f_2(y, \hat{u}_1(y), \hat{u}_2(y)) = 0$; *for all* u_2.

(iii) $\lim_{t \to \tau_s^-} \varphi_i(Y_{u_1, u_2}(t)) = g_i(Y_{u_1, u_2}(\tau_s))\mathbf{1}_{\tau_s < \infty}$ *for* $i = 1, 2$ *and for all* u_1, u_2

(iv) *"growth conditions".*

Then $(\hat{u}_1, \hat{u}_2)$ *is a Nash equilibrium and*

$$\varphi_1(y) = \sup_{u_1 \in \mathcal{A}} J_1^{u_1, \hat{u}_2}(y) = J_1^{\hat{u}_1, \hat{u}_2}(y) \tag{4.30}$$

$$\varphi_2(y) = \sup_{u_2 \in \mathcal{A}_2} J_2^{\hat{u}_1, u_2}(y) = J_2^{\hat{u}_1, \hat{u}_2}(y). \tag{4.31}$$

Stochastic Control (2): The Maximum Principle Approach

We have mentioned that the dynamic programming approach to stochastic control only works if the system is Markovian. However, there is another method, called the *maximum principle* approach, which also works for non-Markovian systems. In this section we describe this method.

Consider a controlled Itô-Lévy process of the form

$$\begin{cases} \mathrm{d}X(t) & = b(X(t), u(t), \omega)\mathrm{d}t + \sigma(t, X(t), u(t), \omega)\mathrm{d}B(t) \\ & + \int_{\mathbb{R}} \gamma(t, X(t), u(t), \zeta, \omega)\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta); t \geq 0 \\ X(0) = x. \end{cases}$$

Here $b(t, x, u, \omega)$ is a given $\mathcal{F}_t$-adapted process, for each x and u and similarly with σ and γ. So this system is not necessarily Markovian.

The *performance functional* has the form:

$$J(u) = \mathbb{E}[\int_0^T f(t, X(t), u(t), \omega)\mathrm{d}t + g(X(T), \omega)]$$

where $T > 0$ is a fixed constant. Note that this performance is also non-Markovian.

Problem: Find $u^* \in \mathcal{A}$ so that $\sup_{u \in \mathcal{A}} J(u) = J(u^*)$.

To solve the problem above, we first define the *Hamiltonian* as follows:

$$\begin{aligned} H(t, x, u, p, q, r(\cdot)) &= f(t, x, u) + b(t, x, u)p + \sigma(t, x, u)q \\ &\quad + \int_{\mathbb{R}} \gamma(t, x, u, \zeta) r(\zeta) \nu(\mathrm{d}\zeta). \end{aligned} \tag{5.1}$$

Here p, q and r are *adjoint variables*, $r = r(\cdot)$ is a real function on $\mathbb{R}$.

The *backward stochastic differential equation (BSDE)* in the adjoint processes $p(t), q(t), r(t,\zeta)$ is defined as follows:

(5.2)

$$\begin{cases} \mathrm{d}p(t) & = -\frac{\partial H}{\partial x}(t, X(t), u(t), p(t), q(t), r(t,\cdot))\mathrm{d}t + q(t)\mathrm{d}B(t) \\ & \quad + \int_{\mathbb{R}} r(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta); \quad 0 \leq t \leq T \\ p(T) & = g'(X(T)). \end{cases}$$

This equation is called *backward* because we are given the terminal value $p(T)$, not the initial value $p(0)$. One can prove in general that under certain conditions on the drift term there exists a unique solution (p,q,r) of such equations. Note that this particular BSDE is linear in p,q and r and hence easy to solve (if we know $X(T)$ and u). See [11], [12] and [13] for more information about BSDEs.

Theorem

(The Mangasarian (sufficient) maximum principle)
Suppose $\hat{u} \in \mathcal{A}$, *with corresponding* $\hat{X}(t) = X_{\hat{u}}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)$. *Suppose the functions*

$x \to g(x)$ *and* $(x,u) \to H(t,x,u,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))$ *are* <u>*concave*</u> *for*

each t *and* ω *and that, for all* t,
(5.3)

$$\max_{v\in\mathcal{V}} H(t,\hat{X}(t),v,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) = H(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)),$$

where $\mathcal{V}$ *is the set of all possible control values.*
Moreover, suppose that some growth conditions are satisfied.

Then $\hat{u}$ *is an optimal control.*

Application to the optimal portfolio problem

We want to maximize $\mathbb{E}[U(X_\pi(T))]$ over all admissible portfolios π, where as before $\pi(t)$ represents the fraction of the wealth invested in the risky asset at time t. The corresponding wealth process $X_\pi(t)$ generated by π is given by

$$\begin{cases} dX(t) = \pi(t)X(t)[\alpha(t,\omega)\mathrm{d}t + \beta(t,\omega)\mathrm{d}B(t) + \int_{\mathbb{R}} \gamma_0(t,\zeta,\omega)\tilde{N}(dt,d\zeta)] \\ X(0) = x \end{cases}$$

which has the solution

$$X(t) = x \exp[\int_0^t \{\pi(s)\alpha(s) - \frac{1}{2}\pi^2(s)\beta^2(s) + \int_{\mathbb{R}} (\ln(1+\pi(s)\gamma_0(s,\zeta)) - \pi(s)\gamma_0(s,\zeta))\nu(d\zeta)\} ds$$

(5.4)

$$+ \int_0^t \pi(s)\beta(s)dB(s) + \int_0^t \int_{\mathbb{R}} \ln(1+\pi(s)\gamma_0(s,\zeta))\tilde{N}(ds,d\zeta)]$$

In this case the coefficients are

$$\begin{aligned} b(t,x,\pi) &= \pi x \alpha(t), \\ \sigma(t,x,\pi) &= \pi x \beta(t), \\ \gamma(t,x,\pi,\zeta) &= \pi x \gamma_0(t,\zeta), \end{aligned} \tag{5.5}$$

and the Hamiltonian is

$$H = \pi x \alpha(t) p + \pi x \beta(t) q + \pi x \int_{\mathbb{R}} \gamma_0(t,\zeta) r(\zeta) \nu(\mathrm{d}\zeta). \tag{5.6}$$

The BSDE (5.2) becomes

$$\begin{cases} \mathrm{d}p(t) & = -\pi(t)[\alpha(t)p(t) + \beta(t)q(t) + \int_{\mathbb{R}} \gamma(t,\zeta)r(t,\zeta)\nu(\mathrm{d}\zeta)]dt \\ & \quad +q(t)\mathrm{d}B(t) + \int_{\mathbb{R}} r(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta); \quad 0 \leq t \leq T \\ p(T) & = U'(X_\pi(T)). \end{cases}$$

Note that π appears linearly in H. Therefore we guess that if π is optimal, the coefficient of π in H must be 0. Otherwise one could make H arbitrary big by choosing π suitably.

Hence we obtain the following two equations that must be satisfied for an optimal triple $(p(t), q(t), r(t, \cdot))$:

$$\alpha(t)p(t) + \beta(t)q(t) + \int_{\mathbb{R}} \gamma(t,\zeta)r(t,\zeta)\nu(\mathrm{d}\zeta) = 0 \tag{5.7}$$

$$\begin{cases} \mathrm{d}p(t) & = q(t)\mathrm{d}B(t) + \int_{\mathbb{R}} \gamma(t,\zeta)r(t,\zeta)\nu(\mathrm{d}\zeta) \\ p(T) & = U'(X_\pi(T)). \end{cases} \tag{5.8}$$

In addition we have the (forward) SDE (5.4) for $X(t)$. So we really have a *coupled system of a forward and backward SDEs* (FBSDE for short). By using a necessary version of the maximum principle we can prove that these two conditions are both necessary and sufficient for a control π to be optimal. We formulate this as follows:

Theorem

A control π is optimal for the utility maximization problem in Example 5 if and only if the solution $(X_\pi(t), p(t), q(t), r(t, \cdot))$ of the FBSDE (5.4) &(5.8) *satisfies the equation* (5.7).

This result can be used to find the optimal portfolio in some cases. To illustrate this, we proceed as follows:

To solve (5.8), we try to put

$$q(t) = p(t)\theta_0(t), r(t,\zeta) = p(t)\theta_1(t,\zeta),$$

for suitable processes θ_0, θ_1 (to be determined). Then (5.8) becomes

$$(5.9) \qquad \begin{cases} \mathrm{d}p(t) & = p(t)[\theta_0(t)\mathrm{d}B(t) + \int_{\mathbb{R}} \theta_1(t,\zeta)\tilde{N}(dt,\mathrm{d}\zeta)] \\ p(T) & = U'(X_\pi(T)). \end{cases}$$

This SDE has the solution

$$p(t) = p(0)\exp\Big(\int_0^t \{-\frac{1}{2}\theta_0^2(s) + \int_{\mathbb{R}} (\ln(1+\theta_1(s,\zeta)) - \theta_1(s,\zeta))\nu(d\zeta)\} ds$$

(5.10)

$$+ \int_0^t \theta_0(s) dB(s) + \int_0^t \int_{\mathbb{R}} \ln(1+\theta_1(s,\zeta)) \tilde{N}(ds,\zeta)\Big).$$

In particular, putting $t = T$ this gives

$$p(T) =$$

$$p(0)\exp\Big(\int_0^T \{-\frac{1}{2}\theta_0^2(s) + \int_{\mathbb{R}} (\ln(1+\theta_1(s,\zeta)) - \theta_1(s,\zeta))\nu(d\zeta)\} ds$$

(5.11)

$$+ \int_0^T \theta_0(s) dB(s) + \int_0^T \int_{\mathbb{R}} \ln(1+\theta_1(s,\zeta)) \tilde{N}(ds,\zeta)\Big).$$

Combining this with the terminal condition in (5.8) we have

$$X_\pi(T) = I(p(T))$$
$$= I\Big(p(0)\exp\Big[\int_0^T \{-\frac{1}{2}\theta_0^2(s) + \int_{\mathbb{R}} (\ln(1+\theta_1(s,\zeta)) - \theta_1(s,\zeta))\nu(d\zeta)\}ds$$

(5.12)

$$+ \int_0^T \theta_0(s)dB(s) + \int_0^T \int_{\mathbb{R}} \ln(1+\theta_1(s,\zeta))\tilde{N}(ds,\zeta)\Big]\Big),$$

where

$$I = (U')^{-1}.$$

The logarithmic utility case

Now let us assume that

$$U(x) = \ln x$$

Then $I(y) = \frac{1}{y}$ and combining (5.12) with (5.4) we get the identity

$$\frac{1}{p(0)} \exp\Big[\int_0^T \{\frac{1}{2}\theta_0^2(s) - \int_{\mathbb{R}} (\ln(1+\theta_1(s,\zeta)) - \theta_1(s,\zeta))\nu(d\zeta)\}ds$$

$$- \int_0^T \theta_0(s)dB(s) - \int_0^T \int_{\mathbb{R}} \ln(1+\theta_1(s,\zeta))\tilde{N}(ds,\zeta)\Big]$$

$$= x\exp\Big[\int_0^T \{\pi(s)\alpha(s) - \frac{1}{2}\pi^2(s)\beta^2(s)$$

$$+ \int_{\mathbb{R}} (\ln(1+\pi(s)\gamma_0(s,\zeta)) - \pi(s)\gamma_0(s,\zeta))\nu(d\zeta)\}ds$$

(5.13)

$$+ \int_0^T \pi(s)\beta(s)dB(s) + \int_0^T \int_{\mathbb{R}} \ln(1+\pi(s)\gamma_0(s,\zeta))\tilde{N}(ds,d\zeta)\Big].$$

If we require that the integrands of the ds-integrals, the $dB(s)$-integrals and the $\tilde{N}(ds, d\zeta)$-integrals, respectively, in the two expressions are identical, we get the following 3 equations in the 3 unknowns $\pi(s), \theta_0(s), \theta_1(s,\zeta)$:

$$\pi(s)\beta(s) = -\theta_0(s) \tag{5.14}$$

$$\ln(1 + \pi(s)\gamma_0(s,\zeta)) = -\ln(1 + \theta_1(s,\zeta)) \tag{5.15}$$

and

$$\pi(s)\alpha(s) - \frac{1}{2}\pi^2(s)\beta^2(s) + \int_{\mathbb{R}} (\ln(1 + \pi(s)\gamma_0(s,\zeta)) - \pi(s)\gamma_0(s,\zeta))\nu(d\zeta)$$

(5.16)

$$= \frac{1}{2}\theta_0^2(s) - \int_{\mathbb{R}} (\ln(1 + \theta_1(s, \theta(s,\zeta)) - \theta_1(s,\zeta))\nu(d\zeta).$$

If we substitute the two first equations into the third, we get the following equation for the optimal portfolio $\pi(s) = \pi^*(s)$:

(5.17)

$$\alpha(s) - \frac{1}{2}\pi^*(s)\beta^2(s) - \pi^*(s)\int_{\mathbb{R}} \frac{\gamma_0^2(s,\zeta)}{1 + \pi^*(s)\gamma_0(s,\zeta)}\nu(d\zeta) = 0.$$

This is the same solution that we found by using dynamic programming and the HJB equation (see (3.11)). But note that now we are dealing with a more general, non-Markovian system, allowing the coefficients $\alpha(s), \beta(s)$ and $\gamma_0(s,\zeta)$ to be general stochastic processes (not necessarily deterministic).

General utility case

Using the maximum principle, we can also deal with the general utility case, but then we cannot get explicit solutions. For simplicity, assume in the following that $\nu = 0$ from now on (i.e., that there are no jumps). Then (5.7) becomes:

$$\alpha(t) + \beta(t)\theta_0(t) = 0$$

i.e.

$$\theta_0(t) = -\frac{\alpha(t)}{\beta(t)}.$$

This determines the value of θ_0 and hence by (5.18) the value of $p(t)$, except for the constant $p(0)$:

$$p(t) = p(0)\exp\Big(\int_0^t -\frac{1}{2}\theta_0^2(s)ds + \int_0^t \theta_0(s)dB(s)\Big). \tag{5.18}$$

How do we find $p(0)$?

Recall the equation for $X(t) = X_\pi(t)$:

$$\begin{cases} \mathrm{d}X(t) & = \pi(t)X(t)\left[\alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t)\right] \\ X(T) & = I(p(T)) \end{cases} \tag{5.19}$$

If we define

$$Z(t) = \pi(t)X(t)\beta(t), \tag{5.20}$$

then we see that $X(t)$ satisfies the BSDE

$$\begin{cases} \mathrm{d}X(t) = \frac{\alpha(t)}{\beta(t)}Z(t)\mathrm{d}t + Z(t)\mathrm{d}B(t) \\ X(T) = I(p(T)). \end{cases} \tag{5.21}$$

The solution of this linear BSDE is

$$X(t) = \frac{1}{\Gamma(t)} \mathbb{E}\left[I(p(T))\Gamma(T) | \mathcal{F}_t\right] \tag{5.22}$$

where $\mathrm{d}\Gamma(t) = -\Gamma(t)\frac{\alpha(t)}{\beta(t)}\mathrm{d}B(t); \quad \Gamma(0) = 1.$

Now put $t = 0$ and take expectation to get

$$X(0) = x = \mathbb{E}\left[I(p(T))\Gamma(T)\right]. \tag{5.23}$$

This equation determines (implicitly) the constant $p(0)$ and hence by (5.21) the optimal terminal wealth $X_\pi(T)$. Then, when the optimal terminal wealth $X_\pi(T)$ is known, one can find the corresponding optimal portfolio π by solving the BSDE (5.21) above for $X(t), Z(t)$ and then using that $Z(t) = \pi(t)X(t)\beta(t)$. We omit the details.

We have obtained the following:

Theorem

The optimal portfolio π^* for the general utility and with no jumps is given by

$$\pi^*(t) = \frac{Z(t)}{X_{\pi^*}(t)\beta(t)} \tag{5.24}$$

where $(X_{\pi^*}(t), Z(t))$ solves the BSDE (5.21), with $p(0)$ given by (5.23) and $p(T)$ given by (5.18).

Remark: The advantage of this approach is that it applies to a general non-Markovian setting, which is inaccessible for dynamic programming.

Moreover, this approach can be extended to case when the agent has only *partial information* to her disposal, which means that her decisions must be based on an information flow which is a subfiltration of $\mathcal{F}$.

A suitably modified version can also be applied to study optimal control under *inside information*, i.e. information about the future value of the system.
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