Regenerative process Monte Carlo methods

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Ongoing work with Krishna B. Athreya, Raoul Normand, & Vivekananda Roy

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Introduction

Let $(S, \mathcal{S}, \pi)$ be a measure space.

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Problem

An important problem in the theory and application of statistical methods to many areas of science and humanities is the estimation of means of functions with respect to some well-defined target distribution. i.e., to estimate $\lambda \equiv \int_S f d\pi$.

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An important problem in the theory and application of statistical methods to many areas of science and humanities is the estimation of means of functions with respect to some well-defined target distribution. i.e., to estimate $\lambda \equiv \int_S f d\pi$.

If π is a probability measure (i.e. $\pi(S) = 1$), there are currently two well-known statistical procedures for this.

Monte Carlo methods

(I) One is based on iid sampling from π, also called IID Monte Carlo.

- generate a sequence of iid random variable $\{X_n\}_{n\geq 0}$ with distribution π, and then use the sample mean $\hat{\lambda}_n \equiv \frac{1}{n}\sum_{i=1}^{n} f(X_i)$ to estimate λ. This, by the law of large number, converges with probability 1 to λ as $n \to \infty$.

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(II) An alternative statistical procedure is known as Markov chain Monte Carlo (MCMC).

- generate a Markov chain $\{X_n\}_{n\geq 0}$ with π as its invariant distribution. Then the law of large numbers for such chains says that for any initial value of X_0, the "time average" $\sum_{j=1}^{n} f(X_j)/n$ converges almost surely to the "space average" $\lambda = \int_S f d\pi$.

Limit theorem

In (I) and (II), in addition to $\int_S |f| d\pi < \infty$, if we also assume $\int_S f^2 d\pi < \infty$, then a central limit theorem (CLT) holds asserting that

$$\frac{\sqrt{n}(\hat{\lambda}_n - \lambda)}{\hat{\sigma}_n} \xrightarrow{\text{d}} \mathcal{N}(0,1) \text{ as } n \to \infty$$

where

$$\hat{\sigma}_n^2 \equiv \frac{1}{n} \sum_{i=1}^{n} f^2(X_i) - \hat{\lambda}_n^2.$$

$\mathcal{N}(0,1)$ denotes the standard normal distribution.

Monte Carlo methods

Both the IID Monte Carlo (IIDMC) and Markov chain Monte Carlo (MCMC) require that the target distribution π is a probability distribution i.e. $\pi(S) = 1$, or at least a totally finite measure.

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Question

Are there such Monte Carlo methods, i.e., statistical tools that could provide satisfactory (i.e. consistent) estimator of means with respect to measures π that are infinite?

Monte Carlo methods

- It is to be noted that MCMC methods can be implemented even when π is not a finite measure as they depend only on conditional distributions of π (i.e. $\pi(\cdot|x)$) which could be well-defined probability distributions.

- Recently, Athreya & Roy have shown that the standard Monte Carlo methods are not applicable for estimating λ in the case of improper targets, that is, when $\pi(S) = \infty$. In particular, they showed

$$\sum_{i=1}^{n} f(X_i)/n \longrightarrow 0 \text{ with probability 1.}$$

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$$\sum_{i=1}^{n} f(X_i)/n \longrightarrow 0 \text{ with probability } 1.$$

- Athreya & Roy provided consistent estimators of λ based on *regenerative sequences* of random variables whose canonical measure π can be finite or infinite.

Regenerative sequence

Definition

A sequence of random variables $\{X_n\}_{n\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with values in a measurable space $(S, \mathcal{S})$ is called *regenerative* if there exists integer valued random times $T_0 = 0 < T_1 < T_2 < T_3 < \cdots$ such that the excursions $\eta_j \equiv \{X_i : T_j \leq i < T_{j+1},\ T_{j+1} - T_j\}$ for $j \geq 1$ are iid.

The random times $\{T_n\}_{n\geq 1}$ are called *regeneration times*.

Regenerative sequence

Definition

The *canonical measure* or *occupation measure* for such a regenerative sequence is defined as:

$$\pi(A) \equiv \mathbb{E}\left(\sum_{j=T_0}^{T_1-1} \mathbb{1}_A(X_j) \right) \text{ for } A \in \mathcal{S}.$$

Regenerative sequence Monte Carlo (RSMC)

Theorem 1 (Athreya-Roy)

Let π *be the occupation measure for regenerative sequence* $\{X_n\}_{n\geq 0}$ *with regeneration times* $\{T_n\}_{n\geq 0}$. *Let* $N_n = k$ *if* $T_k \leq n < T_{k+1}$, *for* $n \geq 0, k \geq 0$, *that is,* N_n *is the number of regenerations by time* n. *Suppose*

$$\mathbb{E}\Big(\sum_{j=T_0}^{T_1-1} |f(X_j)|\Big) = \int_S |f|\mathrm{d}\pi < \infty \tag{1}$$

then

$$\lambda_n := \frac{\sum_{j=0}^{n} f(X_j)}{N_n} \xrightarrow{a.s.} \lambda := \int_S f\mathrm{d}\pi$$

Theorem 2 (Athreya-Roy)

Assume that additionally to (1):

$$\mathbb{E}\Big(\sum_{j=T_0}^{T_1-1} |f(X_j)|\Big) < \infty \qquad (1)$$

that

$$\mathbb{E}\left(\sum_{j=T_0}^{T_1-1} |f(X_i)|\right)^2 < \infty. \qquad (2)$$

Then we have a CLT for the estimator

$$\frac{\lambda_n - \lambda}{\sigma_{N_n}}\sqrt{N_n} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1),$$

where $\sigma_k^2 = \frac{1}{k}\sum_{j=1}^{k} U_j^2 - \left(\frac{1}{k}\sum_{j=1}^{k} U_j\right)^2$ and $U_i = \sum_{j=T_i}^{T_{i+1}-1} f(X_j)$

Theorem 1 & 2 (Athreya-Roy)

Suppose $\{X_n\}_{n\geq 0}$ is a regenerative sequence with occupation measure π, and N_n is the number of regenerations by time n.

- If
$$\mathbb{E}\Big(\sum_{j=T_0}^{T_1-1} |f(X_j)|\Big) < \infty, \text{ then} \tag{1}$$
$$\lambda_n := \frac{\sum_{j=0}^{n} f(X_j)}{N_n} \xrightarrow{\text{a.s.}} \lambda := \int_S f \mathrm{d}\pi$$
- If
$$\mathbb{E}\left(\sum_{j=T_0}^{T_1-1} |f(X_i)|\right)^2 < \infty, \text{ then} \tag{2}$$
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Hence, it would be interesting to get sufficient conditions on f for (1) and (2) to hold.

Simple symmetric random walk on $\mathbb{Z}$ (Athreya-Roy)

Take X_i to be the simple symmetric random walk on $\mathbb{Z}$ starting at $X_0 = 0$. That is,

$$X_{n+1} = X_n + \delta_{n+1}, \quad n \geq 0$$

where $\{\delta_n\}_{n\geq 1}$ are iid with distribution $\mathbb{P}(\delta_1 = +1) = 1/2 = \mathbb{P}(\delta_1 = -1)$.

$N_n = \sum_{j=0}^{n} \mathbb{1}_{(X_j=0)}$ is the number of visits to zero by $\{X_j\}_{j=0}^{n}$.

π is the counting measure: $\{\pi(i) = 1 : i \in \mathbb{Z}\}$.

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Let $f : \mathbb{Z} \to \mathbb{R}$

(1) and (2) read

$$\sum_{j\in\mathbb{Z}} |f(j)|\pi(j) < +\infty \tag{3}$$

$$\sum_{j\in\mathbb{Z}} |f(j)|\sqrt{|j|}\pi(j) < +\infty \tag{4}$$

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In other words,

- under (3),

$$\lambda_n := \frac{\sum_{j=0}^{n} f(X_j)\pi(X_j)}{N_n} \xrightarrow{\text{a.s.}} \lambda := \sum_{i\in\mathbb{Z}} f(i)\pi(i)$$

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- under (4), the general result shows that we have a CLT

$$\frac{\lambda_n - \lambda}{\sigma}\sqrt{N_n} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1)$$

where $\sigma^2 = \mathbb{E}\left[\sum_{j=0}^{T_1-1} f(X_j)\pi(X_j)\right]^2 - \lambda^2$, and
$T_1 = \inf\{n : n \geq 1, X_n = 0\}$.

From discrete space to continuous space

Definition

Let $\{X(t) : t \geq 0\}$ be a continuous time regenerative stochastic process with regenerative times $T_0 = 0 < T_1 < T_2 < T_3 < \cdots$. Let

$$\pi(A) = \mathbb{E}\Big(\int_0^{T_1} \mathbb{1}_A(X(u))\, du \Big)$$

where $A \in \mathcal{S}$. Then $\pi(\cdot)$ is called the *canonical measure* or *occupation measure* of the process.

Brownian motion

We can do the same with the Brownian motion as in SSRW.

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Let $X(t) \equiv B(t), t \geq 0$ where $\{B(t) : t \geq 0\}$ is a standard Brownian motion. Let $T_0 = 0$, and

$$
\begin{aligned}
T_1 &= \inf\{t : t > 0, \exists\ 0 < s < t, \text{ such that } B(s) = 1, B(t) = 0\} \\
T_{k+1} &= \inf\{t : t > T_k, \exists\ T_k < s < t, \text{ such that } B(s) = 1, B(t) = 0\}.
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Then $\{B(t) : t \geq 0\}$ is regenerative with regenerative times $\{T_k\}_{k\geq 0}$.

Brownian motion

The assumptions corresponding to (1) and (2) are

$$\mathbb{E}\left(\int_0^{T_1} |f(B_s)|\,\mathrm{d}s\right) < +\infty \tag{5}$$

and

$$\mathbb{E}\left[\left(\int_0^{T_1} |f(B_s)|\,\mathrm{d}s\right)^2\right] < +\infty. \tag{6}$$

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Local time of Brownian motion

Let $\{L_t^0(x) \equiv L_t(x) : t \geq 0, x \in \mathbb{R}\}$ be the local time process of Brownian motion $B(t)$ (starting at 0) at level x up to time t.

That is, with probability 1, for any Borel set $A \subset \mathbb{R}$ and $t \geq 0$,

$$\int_0^t \mathbb{1}_A\big(B(s)\big)\,\mathrm{d}s = \int_A L_t(x)\mathrm{d}x$$

Local time of Brownian motion

The occupation time formula provides

$$\int_0^{T_1} |f(B_s)| \,\mathrm{d}s = \int_{\mathbb{R}} |f(x)| L_{T_1}(x) \,\mathrm{d}x.$$

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For the p^{th} moment, $1 \leq p < \infty$, we have

$$\begin{aligned}\left[\mathbb{E}\left(\int_0^{T_1} |f(B_s)|\,\mathrm{d}s\right)^p\right]^{1/p} &= \left[\mathbb{E}\left(\int_{\mathbb{R}} |f(x)| L_{T_1}(x)\,\mathrm{d}x\right)^p\right]^{1/p} \\ &\leq \int_{\mathbb{R}} |f(x)| \left[\mathbb{E}(L_{T_1}(x))^p\right]^{1/p}\,\mathrm{d}x\end{aligned}$$

where the inequality is Minkowski's inequality for integrals.

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where the inequality is Minkowski's inequality for integrals.
Hence, we need to compute the first two moments of $L_{T_1}(x)$.

Marginal distribution for $L_{T_1}(x)$

Theorem

Let Y_1 and Y_2 be independent chi-squared random variables with two degrees of freedom, i.e. $Y_1 \overset{(d)}{=} Y_2 \overset{(d)}{=} Z_1^2 + Z_2^2$ for independent standard normal variables Z_1 and Z_2. Assume forever that Y_1 and Y_2 are independent of B. Then the following hold.

1. *For $x > 1$, $L_{T_1}(x) \overset{(d)}{=} \mathbb{1}_{\{\tau_x \le T_1\}} x Y_1$.*
2. *For $x \in [0,1]$, $L_{T_1}(x) \overset{(d)}{=} xY_1 + (1-x)Y_2$.*
3. *For $x < 0$, $L_{T_1}(x) \overset{(d)}{=} \mathbb{1}_{\{\tau_x \le \tau_1\}} (1-x) Y_1$.*

where $\tau_x^0 = \tau_x = \inf\{t \ge 0, B(t) = x\}$ is the hitting time of x by Brownian motion starting at 0.

Case ① : $x > 1$

- If $\{\tau_x > T_1\}$, then $L_{T_1}(x) = 0$. This happens if, starting from 1, the BM returns to 0 before hitting x.

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- Now, condition on $\{\tau_x \le T_1\}$. Then

$$L_{T_1}(x) \overset{(d)}{=} L^x_{\tau^x_0}(x) \overset{(d)}{=} L_{\tau_{-x}}(0) \overset{(d)}{=} L_{\tau_x}(0) \overset{(d)}{=} xL_{\tau_1}(0)$$

where the 1st equality is by the Markov property, the 2nd is shift-invariance, the 3rd symmetry, the 4th the scaling property.

Case ① : $x > 1$ continued...

Further, the first Ray-Knight theorem asserts that the process $(L_{\tau_1}(1-a))_{a\in[0,1]}$ is a squared Bessel(2) process, i.e. has the distribution of

$$\left(B_1(a)^2 + B_2(a)^2, a \in [0,1]\right),$$

where B_1 and B_2 are independent standard Brownian motions. In particular,

$$L_{\tau_1}(0) \overset{(d)}{=} B_1(1)^2 + B_2(1)^2 \overset{(d)}{=} Y_1.$$

We can thus conclude that, conditionally on $\{\tau_x \leq T_1\}$,

$$L_{T_1}(x) \overset{(d)}{=} xL_{\tau_1}(0) \overset{(d)}{=} xY_1.$$

We obtain the first case of Theorem.

Case ②: $x \in [0, 1]$

By continuity, $\tau_x \leq \tau_1 \leq T_1$ and thus the local time accumulated on $[0, T_1]$ is the sum of

(i) the local time $L_{\tau_1}(x)$ accumulated on $[0, \tau_1]$,

(ii) the local time accumulated on $[\tau_1, T_1]$.

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(ii) the local time accumulated on $[\tau_1, T_1]$.

By the Markov property, (ii) is independent of (i) and has the distribution of $L^1_{\tau^1_0}(x)$, and by translation invariance and symmetry, we have:

$$L^1_{\tau^1_0}(x) \overset{(d)}{=} L_{\tau_{-1}}(x-1) \overset{(d)}{=} L_{\tau_1}(1-x)$$

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$$L^1_{\tau^1_0}(x) \overset{(d)}{=} L_{\tau_{-1}}(x-1) \overset{(d)}{=} L_{\tau_1}(1-x)$$

Thus,

$$L_{T_1}(x) \overset{(d)}{=} \tilde{L}_{\tau_1}(x) + \hat{L}_{\tau_1}(1-x)$$

for $\tilde{L}, \hat{L}$ independent copies of L. By the first Ray-Knight theorem again, we conclude

$$L_{T_1}(x) \overset{(d)}{=} (1-x)Y_1 + xY_2$$

and this proves the second case.

Case ③: $x < 0$

- On the event $\{\tau_x > \tau_1\}$, we have $\tau_x > T_1$ by continuity, and thus $L_{T_1}(x) = 0$.

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- On the event $\{\tau_x > \tau_1\}$, we have $\tau_x > T_1$ by continuity, and thus $L_{T_1}(x) = 0$.
- On the other hand, on $\{\tau_x \leq \tau_1\}$, we have

$$L_{T_1}(x) = L_{\tau_1}(x) \overset{(d)}{=} L^x_{\tau_1^x}(x) \overset{(d)}{=} L_{\tau_{1-x}}(0) \overset{(d)}{=} (1-x)L_{\tau_1}(0) \overset{(d)}{=} (1-x)Y_1,$$

where the second equality is the Markov property, the third translation invariance, then the scaling property and the Ray-Knight theorem. We conclude as in the first case.

Moments of $L_{T_1}(x)$

Corollary

For any $x \in \mathbb{R}$, it holds that

$$\mathbb{E}(L_{T_1}(x)) = 2$$

and

$$\mathbb{E}\left(L_{T_1}(x)^2\right) = \begin{cases} 8(1-x) & \text{for } x < 0, \\ 8(x^2 - x + 1) & \text{for } x \in [0,1], \\ 8x & \text{for } x > 1. \end{cases}$$

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Therefore, $\mathbb{E}\left(\int_0^{T_1} |f(B_s)|\, \mathrm{d}s\right) \leq \int_{\mathbb{R}} |f(x)| \mathbb{E}\left(L_{T_1}(x)\right)\, \mathrm{d}x = 2\int_{\mathbb{R}} |f(x)|\, \mathrm{d}x,$

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Therefore, $\mathbb{E}\left(\int_0^{T_1} |f(B_s)|\,\mathrm{d}s\right) \leq \int_{\mathbb{R}} |f(x)|\mathbb{E}\left(L_{T_1}(x)\right)\,\mathrm{d}x = 2\int_{\mathbb{R}} |f(x)|\,\mathrm{d}x$,

$$\left[\mathbb{E}\left(\int_0^{T_1} |f(B_s)|\,\mathrm{d}s\right)^2\right]^{\frac{1}{2}} \leq \int_{\mathbb{R}} |f(x)|\left[\mathbb{E}(L_{T_1}(x))^2\right]^{\frac{1}{2}}\,\mathrm{d}x \leq C\int_{\mathbb{R}} |f(x)|\sqrt{|x|}dx.$$

- Hence, to have $\mathbb{E}\left(\int_0^{T_1}|f(B_s)|\,\mathrm{d}s\right) < +\infty$, it is necessary and sufficient to have

$$\int_{\mathbb{R}} |f(x)|\,\mathrm{d}x < +\infty.$$

- To have $\mathbb{E}\left[\left(\int_0^{T_1}|f(B_s)|\,\mathrm{d}s\right)^2\right] < +\infty$, it suffices to have

$$\int_{\mathbb{R}} |f(x)|\sqrt{|x|}\,\mathrm{d}x < +\infty.$$

Limit theorem for the Brownian motion case

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$, $\int_{\mathbb{R}} |f(x)| \,\mathrm{d}x < +\infty$, *and* $\lambda = \int_{\mathbb{R}} f(x) \,\mathrm{d}x$. *Assume moreover that* $\int_{\mathbb{R}} |f(x)| \sqrt{|x|} \,\mathrm{d}x < +\infty$, *and define*

$$\lambda^*(t) = \frac{1}{N(t)} \int_0^t f(B(s)) \,\mathrm{d}s.$$

Then

1. $\lambda^*(t) \to \lambda$ *a.s.;*
2. $$\frac{\lambda^*(t) - \lambda}{\sigma} \sqrt{N(t)} \xrightarrow{\mathrm{d}} \mathcal{N}(0, 1), \quad t \to +\infty;$$

where $\sigma^2 = \mathbb{E}\left(\int_0^{T_1} f(B(s)) \,\mathrm{d}s\right)^2 - \lambda^2$.

bayarlalaa !!!