Optimality of Refraction Strategies for Spectrally Negative Lévy Processes

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Inventory Control Problems

Solving the tradeoff b/w controlling costs and inventory costs.

- inventory costs: shortage costs and surplus costs
  - want the inventory to be not too much and not too little,
  - typically modeled by a v-shaped function.
- controlling costs:
  - some variations – fixed and/or proportional – one-sided or two-sided etc.

Applications

- inventory management, cash management, currency control, international reserve etc.

Optimality in Existing Literature

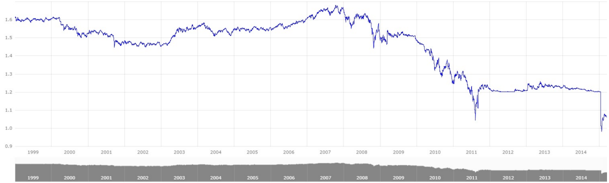
Essentially all papers show that the barrier strategy (or its variation) is optimal.

- one-sided control w/o fixed costs – reflection strategy is optimal.
- one-sided control w/ fixed costs – (s, S)-strategy (policy) is optimal.
- two-sided control w/o fixed costs – double reflection strategy is optimal.
- two-sided control w/ fixed costs – (d, D, U, u)-band strategy is optimal.

Barrier Strategy Is NOT Realistic

In reality, barrier strategies are hard to implement.

Euro vs Chinese yuan



Euro vs Swiss franc

Under Absolutely Continuous Assumptions

- The typical inventory control (one-sided w/o fixed costs) minimizes

$$v^{\pi}(x) := \mathbb{E}_x\Big[\int_0^{\infty} e^{-qt} h(X_t \pm L_t^{\pi})\mathrm{d}t + \int_{[0,\infty)} e^{-qt}\beta \mathrm{d}L_t^{\pi}\Big],$$

  by choosing the optimal control $\pi^* = (L_t^*)$ among the set of nonincreasing, cadlag, adapted processes $\pi = (L_t^{\pi})$.
- In this talk, we restrict the set of admissible strategies to be absolutely continuous (w.r.t. the Lebesgue measure):

$$L_t^{\pi} = \int_0^t \ell_s^{\pi}\mathrm{d}s, \quad t \geq 0,$$

  with ℓ^{π} restricted to take values in $[0, \delta]$ uniformly in time.

Spectrally Negative Lévy Processes

Defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $X = \{X_t; t \geq 0\}$ be a spectrally negative Lévy process, i.e.

1. The paths are almost surely right continuous with left limits.
2. For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.
3. For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.
4. Jumps are almost surely negative (spectrally negative).
5. Not the negative of a subordinator.

Examples include Brownian motion, (compound) Poisson, stable processes, CGMY, NIG, variance gamma, meromorphic Lévy processes etc.

Reflected & Refracted Lévy Processes

- Reflected Lévy processes: $U_t^s := X_t - L_t^s$ where

$$L_t^s := \sup_{0 \le t' \le t} (X_{t'} - s) \vee 0, \quad t \ge 0.$$

- Refracted Lévy processes (Kyprianou and Loeffen, Annales de l'Instut Henri Poincaré, 2009)
  - A strong Markov process given by the unique strong sol'n to the SDE

$$\mathrm{d}U_t^b = \mathrm{d}X_t - \delta \mathbf{1}_{\{U_t^b > b\}} \mathrm{d}t, \quad t \ge 0.$$

  - Namely, U^b progresses like X below the boundary b while it does like

$$Y_t := X_t - \delta t, \quad t \ge 0,$$

above b.

Objective

We show the optimality of a "refraction strategy" under the absolutely continuous condition.

- We minimize

$$v^{\pi}(x) := \mathbb{E}_x\Big[\int_0^{\infty} e^{-qt} h(X_t \pm L_t^{\pi})\mathrm{d}t + \int_{[0,\infty)} e^{-qt}\beta \mathrm{d}L_t^{\pi}\Big],$$

  over the set Π_δ given by

$$L_t^{\pi} = \int_0^t \ell_s^{\pi}\mathrm{d}s, \quad t \geq 0,$$

  with ℓ^{π} restricted to take values in $[0,\delta]$ uniformly in time.
- The optimally controlled process becomes the refracted Lévy process $U_t^{b^*}$, with a suitable choice b^*,

$$\mathrm{d}U_t^{b^*} = \mathrm{d}X_t - \delta \mathbf{1}_{\{U_t^{b^*} > b^*\}}\mathrm{d}t, \quad t \geq 0.$$

SN Lévy Processes and Laplace Exponents

- Given a SN Lévy process $X = \{X_t; t \geq 0\}$, the Laplace exponent is

$$\psi(\theta) := \log \mathbb{E}[e^{\theta X_1}] = \gamma\theta + \frac{\sigma^2}{2}\theta^2 + \int_{(-\infty,0)} (e^{\theta z} - 1 - \theta z \mathbf{1}_{\{z>-1\}})\nu(dz), \quad \theta \geq 0.$$

- ν is a Lévy measure such that $\int_{(-\infty,0)}(1 \wedge z^2)\nu(dz) < \infty$.
- It has paths of bounded variation if and only if $\sigma = 0$ and $\int_{(-1,0)} |z| \, \nu(dz) < \infty$.
- For the case of bounded variation, we can write

$$\psi(\theta) = \tilde{\gamma}\theta + \int_{(-\infty,0)} (e^{\theta z} - 1)\nu(dz), \quad \theta \geq 0,$$

with $\tilde{\gamma} := \gamma - \int_{(-1,0)} z \, \nu(dz)$.

Scale Functions

- Recall that X is a spectrally negative Lévy process with Laplace exponent $\psi(s) = \log \mathbb{E}\left[e^{sX_1}\right]$.
- Fix any $q > 0$, there exists a function called the q-scale function

$$W^{(q)} : \mathbb{R} \to [0, \infty),$$

which is zero on $(-\infty, 0)$, continuous and strictly increasing on $[0, \infty)$, and is characterized by the Laplace transform:

$$\int_0^\infty e^{-sx} W^{(q)}(x) \mathrm{d}x = \frac{1}{\psi(s) - q}, \qquad s > \Phi(q),$$

where

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}.$$

Scale Functions (Cont'd)

Let us define the first down- and up-crossing times, respectively, by

$$\tau_a^- := \inf\{t \ge 0 : X_t < a\},$$
$$\tau_b^+ := \inf\{t \ge 0 : X_t > b\}.$$

Then we have for any $b > 0$

$$\mathbb{E}_x\left[e^{-q\tau_b^+}1_{\{\tau_b^+<\tau_0^-\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(b)},$$
$$\mathbb{E}_x\left[e^{-q\tau_0^-}1_{\{\tau_b^+>\tau_0^-\}}\right] = Z^{(q)}(x) - Z^{(q)}(b)\frac{W^{(q)}(x)}{W^{(q)}(b)},$$

where

$$\overline{W}^{(q)}(x) := \int_0^x W^{(q)}(y)\mathrm{d}y,$$
$$Z^{(q)}(x) := 1 + q\overline{W}^{(q)}(x).$$

Back to the Problem

- Define Π_δ as the set of absolutely continuous strategies π given by adapted processes

$$L_t^\pi = \int_0^t \ell_s^\pi \mathrm{d}s, \quad t \geq 0,$$

  with ℓ^π restricted to take values in $[0, \delta]$ uniformly in time.
- The objective is to minimize the net present value (NPV) of the expected total costs

$$v^\pi(x) := \mathbb{E}_x \Big[\int_0^\infty \mathrm{e}^{-qt} (h(U_t^\pi) + \beta \ell_t^\pi) \mathrm{d}t \Big],$$

  where

$$U_t^\pi := X_t - L_t^\pi, \quad t \geq 0.$$

- Remark: β can be negative – the case $U_t^\pi := X_t + L_t^\pi$ is also covered.

Assumptions

Key Assumption: We assume h is convex

We need the following so the refracted process
will not be a subordinator above and below the threshold.

- For the case X is of bounded variation, we assume that $\tilde{\gamma} - \delta > 0$.

We want the following properties to exchange derivatives over integrals and also to take limits in verification.

- We assume that there exists $\bar{\theta} > 0$ such that $\int_{(-\infty,-1]} \exp(\bar{\theta}|z|)\nu(\mathrm{d}z) < \infty$ – the jump size should not have a heavy tail.
- We assume h has at most polynomial growth in the tail.

Refracted Lévy Processes

- A strong Markov process given by the unique strong sol'n to the SDE

$$\mathrm{d}U_t^b = \mathrm{d}X_t - \delta \mathbf{1}_{\{U_t^b > b\}} \mathrm{d}t, \quad t \geq 0.$$

- Namely, U^b progresses like X below the boundary b while it does like

$$Y_t := X_t - \delta t, \quad t \geq 0,$$

above b.

- The corresponding NPV of the total costs

$$v_b(x) := \mathbb{E}_x \Big[\int_0^\infty \mathrm{e}^{-qt} (h(U_t^b) + \beta\delta \mathbf{1}_{\{U_t^b > b\}}) \mathrm{d}t \Big], \quad x \in \mathbb{R},$$

can be written using the scale functions of X and Y.

Scale Functions

- We use $W^{(q)}$ and $\mathbb{W}^{(q)}$ for the scale functions of X and Y, respectively. Namely, these are defined by

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q),$$
$$\int_0^\infty \mathrm{e}^{-\theta x} \mathbb{W}^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\theta) - \delta\theta - q}, \quad \theta > \varphi(q),$$

  where

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\},$$
$$\varphi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) - \delta\lambda = q\}.$$

- By the strict convexity of ψ, we have the strict inequality

$$\varphi(q) > \Phi(q) > 0.$$

NPV under Refraction Strategies

By the resolvent measure obtained by Kyprianou and Loeffen (2010),

$$
\begin{aligned}
v_b(x) &= v_b^{(1)}(x) + v_b^{(2)}(x)\mathbf{1}_{\{x>b\}}, \\
v_b^{(1)}(x) &:= \mathrm{e}^{\Phi(q)(x-b)}\frac{\varphi(q)-\Phi(q)}{\delta\Phi(q)}\Big[\int_0^\infty h(y+b)\mathrm{e}^{-\varphi(q)y}\mathrm{d}y + \frac{\beta\delta}{\varphi(q)}\Big] \\
&\quad + \int_{-\infty}^0 h(y+b)\Big[\mathrm{e}^{\Phi(q)(x-b)}\frac{\varphi(q)-\Phi(q)}{\Phi(q)} \\
&\qquad \times \int_0^\infty \mathrm{e}^{-\varphi(q)z} W^{(q)\prime}(z-y)\mathrm{d}z - W^{(q)}(x-b-y)\Big]\mathrm{d}y, \\
v_b^{(2)}(x) &:= \int_0^\infty (h(y+b)+\beta\delta)\Big\{\mathrm{e}^{-\varphi(q)y}M(x;b) - \mathbb{W}^{(q)}(x-b-y)\Big\}\mathrm{d}y \\
&\quad + \delta\int_{-\infty}^0 h(y+b)\Big\{M(x;b)\int_0^\infty \mathrm{e}^{-\varphi(q)z}W^{(q)\prime}(z-y)\mathrm{d}z \\
&\quad - \int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z-b-y)\mathrm{d}z\Big\}\mathrm{d}y, \\
M(x;b) &:= (\varphi(q)-\Phi(q))\mathrm{e}^{-\Phi(q)b}\int_b^x \mathrm{e}^{\Phi(q)z}\mathbb{W}^{(q)}(x-z)\mathrm{d}z, \quad x>b.
\end{aligned}
$$

Candidate Threshold b^*

- We see (once we confirm that $\partial/\partial b$ and $\partial/\partial x$ can go into the integrals):

$$\frac{\partial}{\partial b} v_b(x) = u_b(x)$$

where

$$u_b(x) := \mathbb{E}_x\Big[\int_0^\infty \mathrm{e}^{-qt} h'(U_t^b)\mathrm{d}t\Big] - v_b'(x), \quad x, b \in \mathbb{R}.$$

- We shall pursue b^* such that $u_{b^*}(x)$ vanishes or equivalently

$$v_{b^*}'(x) = \mathbb{E}_x\Big[\int_0^\infty \mathrm{e}^{-qt} h'(U_t^{b^*})\mathrm{d}t\Big].$$

Candidate Threshold (Cont'd)

After tedious calculation, for all $x, b \in \mathbb{R}$,

$$u_b(x) = \Big[\frac{\varphi(q) - \Phi(q)}{\delta\Phi(q)} \mathrm{e}^{\Phi(q)(x-b)} + \mathbf{1}_{\{x>b\}}(M(x;b) - \mathbb{W}^{(q)}(x-b))\Big] I(b),$$

where

$$\begin{aligned} I(b) = &\frac{\varphi(q) - \Phi(q)}{\varphi(q)} \int_0^\infty h'(y+b)\mathrm{e}^{-\varphi(q)y}\mathrm{d}y \\ &+ \frac{\delta}{\varphi(q)}\Big[\int_{-\infty}^0 h'(y+b)\Big\{(\varphi(q) - \Phi(q)) \\ &\times \int_0^\infty \mathrm{e}^{-\varphi(q)z} W^{(q)\prime}(z-y)\mathrm{d}z - \Phi(q)W^{(q)}(-y)\Big\}\mathrm{d}y - \beta\Phi(q)\Big]. \end{aligned}$$

Candidate Threshold (Cont'd)

By the convexity of h, the function I is *nondecreasing*.

- Hence we can define the limits $I(\infty) := \lim_{b\uparrow\infty} I(b)$ and $I(-\infty) := \lim_{b\downarrow-\infty} I(b)$ – we set our candidate optimal threshold level b^* to be the *largest root* of $I(b) = 0$ if $I(-\infty) < 0 < I(\infty)$.
- If $I(\infty) \leq 0$, we let $b^* = \infty$
- If $I(-\infty) \geq 0$, we let $b^* = -\infty$.

For example, for the case $h(y) := \alpha y^2$ for some $\alpha > 0$,

$$b^* = \beta q/(2\alpha) + \mathbb{E}(-\underline{X}_{e_q}) - \varphi(q)^{-1}.$$

Verification Lemma

Lemma (Verification lemma)

Suppose $\hat{\pi} \in \Pi_\delta$ *is such that* $v_{\hat{\pi}}$ *is sufficiently smooth on* $\mathbb{R}$ *and satisfies*

$$\begin{cases} (\Gamma - q)v_{\hat{\pi}}(x) + h(x) \geq 0 & \text{if } v'_{\hat{\pi}}(x) \leq \beta, \\ (\Gamma - q)v_{\hat{\pi}}(x) - \delta(v'_{b^*}(x) - \beta) + h(x) \geq 0 & \text{if } v'_{\hat{\pi}}(x) > \beta. \end{cases}$$

Then $\hat{\pi}$ *is an optimal strategy and* $v(x) = v_{\hat{\pi}}(x)$ *for all* $x \in \mathbb{R}$.

Main Results

Theorem

v_{b^*} satisfies the above conditions, and hence the refraction strategy w/ refraction trigger level b^* is optimal.

Convergence to Reflection Strategy

We have

$$\tilde{v}(x;\delta) := \inf_{\pi \in \Pi_\delta} \mathbb{E}_x\Big[\int_0^\infty \mathrm{e}^{-qt}(h(Y_t + L_t^\pi) + \tilde{\beta}\ell_t^\pi)\mathrm{d}t\Big]$$
$$= v(x;\delta,-\tilde{\beta}) + \frac{\tilde{\beta}\delta}{q},$$

where $v(x;\delta,-\tilde{\beta})$ is the value function obtained above with X_t replaced with $X_t^{(\delta)} := Y_t + \delta t$ and β with $-\tilde{\beta}$.

Convergence to Reflection Strategy (Cont'd)

- Let Π_∞ be the set of admissible strategies w/o restrictions on the absolutely continuity. It is known as in Y. (arXiv, 2013) that

$$\tilde{v}(x;\infty) := \inf_{\pi\in\Pi_\infty} \mathbb{E}_x\Big[\int_{[0,\infty)} \mathrm{e}^{-qt}(h(Y_t + L_t^\pi)\mathrm{d}t + \tilde{\beta}\mathrm{d}L_t^\pi)\Big]$$
$$= -\tilde{\beta}\left(\overline{\mathbb{Z}}^{(q)}(x - b^*(\infty)) + \frac{\psi_Y'(0+)}{q}\right)$$
$$- \int_{b^*(\infty)}^{x} \mathbb{W}^{(q)}(x-y)h(y)\mathrm{d}y + \mathbb{Z}^{(q)}(x - b^*(\infty))\times$$
$$\left(\frac{\varphi(q)}{q}\int_0^\infty \mathrm{e}^{-\varphi(q)y}h(y + b^*(\infty))\mathrm{d}y + \frac{\tilde{\beta}}{\varphi(q)}\right).$$

- This is attained by the *reflected Lévy process* $Y_t + L_t^{b^*(\infty)}$ with

$$L_t^{b^*(\infty)} := \sup_{0\le t'\le t}((b^*(\infty)) - Y_{t'}) \vee 0, \quad t \ge 0.$$

Convergence to Reflection Strategy (Cont'd)

Proposition

We have $b^*(\delta) \xrightarrow{b\uparrow\infty} b^*(\infty)$.

Theorem

Uniformly in x *in compacts,* $\tilde{v}(x;\delta) \xrightarrow{\delta\uparrow\infty} \tilde{v}(x;\infty)$.

Numerical Results

- Let X be a spectrally negative process with i.i.d. phase-type distributed jumps of the form

$$X_t - X_0 = \tilde{\gamma} t + \sigma B_t - \sum_{n=1}^{N_t} Z_n, \quad 0 \le t < \infty,$$

  for some $\tilde{\gamma} \in \mathbb{R}$ and $\sigma \ge 0$. Here $B = \{B_t; t \ge 0\}$ is a standard Brownian motion, $N = \{N_t; t \ge 0\}$ is a Poisson process with arrival rate κ, and $Z = \{Z_n; n = 1, 2, \ldots\}$ is an i.i.d. sequence of phase-type-distributed random variables with representation $(m, \boldsymbol{\alpha}, \boldsymbol{T})$.
- For Z, we choose such that it appoximates the Weibull random variable with parameter $(2, 1)$.

Numerical Results (Cont'd)

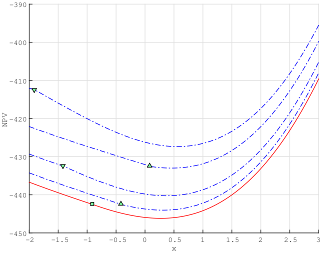


Figure: Plots of $v_b(x)$ for the case $\beta = -5$ (left) and $\beta = 5$ (right). Each panel shows $v_{b^*}(x)$ (solid) in comparison to $v_b(x)$ (dotted) for $b \in \{b^* - 1, b^* - 0.5, b^* + 0.5, b^*, b^* + 1\}$.

Convergence as $\delta \uparrow \infty$ (Cont'd)

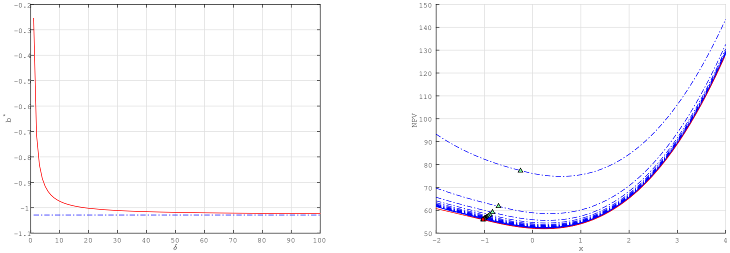


Figure: Plots of convergence as $\delta \uparrow \infty$. The left panel shows $b^*(\delta)$ for δ running from 1 to 100. The value of $b^*(\infty)$ is indicated by the dotted line. On the right panel, the functions $\tilde{v}(x;\delta)$ are shown as dotted lines for $\delta \in \{1, 2, \ldots, 20, 40, 60, 80, 100\}$.

Thank you. – kyamazak@kansai-u.ac.jp –