Why is Andreas Kyprianou so obsessed with Lévy processes?

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Outline of the talk:

- Poisson processes.
- Martingales in continuous time.
- Poisson random measures.
- Lévy processes.
- Lévy-Itô decomposition.
- Strong Markov property.

Poisson processes

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Let us explain some details of the above definition.

We first recall that S_n has the same law as a Gamma distribution with parameters c and n.

Therefore, for any fixed $k \in \mathbb{N}$ and $t \in \mathbb{R}_+$

$$\mathbb{P}(N_t = k) = \mathbb{E}\left[\mathbf{1}_{\{S_k \leq t < S_{k+1}\}}\right] =$$

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The lack of memory property of the exponential law implies that for every $0 \le s \le t$, the increment $N_{t+s} - N_t$ is a Poisson r.v. with parameter cs and is independent of the σ-field generated by $(N_u, 0 \le u \le t)$.

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Therefore, for any fixed $k \in \mathbb{N}$ and $t \in \mathbb{R}_+$

$$\begin{aligned}\mathbb{P}(N_t = k) &= \mathbb{E}\left[\mathbf{1}_{\{S_k \leq t < S_{k+1}\}}\right] = \mathbb{E}\left[\mathbf{1}_{\{S_k \leq t\}}\,\mathbb{P}(\tau_{n+1} \geq t - S_k)\right] \\ &= \frac{1}{\Gamma(k)}\int_0^t e^{-c(t-x)}\,c^k e^{-cx} x^{k-1}\mathrm{d}x = \frac{c^k t^k}{k!}e^{-ct}.\end{aligned}$$

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In other words, the Poisson process $N = (N_t, t \geq 0)$ is an increasing process with independent and homogeneous increments and jumps of size one.

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We say that M is right-continuous if its paths are right-continuous a.s.

Maximal inequality (Doob): Let M be a right-continuous martingale such that for $p > 1$ and $T > 0$, we have $\sup_{s\in[0,T]} \mathbb{E}[|M_s|^p] < \infty$, then

$$\mathbb{E}\left[\sup_{s\in[0,T]} |M_s|^p\right] \leq \left(\frac{p}{p-1}\right)^p \sup_{s\in[0,T]} \mathbb{E}[|M_s|^p].$$

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Very important example: Let $X = (X_t, t \geq 0)$ be a real-valued stochastic process which is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, i.e. X_t is $\mathcal{F}_t$-measurable. If X is right-continuous and $A \in \mathcal{B}(\mathbb{R})$ open or closed, then

$$\tau_A = \inf\{t \geq 0 : X_t \in A\}$$

is a stopping time.

Optimal stopping theorem: Suppose that τ is an a.s. finite stopping time, then

i) The process $(M_{\tau\wedge t}, t \geq 0)$ is a martingale.

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$$0 = \mathbb{E}[B_{\tau_{a,b}}] = a\,\mathbb{P}(B_{\tau_{a,b}} = a) + b\,\mathbb{P}(B_{\tau_{a,b}} = b).$$

On the other hand $\mathbb{P}(B_{\tau_{a,b}} = a) + \mathbb{P}(B_{\tau_{a,b}} = b) = 1$, implying

$$\mathbb{P}(B_{\tau_{a,b}} = b) = -\frac{a}{b-a}.$$

Convergence theorem: Suppose that M is an uniformly integrable martingale. Then $\lim_{t\to\infty} M_t = M_\infty$ exists a.s., and in $L^1(\mathbb{P})$. Moreover, we have

$$M_t = \mathbb{E}[M_\infty|\mathcal{F}_t] \qquad \text{for all } t \geq 0.$$

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Let us define

$$M_t = N_t - ct, \quad \text{and} \quad \xi_t^q = \exp\left\{-qN_t + ct(1-e^{-q})\right\}, \qquad t \geq 0, \quad q > 0.$$

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From the independence and the homogeneity of the increments, we get

$$\mathbb{E}[N_{t+s}|\mathcal{F}_t] = \mathbb{E}[N_{t+s} - N_t + N_t|\mathcal{F}_s] = N_t + cs,$$

then subtracting $c(t+s)$ in both sides, we deduce that $M = (M_t, t \geq 0)$ is a martingale related to $(\mathcal{F}_t)$.

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An interesting example Let $H = (H_t, t \geq 0)$ be a left-continuous adapted process. Now, let us introduce the stochastic integral related to the Poisson process N by

$$\int_0^t H_s \mathrm{d}N_s = \sum_{s \leq t} H_s \Delta N_s, \quad \text{where} \quad \Delta N_s = N_s - N_{s-}.$$

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Observe that from the definition of N, we have the following identity

$$\int_0^t H_s \mathrm{d}N_s = \sum_{n=1}^{\infty} H_{\tau_n} \mathbb{I}_{\{\tau_n \leq t\}}.$$

Proposition

Let H be a left-continuous and adapted process with

$$\mathbb{E}\left(\int_0^t |H_s|\mathrm{d}s\right) < \infty, \qquad \textit{for all} \quad t \geq 0,$$

then the compensated integral

$$\int_0^t H_s \mathrm{d}N_s - c\int_0^t H_s \mathrm{d}s, \qquad t \geq 0,$$

is a martingale related to $(\mathcal{F}_t)$. Moreover, we have the well-known **compensation formula**

$$\mathbb{E}\left[\int_0^t H_s \mathrm{d}N_s\right] = c\,\mathbb{E}\left[\int_0^t H_s \mathrm{d}s\right].$$

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Construction: Let us suppose $c = \mu(E) < \infty$. Let $(\xi_i)_{i\geq 1}$ be a sequence of i.i.d. r.v's with common distribution $c^{-1}\mu$ and N a Poisson r.v. with parameter c which is independent of (ξ_i). The random measure

$$M(\cdot) = \sum_{j=1}^{N} \delta_{\xi_j}(\cdot),$$

is a Poisson random measure with intensity $\mu(\cdot)$.

Since μ is σ-finite, then there exist a partition $(E_n)_{n\geq 0}$ of E such that $\mu(E_n) < \infty$, for each n. Therefore, one can construct a sequence M_n of Poisson random measures with intensity $\mu(\cdot \cap E_n)$ and such that $M = \sum M_n$ is a Poisson random measure with intensity μ.

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Now, we consider the space $E \times [0, \infty)$, the measure $\mu \otimes \mathrm{d}x$ and a Poisson random measure M with intensity $\mu \otimes \mathrm{d}x$. One can show that a.s. for all $t \geq 0$,

$$M(E \times \{t\}) = 0 \text{ or } 1.$$

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This allow us to define a process $(e(t), t \geq 0)$, that we will call Poisson point process taking values on $E \cup \Upsilon$, where Υ is an additional isolated point, such that $M(E \times \{t\}) = 0$, then $e(t) = \Upsilon$ and if $M(E \times \{t\}) = 1$, then the restriction of M to $E \times \{t\}$ is the Dirac measure on the point (ε, t) and we define $e(t) = \varepsilon$. In other words, we can write the Poisson random measure M as follows

$$M = \sum_{t \geq 0} \delta_{(e(t), t)}.$$

Proposition

Let B *such that* $\mu(B) < \infty$, *and define the counting process*

$$N_t^B = \#\{s \leq t : e(s) \in B\}, \qquad t \geq 0,$$

and the first hitting time

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i) N^B *is a Poisson process with parameter* $\mu(B)$ *adapted to the filtration* $(\mathcal{G}_t)_{t \geq 0}$ *generated by* e*. The time* T_B *is a stopping time with respect to* $(\mathcal{G}_t)$ *and it is an exponential r.v. with parameter* $\mu(B)$*.*

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- **ii)** *$e(T_B)$ and T_B are independent, and for $A \in \mathcal{E}$*

$$\mathbb{P}(e(T_B) \in A) = \frac{\mu(A \cap B)}{\mu(B)}.$$

Proof: The first part of (i) is straightforward, since

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The r.v. T_B is the first jump of N^B, implying that it is exponential with parameter $\mu(B)$. In order to prove the second part of the proposition, lets take $A \subset B$ and observe that

$$\mathbb{P}(T_B \le t, e(T_B) \in A) = \mathbb{P}(T_A < T_{B\setminus A}, T_A \wedge T_{B\setminus A} \le t),$$

where T_A denotes the first jump of the Poisson process N^A.

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where T_A denotes the first jump of the Poisson process N^A. Since A and $B \setminus A$ are disjoint, T_A and $T_{B \setminus A}$ are independent and exponentially distributed with parameter $\mu(A)$ and $\mu(B) - \mu(A)$, respectively. Therefore,

$$\mathbb{P}(T_B \le t, e(T_B) \in A) = \frac{\mu(A)}{\mu(B)}(1 - e^{-t\mu(B)}).$$

Let $B \in \mathcal{E}$, $0 \leq t_1 < t_2$ and define

$$H_t(y) = \mathbf{1}_{B \times (t_1, t_2]}(y, t),$$

in this case, it is clear

$$\mathbb{E}\left[\sum_{0 \leq t < \infty} H_t(e(t))\right] = (t_2 - t_1)\mu(B).$$

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From this fact, we deduce the compensation formula for Poisson point processes: Let $H = (H_t, t \geq 0)$ be a right-continuous process taking values on $E \cup \Upsilon$ and such that $H_t(\Upsilon) = 0$. Then

$$\mathbb{E}\left[\sum_{0\leq t<\infty} H_t(e(t))\right] = \mathbb{E}\left[\int_0^\infty \mathrm{d}t \int_E H_t(y)\mu(\mathrm{d}y)\right].$$

In a similar way, one can obtain the exponential formula: Let f be a positive measurable function defined on $E \cup \Upsilon$ with $f(\Upsilon) = 0$ and such that

$$\int_E |1 - e^{-f(y)}|\mu(\mathrm{d}y) < \infty.$$

Then for all $t \geq 0$,

$$\mathbb{E}\left[\exp\left\{-\sum_{0\leq s\leq t} f(e(s))\right\}\right] = \exp\left\{-t\int_E (1 - e^{-f(y)})\mu(\mathrm{d}y)\right\}.$$

Lévy processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We say that $X = (X_t, t \geq 0)$ is a real-valued Lévy process if for $0 \leq s \leq t$, the increment $X_{t+s} - X_t$ is independent from $(X_u, 0 \leq u \leq t)$ and has the same distribution as X_s.

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From the decomposition

$$X_1 = X_{1/n} + \left(X_{2/n} - X_{1/n}\right) + \cdots + \left(X_{n/n} - X_{(n-1)/n}\right),$$

we see that X_1 is infinitely divisible. The law of an infinitely divisible r.v. is characterized by the so-called Lévy-Khintchine formula which implies the following result.

Theorem

Let X be a Lévy processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$, then

$$\mathbb{E}\left[\exp\{i\lambda X_t\}\right] = \exp\{-t\Psi(\lambda)\}, \qquad t \geq 0, \lambda \in \mathbb{R},$$

where

$$\Psi(\lambda) = ia\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{(-\infty,\infty)} \left(1 - e^{i\lambda x} + i\lambda x \mathbf{1}_{\{|x|<1\}}\right)\Pi(\mathrm{d}x),$$

for $a \in \mathbb{R}$, $\sigma \geq 0$ and a measure Π on $\mathbb{R} \setminus \{0\}$ such that

$$\int_{(-\infty,\infty)} (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty.$$

Examples:

- In the case of the standard Brownian motion, $a = 0$, $\sigma = 1$, $\Pi \equiv 0$ and
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- In the case of the compound Poisson process with jump rate c and jump distribution F, we have
$$a = -c\int_{\{|x|<1\}} xF(\mathrm{d}x), \quad \sigma = 0, \quad \Pi(\mathrm{d}x) = cF(\mathrm{d}x),$$
and
$$\Psi(\lambda) = c\int_{(-\infty,\infty)} \left(1 - e^{i\lambda x}\right) F(\mathrm{d}x).$$

- In the case of the Gamma process,

$$a = c(e^{-1} - 1), \quad \sigma = 0, \quad \Pi(\mathrm{d}x) = c\frac{e^{-x}}{x}\mathbf{1}_{\{x>0\}}\mathrm{d}x,$$

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- In the case of the strictly stable process with index $\alpha \in (0,1) \cup (1,2)$, we have

$$\sigma = 0, \quad \Pi(\mathrm{d}x) = \left(\frac{c_+}{x^{1+\alpha}}\mathbf{1}_{\{x>0\}} + \frac{c_-}{|x|^{1+\alpha}}\mathbf{1}_{\{x<0\}}\right)\mathrm{d}x$$

with $c_+ \geq 0$, $c_- \geq 0$, and

$$\Psi(\lambda) = c|\lambda|^{\alpha}(1 - i\beta\mathrm{sgn}(\lambda)\tan(\pi\alpha/2)) + ia\lambda$$

where $\beta = (c_+ - c_-)/(c_+ + c_-)$. If $\alpha = 1$, $c_+ = c_- \geq 0$ and $\Psi(\lambda) = c_+|\lambda| + ia\lambda$. The latter case is known as the Cauchy process with drift.

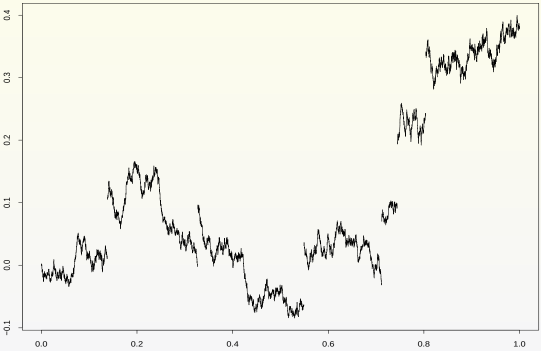

Figure : A sample path of the independent sum of a Brownian motion and a compound Poisson process.

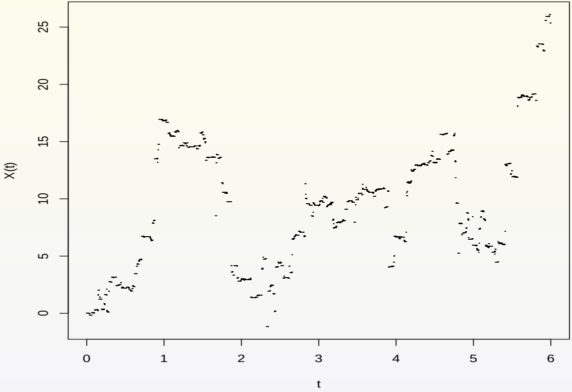

Figure : $a = 0$, $\sigma = 0$ and
$\Pi(dx) = \left(\frac{e^x}{(e^x-1)^{3/2}}\mathbf{1}_{\{x>0\}} + \frac{e^{-x}}{(e^{-x}-1)^{3/2}}\mathbf{1}_{\{x<0\}}\right)\mathrm{d}x.$

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Let $(e(t), t \geq 0)$ be a Poisson point process on $\mathbb{R} \times [0, \infty)$ with intensity Π. Observe that $\Pi(\{x : |x| \geq 1\}) < \infty$, then

$$\sum_{s \leq t} \mathbf{1}_{\{|e(s)| \geq 1\}} |e(s)| < \infty, \quad \text{a.s.}$$

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$$\sum_{s \leq t} \mathbf{1}_{\{|e(s)| \geq 1\}} |e(s)| < \infty, \quad \text{a.s.}$$

Let us define

$$Y_t^{(1)} = \sum_{s \leq t} \mathbf{1}_{\{|e(s)| \geq 1\}} e(s), \qquad t \geq 0.$$

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If

$$I = \int_{(-\infty,\infty)} (1 \wedge |x|)\Pi(\mathrm{d}x) < \infty,$$

one can prove for all t

$$\sum_{s\leq t} \mathbf{1}_{\{\epsilon<|e(s)|<1\}}|e(s)| \xrightarrow[\epsilon\to 0]{} \sum_{s\leq t} \mathbf{1}_{\{|e(s)|<1\}}|e(s)| < \infty,$$

In this case $Y_t = Y_t^{(1)} + Y_t^{(2)}$, where

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The process Y has paths of bounded variation (on finite intervals) and

$$\Psi(\lambda) = -i\lambda \mathtt{d} + \frac{\sigma^2}{2}\lambda^2 + \Psi^{(1)}(\lambda) + \Psi^{(2)}(\lambda),$$

where

$$\mathtt{d} = -a - \int_{\{|x<1|\}} x\Pi(\mathrm{d}x).$$

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If $I = \infty$, then a.s. $\sum_{s \leq t} |e(s)| = \infty$ for $t > 0$. In this case, we define $Y^{(2)}$ as the limit when $\epsilon \to 0$ of

$$Y_t^{(2),\epsilon} = \sum_{s \leq t} \mathbf{1}_{\{\epsilon < |e(s)| < 1\}} e(s) - t \int_{\{\epsilon < |x| < 1\}} x \Pi(\mathrm{d}x).$$

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The process $Y^{(2),\epsilon}$ is a compensated compound Poisson process, i.e.

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Using Doob's maximal inequality, for $p = 2$, we see that for all $t \geq 0$ and $\eta \in (0, \epsilon)$

$$\mathbb{E}\left[\sup_{s \leq t}\left| Y_s^{(2),\epsilon} - Y_s^{(2),\eta}\right|^2\right] \leq 4t \int_{\{\eta < |x| < \epsilon\}} |x|^2 \Pi(\mathrm{d}x),$$

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Lema

A Lévy process is a subordinator if and only if $\Pi(-\infty, 0) = 0$, $I < \infty$, $\sigma = 0$ *and* $\mathtt{d} \geq 0$.

Strong Markov property.

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We say that the process $Y = (Y_t, t \geq 0)$ satisfies the Markov property if for $B \in \mathcal{B}(\mathbb{R})$ and $s, t \geq 0$,

$$\mathbb{P}(Y_{t+s} \in B | \mathcal{F}_t) = p(Y_t, s, B),$$

where $x \in \mathbb{R}$ and $s \geq 0$, $p(x, s, B) = \mathbb{P}(Y_s \in B | Y_0 = x)$.

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Thanks to the property of independent and stationary increments, it is clear that X satisfies the Markov property. In this case

$$p(x, s, B) = \mathbb{P}_x(X_s \in B).$$

Let τ be a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$. Define the stopped σ-algebra

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We say that the process Y satisfies the strong Markov property if for each stopping time τ we have

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Theorem

Suppose that τ is a stopping time. Under $\{\tau < \infty\}$, we define the process $\tilde{X} = (\tilde{X}_t, t \geq 0)$ where

$$\tilde{X}_t = X_{\tau+t} - X_\tau, \quad t \geq 0.$$

Then, under $\{\tau < \infty\}$, the process $\tilde{X}$ is independent of $\mathcal{F}_\tau$, and has the same law as X and in particular is a Lévy process.

Proof: Suppose that τ is finite a.s., and let $A \in \mathcal{F}_\tau$, $0 \le t_1 \le \cdots \le t_n$ and $F : \mathbb{R}^n \to \mathbb{R}$ a continuous and bounded function. It is enough to see

$$\mathbb{E}\left[\mathbf{1}_A F(\tilde{X}_{t_1}, \cdots \tilde{X}_{t_n})\right] = \mathbb{P}(A)\, \mathbb{E}\left[F(X_{t_1}, \cdots, X_{t_n})\right].$$

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Then now, we observe

$$\sum_{i=1}^{\infty} \mathbf{1}_{\{\frac{i-1}{2^m} < \tau \leq \frac{i}{2^m}\}} F\left(X_{\frac{i}{2^m}+t_1} - X_{\frac{i}{2^m}}, \cdots, X_{\frac{i}{2^m}+t_n} - X_{\frac{i}{2^m}}\right),$$

converge, as $m \to \infty$, to $F\left(\tilde{X}_{t_1}, \cdots, \tilde{X}_{t_n}\right)$.

From the Monotone Convergence Theorem and the Markov property, we deduce

$$\mathbb{E}\left[\mathbf{1}_A F\left(\widetilde{X}_{t_1}, \cdots, \widetilde{X}_{t_n}\right)\right]$$
$$= \lim_{m\to\infty} \sum_{i=0}^{\infty} \mathbb{P}\left(A \cap \left\{\frac{i-1}{2^m} < \tau \leq \frac{i}{2^m}\right\}\right) \mathbb{E}\left[F\left(X_{t_1}, \cdots, X_{t_n}\right)\right]$$
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The general case follows from similar arguments,

$$\mathbb{E}\left[\mathbf{1}_{A\cap\{\tau<\infty\}} F\left(\widetilde{X}_{t_1}, \cdots, \widetilde{X}_{t_n}\right)\right] = \mathbb{P}\left(A\cap\{\tau<\infty\}\right)\mathbb{E}\left[F\left(X_{t_1}, \cdots, X_{t_n}\right)\right].$$

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