Branching diffusions on the boundary and the interior of balls

submitted by

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for the degree of Doctor of Philosophy

of the

University of Bath

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September 2013

Für Mama und Papa

Summary

The object of study in this thesis are branching diffusions which arise as stochastic models for evolving populations.
Our focus lies on studying branching diffusions in which particles or, more generally, mass gets killed upon exiting a ball. In particular, we investigate the way in which populations can survive within a ball and how the mass evolves upon its exit from an increasing sequence of balls.

Acknowledgements

First and foremost, I would like to thank my supervisors Andreas Kyprianou and Simon Harris. Without their guidance and advice throughout the years this thesis would not have been possible. I am grateful for their continuous encouragement which kept me going through the difficult times - thank you so much for your support.

I would like to thank everyone who has made my time in Bath unforgettable: Andrea, Melina, Tania and Jane for lots of lovely Sunday afternoons, my office mate Christoph, and Alex whose friendship throughout the last three year has been invaluable to me. I would also like to thank my friends in Cambridge who have made my stays there so enjoyable. And finally a huge thank you to my friends at home for all the good times we have shared together: to my best friend Angi, to Sonja, Ricky, Kerstin, Tina, Ann Katrin, Schlucker and Joachim.

A special thank you goes to Bati. Thank you for your love and support, for many 'scientific discussions', and for cheering me up when I need it most.

Finally, I would like to thank my family, my parents and my sisters Carolin, Sandra and Julia, who have always encouraged and supported me. None of this would have been possible without you. This thesis is dedicated to you - thank you.

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Chapter 1

Introduction

Branching processes are stochastic population models which rely on an explicit description of the individuals' lifetime and reproduction law. The underlying assumption is that individuals live and reproduce independently of each other and identically in distribution. Spatial branching processes generalise this basic idea by allowing particles to move in space, according to some Markov process, during their lifetime and keeping track of their spatial positions.
Spatial branching processes became increasingly popular not only through their obvious connection to population biology but also because of their close relation to non-linear differential equations. There is a vast literature on various aspects of spatial branching processes and we refer the reader to the monographs by Dawson [12], Le Gall [53], Etheridge [27] and Dynkin [21] for an introduction and overview.

The aim of this thesis is to investigate branching diffusions in which individuals move according to a diffusion and get *killed* upon exiting a ball or, in one-dimension, a fixed size interval. This first chapter gives an introduction to branching diffusions with killing. We address some of the problems that will be treated in this thesis, with pointers to the subsequent chapters. We also discuss some related results which motivated our investigations.

1.1 Branching diffusions in balls

In this section, we introduce branching Brownian motion and super-Brownian motion in a ball as a model of an evolving population. We also hint at the relations to differential equations at the end of this section.

1.1.1 Branching Brownian motion in a ball

Branching Brownian motion (BBM) is the canonical example of a branching diffusion. It is easily described as a model of an evolving population. In this population, each individual or particle moves like a Brownian motion in $\mathbb{R}^d$ and reproduces according to a continuous-time Galton-Watson process. The branching Brownian motion in a ball is a variation of this process in which particles get killed upon exiting a fixed size ball. More precisely, we have the following description of its evolution.

Definition 1.1 (Branching Brownian motion in a ball)**.** *Let $\mathbb{B}$ be an open ball in $\mathbb{R}^d$ and $x \in \mathbb{B}$. Let A be a $\mathbb{N}_0$-valued random variable with distribution $\{q_k; k = 0, 1, ...\}$ and finite mean m.*

- *We begin with one initial particle at position x.*
- *Each particle moves according to a Brownian motion in $\mathbb{R}^d$, independently of all other particles.*
- *When a particle exits the ball $\mathbb{B}$, it gets killed instantaneously and removed from the population.*
- *While moving inside $\mathbb{B}$, each particle has an exponentially distributed lifetime with parameter $\beta > 0$.*
- *At the end of its lifetime, a particle dies and is replaced by a random number of offspring particles which has the same distribution as A.*
- *Once born, offspring particles move off independently from their birth position, repeating the stochastic behaviour of their parent.*

Note the distinction in terminology: a particle gets *killed* upon exiting $\mathbb{B}$ and a particle *dies* at the end of its exponential lifetime (at which it is then replaced by a random number of offspring).
We let N_t be the set of and $|N_t|$ be the number of particles which are alive at time t. For any $u \in N_t$ we denote by $x_u(t)$ its spatial position at time t. Denote by $\mathcal{M}_a(\mathbb{B})$ the space of atomic measures on $\mathbb{B}$ with finitely many atoms each carrying unit mass. Then the BBM in the ball $\mathbb{B}$ is the $\mathcal{M}_a(\mathbb{B})$-valued process $X = (X_t, t \geq 0)$ where

$$X_t = \sum_{u \in N_t} \delta_{x_u(t)}.$$

We denote the law of X initiated from one particle at $x \in \mathbb{B}$ by $P_x^{\mathbb{B}}$. Of course, we can start the process from any finite configuration of particles in $\mathbb{B}$ such that each of them initiates an independent BBM in a ball from its spatial position. If X has such an initial configuration $\nu \in \mathcal{M}_a(\mathbb{B})$, we write $P_\nu^{\mathbb{B}}$ for its law (with the simplification $P_x^{\mathbb{B}}$ instead of $P_{\delta_x}^{\mathbb{B}}$).

In short, the BBM in a ball can be characterised by two main features: the *spatial motion* of the particles and the *branching activity*. The latter is governed by a continuous-time Galton-Watson process with a branching mechanism F of the form

$$F(s) = \beta \sum_{k=0}^{\infty} (q_k s^k - s), \quad s \in [0,1]. \tag{1.1.1}$$

For example, if $q_2 = 1$ we are in the case of *binary branching* and $F(s) = \beta(s^2 - s)$. Of course, we can generalise the model above in several ways. For instance, instead of Brownian motion in a ball we can allow the single particle motion to be any diffusion in a domain $D \subset \mathbb{R}^d$ or, even more generally, a Markov process on any Polish space, leading to branching diffusions and Markov branching processes respectively; the branching rate β can depend on the spatial position of a particle and the offspring number A can depend on the spatial position of the branch point which gives a space-dependent branching mechanism of the form

$$F(s,y) = \beta(y) \sum_{k=0}^{\infty} (q_k(y) s^k - s), \quad s \in [0,1], y \in D.$$

In some sense, through the BBM in the ball we have already met an example of a space-dependent branching mechanism since only the particle within $\mathbb{B}$ reproduce. To emphasize this we could write its branching mechanism as

$$F(s,y) = F(s)\mathbf{1}_{(y \in \mathbb{B})}, \quad s \in [0,1] \text{ and } y \in \mathbb{R}^d,$$

where $F(s)$ is as in (1.1.1). The indicator $\mathbf{1}_{(y \in \mathbb{B})}$ ensures that only particles within $\mathbb{B}$ reproduce. However, as particles get killed immediately upon exiting $\mathbb{B}$, this more complicated formulation is not necessary and we simply drop the indicator. If the spatial killing is removed in Definition 1.1 above, meaning that each particle moves like a Brownian motion in $\mathbb{R}^d$ and reproduces according to the branching mechanism F in (1.1.1), then we will call it a standard BBM. We denote its law by P_ν for an initial configuration $\nu \in \mathcal{M}_a(\mathbb{R}^d)$, the space of atomic measures in $\mathbb{R}^d$ with finitely many unit mass atoms.

Two important properties of the BBM in a ball follow from the description above, the *Markov property* and the *branching property*. The Markov property holds since the lifetimes are exponentially distributed and each particle follows a Brownian motion killed upon exiting $\mathbb{B}$ which is itself Markovian.

Definition 1.2 (The branching property). *For two initial distributions $\nu_1, \nu_2 \in \mathcal{M}_a(\mathbb{B})$ and $t \geq 0$, the law of X_t under $P^{\mathbb{B}}_{\nu_1+\nu_2}$ is equal to the law of the independent sum $X^{(1)}_t + X^{(2)}_t$, where $X^{(1)}_t$ and $X^{(2)}_t$ are independent copies of X_t under $P^{\mathbb{B}}_{\nu_1}$ respectively $P^{\mathbb{B}}_{\nu_2}$.*

The branching property follows from our description since the evolution of each particle is independent from all other particles and all particles have the same stochastic behaviour.

1.1.2 Super-Brownian motion in a ball

A super-Brownian motion is a measure-valued process which can be seen as the short lifetime and high density limit of a BBM. Let us sketch the approximation for the super-Brownian motion in a ball $\mathbb{B} \subset \mathbb{R}^d$. Denote by $\mathcal{M}_F(\mathbb{B})$ the space of finite measures on $\mathbb{B}$ and let $\nu \in \mathcal{M}_F(\mathbb{B})$. In the n-th approximation step,

- we start with a $Poisson(n\nu)$ number of initial particles, each of which has mass $1/n$;
- each particle moves according to a Brownian motion in $\mathbb{R}^d$;
- a particles gets killed upon hitting the boundary $\partial\mathbb{B}$ and is removed from the process;
- while a particle stays within $\mathbb{B}$, it has an exponentially distributed lifetime with rate $\beta^{(n)}$, typically $\beta^{(n)} = n\beta$ for a $\beta > 0$;
- when a particle dies it is replaced by a random number of offspring according to a distribution $\{q_k^{(n)}, k \geq 0\}$.

We assume that $\beta^{(n)}$ and $\{q_k^{(n)}, k \geq 0\}$ are such that $\beta^{(n)} \to \infty$, as $n \to \infty$, and, for $\lambda \geq 0$,

$$\lim_{n\to\infty} nF^{(n)}\Big(1 - \frac{\lambda}{n}\Big) = \lim_{n\to\infty} n\beta^{(n)}\Big(\sum_{k\geq 0} q_k^{(n)}\big(1 - \frac{\lambda}{n}\big)^k - \big(1 - \frac{\lambda}{n}\big)\Big) = \psi(\lambda),$$

where ψ is of the form

$$\psi(\lambda) = -a\lambda + b\lambda^2 + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x)\, \Pi(\mathrm{d}x), \tag{1.1.2}$$

for some constants $a \in \mathbb{R}$, $b \geq 0$ and a measure Π concentrated on $(0, \infty)$ satisfying $\int_{(0,\infty)}(x \wedge x^2)\, \Pi(\mathrm{d}x) < \infty$.

In the n-th approximation step, denote by $\nu^{(n)}$ the random initial distribution at time 0. Then $\nu^{(n)}$ is of the form

$$\nu^{(n)} = \frac{1}{n}\sum_{i=1}^{N_0^{(n)}} \delta_{x_i},$$

where x_i, $i = 1, .., N_0^{(n)}$, are the atoms of a Poisson random measure on $\mathbb{B}$ with intensity measure $n\nu$. Let $N_t^{(n)}$ be the set of particles alive at time t, and for any particle $u \in N_t^{(n)}$ we denote by $x_u^{(n)}(t)$ its spatial position at time t. Then we set

$$Y_t^{(n)} = \frac{1}{n} \sum_{u \in N_t^{(n)}} \delta_{x_u^{(n)}(t)}$$

and define $Y^{(n)} = (Y_t^{(n)}, t \geq 0)$ as the branching particle system in the n-th approximation. The sequence of random initial measures $\nu^{(n)}$ converges weakly to ν and it can then be shown that there exists an $\mathcal{M}_F(\mathbb{B})$-valued process $Y = (Y_t, t \geq 0)$ such that $Y^{(n)} \to Y$ in law, in the sense of $\mathcal{M}_F(\mathbb{B})$-valued càdlàg functions, as $n \to \infty$. Rigorous constructions of this fashion are given, for instance, in Ethier and Kurtz [29], Fitzsimmons [32], Dawson [12] and Dynkin [18, 19, 20, 21].
The limit process Y then inherits the Markov property and the branching property from the BBMs in the approximation procedure.

Definition 1.3. *We call the limiting $\mathcal{M}_F(\mathbb{B})$-valued process Y a super-Brownian motion in a ball $\mathbb{B}$ with branching mechanism ψ and initial configuration ν and denote its law by $\mathbf{P}_\nu^{\mathbb{B}}$.*

Let $a \in \mathbb{R}$, $b \geq 0$ and let Π be a measure concentrated on $(0, \infty)$ satisfying $\int_{(0,\infty)} (x \wedge x^2)\ \Pi(\mathrm{d}x) < \infty$. Then for any triplet (a, b, Π), which determines the branching mechanism ψ in (1.1.2), one can choose $\beta^{(n)}$ and $\{q_k^{(n)}, k \geq 0\}$ in such a way that the super-Brownian motion obtained from the approximation construction above has branching mechanism ψ.
As for the BBM in a ball, we can construct so-called superdiffusions and more general measure-valued branching Markov processes by replacing the Brownian motion in $\mathbb{B}$ with a diffusion in $\mathbb{R}^d$ or a Markov process respectively. Under certain regularity assumption, we can further allow the parameters a, b and the measure Π in (1.1.2) to depend on the state space or even on time, thus leading to space- and time-dependent branching mechanisms. The existence and construction of these generalisations are covered in the references given above.

It is often convenient to analytically characterise the distribution of a super-Brownian motion in $\mathbb{B}$ through its Laplace functional. We denote by $\mathcal{H} = (\mathcal{H}_t = \sigma(Y_s, s \leq t), t \geq 0)$ the filtration generated by Y and use the inner product notation $\langle f, \nu \rangle = \int_{\mathbb{B}} f \mathrm{d}\nu$ for a function f on $\mathbb{B}$ and a measure $\nu \in \mathcal{M}_F(\mathbb{B})$. The following proposition is a standard result in the theory of superdiffusions, see for instance Chapter 4 in Dynkin [21].

Proposition 1.4 (Markov branching property)**.** *Let $\nu \in \mathcal{M}_F(\mathbb{B})$. For any positive, bounded, measurable function g on $\mathbb{B}$ and $0 \leq s \leq t$,*

$$\mathbf{E}_\nu^{\mathbb{B}}[e^{-\langle g, Y_t \rangle} | \mathcal{H}_s] = e^{-\langle u_g(\cdot, t), Y_s \rangle}.$$

The Laplace functional u_g is the unique non-negative solution to

$$u(x,t) = \mathbb{E}_x^{\mathbb{B}}[g(\xi_t), t < T_{\mathbb{B}}] - \mathbb{E}_x^{\mathbb{B}} \int_0^{t \wedge T_{\mathbb{B}}} \psi(u(\xi_s, t-s))\, \mathrm{d}s, \qquad (1.1.3)$$

where $((\xi_t, t \geq 0), \mathbb{P}^{\mathbb{B}})$ is a Brownian motion with killing upon exiting $\mathbb{B}$ and $T_{\mathbb{B}} = \inf\{t > 0 : \xi_t \notin \mathbb{B}\}$ denotes its killing time.[1]

This characterisation is sometimes referred to as the *Markov branching property* and, as the name suggests, it is a consequence of the Markov and the branching property of Y. The distribution of any branching diffusion can be characterised by an integral equation similar in fashion to (1.1.3).
The integral equation in (1.1.3) is equivalent to the parabolic partial differential equation

$$\begin{aligned} \frac{1}{2}\Delta u(x,t) - \psi(u(x,t)) &= \frac{\partial}{\partial t} u(x,t) \qquad \text{in } \mathbb{B} \times (0,\infty) \\ u(x,0) &= g(x), \qquad x \in \mathbb{B} \\ u(x,t) &= 0 \qquad x \in \partial\mathbb{B}, t \geq 0 \end{aligned} \qquad (1.1.4)$$

in that solutions to the former also solve the latter equation and vice versa, cf. [17, 18, 21] for similar results; we give an example of how to translate an integral equation similar to (1.1.3) into a PDE in Section 3.4.
The relation between branching diffusions and differential equations, or the corresponding integral equations, was already noted for instance in Sevest'yanov [63], Skorohod [67] and Ikeda, Nagasawa and Watanabe [41, 42]. In the context of branching Brownian motion this connection is often credited to McKean [57]. He observed that the distribution of a one-dimensional standard BBM with binary branching mechanism can be characterised in terms of the semi-linear heat equation

$$\begin{aligned} \frac{1}{2}\frac{\partial^2}{\partial x^2} u(x,t) + \beta(u(x,t)^2 - u(x,t)) &= \frac{\partial}{\partial t} u(x,t) \qquad \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) &= f(x), \qquad x \in \mathbb{R} \end{aligned}$$

for a $[0,1]$-valued, measurable function f on $\mathbb{R}$, which is the celebrated Fisher-Kolmogorov-Petrovskii-Piscunov equation arising in population genetics. We will discover further relationships between branching diffusions and differential equations of this kind in the later chapters.

In the analytic study, it is sometimes more convenient to use Brownian motion with *absorption* upon exiting $\mathbb{B}$ as the underlying motion of Y, instead of

[1] The term $T_{\mathbb{B}}$ in (1.1.3) is included for clarity but is somewhat superfluous since $\mathbb{E}_x^{\mathbb{B}}[g(\xi_t)] = \mathbb{E}_x[g(\xi_t), t < T_{\mathbb{B}}]$ by definition.

Brownian motion with killing upon exiting $\mathbb{B}$. This brings the advantage of working with a conservative diffusion as the underlying motion. Conservativeness is not necessary for the analytical construction of superprocesses, see for instance [19, 21], but often assumed for convenience.
We can construct the super-Brownian motion with absorption upon exiting $\mathbb{B}$ through the approximating particle picture described above. The only difference is that, in this case, a particle gets absorbed upon exiting $\mathbb{B}$ and stops reproducing while it remains part of the population (unlike the killing case in which particles are removed from the population when they get killed). If we construct the super-Brownian motion in this way, using absorption rather than killing, then we can discard the 'absorbed' mass by restricting the resulting measures $Y_t, t \geq 0$, to the interior of $\mathbb{B}$. The restricted process then agrees with the super-Brownian motion in a ball obtained from the approximation construction with killing as described at the beginning of this section. If we work with Brownian motion with absorption as the underlying motion then for the PDE (1.1.4) the boundary condition simplifies to $u(x,t) = g(x)$ on $\{\mathbb{B} \times \{0\}\} \cup \{\partial\mathbb{B} \times [0,\infty)\}$.

1.2 Survival of branching diffusions in balls

A question that arises immediately is whether the BBM in a ball described in Definition 1.1 can survive forever. Before we give an answer to this question we recall some general results on survival and extinction probabilities for non-spatial branching processes.

1.2.1 Survival and extinction of Galton-Watson and continuous state branching processes

Consider an evolving population described by a standard BBM (ignoring the spatial killing in Definition 1.1). Then the process of the number of particles $(|N_t|, t \geq 0)$ is a Galton-Watson process with branching mechanism F given by (1.1.1). Survival of the standard BBM, that is the event $\{\forall t \geq 0 : N_t \neq \emptyset\}$, is thus the same as survival of the underlying Galton-Watson process. It is well known that the latter depends only on the mean offspring number $m = E[A]$ which we assumed to be finite. More precisely, the population becomes extinct in finite time, that is $N_t = \emptyset$ for large t, P-a.s., if $m \leq 1$ and the population has a positive survival probability if $m > 1$. In the former case the offspring distribution is called (sub)critical[2] while it is referred to as supercritical in the latter case.
The probability of extinction, say p^*, is given explicitly as $p^* = \inf\{s \in [0,1] :$

[2] $m = 1$ is the critical case.

$F(s) = 0\}$. We refer to Athreya and Ney [2] for these results and more on Galton-Watson processes.

In the setting of a standard super-Brownian motion (again ignoring spatial killing), the total mass process $(||Y_t||, t \geq 0)$ is a continuous-state branching process (CSBP) with branching mechanism ψ. Grey [34] proved that $||Y_t|| < \infty$ a.s. for all $t \geq 0$, if and only if the non-explosion condition

$$\int_{0+} |\psi(s)|^{-1} \,\mathrm{d}s = \infty \tag{1.2.1}$$

holds. From now on, we assume that (1.2.1) holds and that $\psi(\infty) = \infty$.
The expected total mass at time $t \geq 0$ when we start with unit mass at a point $x \in \mathbb{R}^d$ is given by

$$\mathbf{E}_{\delta_x}(||Y_t||) = e^{-\psi'(0+)t}.$$

Similar to the terminology employed for Galton-Watson processes, we call the branching mechanism ψ (sub)critical if $\psi'(0+) \geq 0$ and supercritical if $\psi'(0+) < 0$. This then refers to whether on average the process will decrease or increase.
The probability of *extinguishing*, namely the event $\{\lim_{t\to\infty} ||Y_t|| = 0\}$, can be expressed in terms of the largest root of ψ, that is $\lambda^* := \inf\{\lambda > 0 : \psi(\lambda) > 0\}$. Then λ^* is the survival rate of X in the sense that, for any $\nu \in \mathcal{M}_F(\mathbb{R}^d)$, the space of finite measures on $\mathbb{R}^d$ with compact support,

$$\mathbf{P}_\nu(\lim_{t\to\infty} ||Y_t|| = 0) = e^{-\lambda^* ||\nu||}.$$

Note that convexity of ψ and the assumption $\psi(\infty) = \infty$ ensure that λ^* is finite and strictly positive in the supercritical case and equal to 0 in the (sub)critical case.
We encounter a slight difference to Galton-Watson processes here in that the event of extinction, that is $\{\exists t \geq 0 : ||Y_t|| = 0\}$, can differ from the event of becoming extinguished. In fact, Grey [34] showed that both events agree a.s if and only if

$$\int^{\infty} |\psi(s)|^{-1} \,\mathrm{d}s < \infty \tag{1.2.2}$$

holds. If this condition fails, then the probability of extinction is equal to 0.
For an overview of the classical results for CSBPs stated here we refer the reader to [50], Chapter 10.

1.2.2 Survival of branching diffusions with spatial killing

The long-term behaviour of spatial branching processes becomes more interesting when we introduce a spatial killing barrier. We consider the case of supercritical branching mechanisms only since in the (sub)critical case the processes cannot survive regardless of the motion.

Branching Brownian motion with absorption

In the model of *branching Brownian motion with absorption*, as studied by Kesten [45], each particle moves like a Brownian motion in $\mathbb{R}_+$ with a drift $-\mu$, $\mu \geq 0$, and gets killed upon hitting the origin. Kesten discovered that this process has a strictly positive survival probability if $\mu < \sqrt{2(m-1)\beta}$ and dies out almost surely if $\mu \geq \sqrt{2(m-1)\beta}$.
Berestycki, Berestycki and Schweinsberg [3] study the survival probability of the BBM with absorption near criticality. They assume that each particle splits into two at rate 1 and thus the critical drift is $\sqrt{2}$. Set $L = \pi(\sqrt{2-\mu^2})^{-1}$. Denote by $p^\mu(x)$ the probability of survival of the BBM with absorption when it has drift $-\mu$ and is initiated from one particle at x. Theorem 3 in [3] states that there exists a constant $C > 0$ such that, for $x \in \mathbb{R}$,

$$p^\mu(x) \sim CLe^{-\mu(L-x)} \sin(\pi x/L) \quad \text{as} \quad \mu \uparrow \sqrt{2}, \tag{1.2.3}$$

where $\sim$ means that the ratio of left- and right-hand side tends to 1 in the limit. The result in [3] is in fact more general than this in that it allows x to be a function of μ provided that $L - x \to \infty$ as $\mu \uparrow \sqrt{2}$.

Branching Brownian motion in a strip

Motivated by the results on BBM with absorption at the origin, we are interested in finding asymptotics for the survival probability of the related model of *branching Brownian motion in a strip.* Here particles get killed when they exit a fixed-size interval. This is simply the one-dimensional version of the BBM in a ball described in Definition 1.1. For notational convenience, we write P^K for its law when the strip is of size K.
We assume again that particles have drift $-\mu$, $\mu \geq 0$. As a first result, we state in Proposition 2.1 in Chapter 2 that survival of the BBM in a strip is possible if and only if $\mu < \sqrt{2(m-1)\beta}$ and the size of the strip exceeds the critical width

$$K_0 := \frac{\pi}{\sqrt{2(m-1)\beta - \mu^2}}. \tag{1.2.4}$$

Subsequently, we study the probability of survival as the width of the interval shrinks to the critical value K_0. Let us denote by $p_K(x)$ the probability of survival of the BBM in the strip of size K initiated from one particle at $x \in (0, K)$. Theorem 2.4 in Chapter 2 says that, as $K \downarrow K_0$,

$$p_K(x) \sim C_K \sin(\pi x/K_0) e^{\mu x}, \tag{1.2.5}$$

uniformly for all x in the critical width strip where the constant C_K tends to 0 as $K \downarrow K_0$. In fact, we are able to find an explicit expression for C_K as

$$C_K = (K - K_0) \frac{(K_0^2\mu^2 + \pi^2)(K_0^2\mu^2 + 9\pi^2)}{12(m-1)\beta\pi K_0^3(e^{\mu K_0} + 1)}, \tag{1.2.6}$$

which is also given in Theorem 2.4. We should remark that it has not been possible so far to find an explicit expression for the constant C in (1.2.3) of the BBM with absorption at 0.

Spines and backbones

The proofs of the critical width in (1.2.4) and the asymptotics for p_K in (1.2.5) employ classical spine techniques which were developed in Chauvin and Rouault [11] and put into use through path-wise arguments in Lyons et al. [56] and Lyons [55]. Our arguments rely on a martingale change of measure on the probability space of the BBM in a strip under which one line of descent, the so-called *spine*, moves like a Brownian motion conditioned to stay in $(0, K)$, see Figure 1-1. The particles in the spine reproduce at an accelerated rate and according to a size-biased offspring distribution, which ensures that at least one child is born at any branch point. When the spine particle dies we choose one of its children to repeat the spine particles stochastic behaviour and to continue the path of the spine. Conditionally on the spine, all other children of the spine initiate a copy of X under the original measure P^K from the space-time position of their birth. Clearly, the process constructed in this way survives almost surely since the spine survives.
The critical width K_0 in (1.2.4) and the asymptotics for p_K in (1.2.5) can then be derived from computations depending essentially on the spine only.

In order to find the expression for C_K in (1.2.6), it is not enough to study only the spine. In fact, we have to take into consideration every infinite line of descent. For this reason, we decompose the BBM in a strip in such a way that we can identify all lines of descent which are infinite. As an illustration, consider a realisation of (X, P^K) and colour in blue all particles with an infinite line of descent and in red all remaining particles. On the event of survival, see Figure 1-2 (a), the blue particles form an infinite blue tree. More precisely, the blue tree is made up of all infinite lines of descent and does not contain any finite ones. The red particles form finite red trees which branch off the blue tree. On the event of extinction we only see one finite red tree as shown in Figure 1-2 (b).

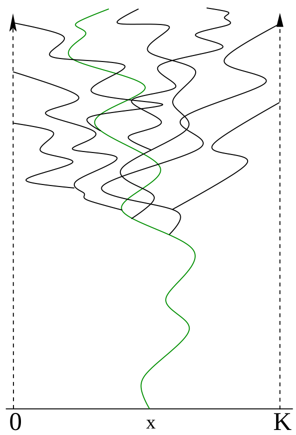
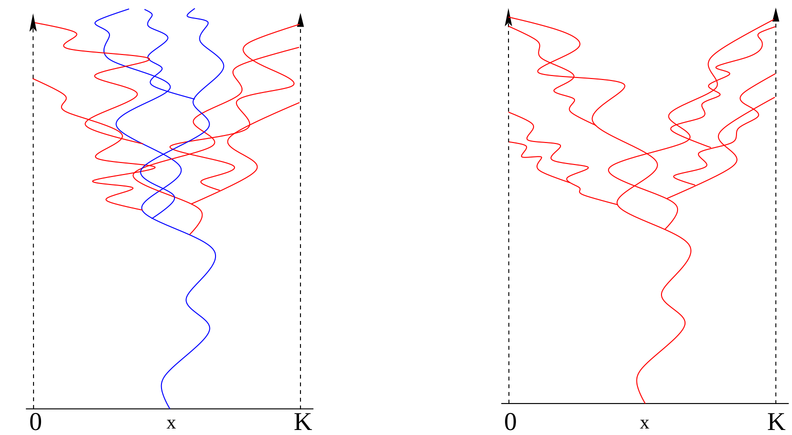


Figure 1-1: Sketch of a realisation of X under a change of measure with a spine (green).

The blue tree is a realisation of what we will call a *blue branching diffusion*, that is, a certain branching diffusion in which each particle's line of descent is infinite. In view of the picture above, we sometimes refer to the blue branching diffusion as the *backbone* of X. Note the following conceptual difference between backbones and spines. The backbone contains all infinite lines of descent and, on the event of survival, its particles make up an infinite subtree of (X, P^K). The previously discussed spine is only a single infinite line of descent of X under a change of measure.
The blue branching diffusion, or backbone, is a branching diffusion with a space-dependent branching mechanism in which the single particle motion is a diffusion with a space-dependent drift term. The single particle motion and the branching mechanism are such that the particles cannot reach the killing boundaries and, when a particle dies, it is replaced by at least two offspring particles. Therefore every line of descent of the backbone is infinite, just as in the colouring picture explained above.
The backbone is our main tool of analysis in the proof of (1.2.6) as it captures enough information about the evolution of (X, P^K) on survival. The explicit expression for C_K can eventually be derived from computations of the expected growth rate of the backbone.

While we highlighted above the conceptual difference between spines and backbones they are, at the same time, intimately connected to one another: Condition (X, P^K) on the event of survival, which is equivalent to conditioning on the presence of a backbone, and let us study the evolution of its backbone as we make the interval smaller, that is we take $K \downarrow K_0$. We see that the branching rate of the backbone slows down and it thins down to a single line of descent at criticality. In the limit, the single particle motion 'becomes' a Brownian motion



(a) On the event of survival

(b) On the event of extinction

Figure 1-2: Sketch of realisations of X with infinite lines of descent coloured in blue (backbone) and finite lines of descent in red.

conditioned to stay in $(0, K)$. Thus, conditional on survival, as we approach criticality, the backbone of X thins down to become a spine. This observation is key to Theorem 2.6 which presents a quasi-stationary limit result for the evolution of the BBM in a strip conditioned on survival as the width approaches the critical value K_0.

Our techniques are robust enough to transfer these results into the setting of the super-Brownian motion in a strip. Further generalisations to higher dimensions and different underlying single particle motions are discussed in the final section of Chapter 2.

1.3 Branching diffusions on the boundary of balls

While the previous section dealt with the particles or, more generally, the mass of a branching diffusion which stayed in the interior of a ball we are now interested in the configuration of particles or mass as it exits a ball. These configurations are described by so-called exit measures.

1.3.1 Exit measures from balls

Ultimately, we will be interested in studying exit measures for super-Brownian motion but, for simplicity, we begin with the discrete particle picture setting. Consider the BBM in a ball $\mathbb{B} \subset \mathbb{R}^d$ as in Definition 1.1. For each particle which gets killed, we assign a unit mass to the point of the boundary $\partial\mathbb{B}$ at which it got killed. The *exit measure* of X from $\mathbb{B}$ is the measure $X_{\mathbb{B}}$ on $\partial\mathbb{B}$ consisting of

these unit masses. Alternatively, we can assume that particles 'freeze', and stop reproducing, when they exit the ball. Then $X_{\mathbb{B}}$ is the measure supported on $\partial\mathbb{B}$ consisting of Dirac masses at the positions of the 'frozen' particles.
Neveu [58] and Chauvin [10] introduced so-called stopping lines to formalise the idea of 'freezing' particles upon exiting a domain. Depending on the terminology used, a stopping line can refer to the set of particles which get 'frozen' or to the set of times at which they 'freeze'. Although the term stopping line is more commonly employed with regard to BBM, we will use the term exit measure here to emphasize that we consider a random measure indicating the spatial positions of the 'frozen' particles, rather than the set of 'frozen' particles or their 'freezing' times.
One way to adopt the concept of exit measures to the setting of super-Brownian motion is to define the exit measure from a ball through the approximating particle picture discussed in Section 1.1.2: the exit measure from a ball $\mathbb{B}$ can be defined for the discrete particle system in the n-th approximation, and then one has to verify that as $n \to \infty$, those discrete exit measures converge to a limit $Y_{\mathbb{B}}$, see Dynkin [20, 21]. Loosely speaking, the measure $Y_{\mathbb{B}}$ consists of the accumulated mass which got 'frozen' upon exiting the ball $\mathbb{B}$. Formally, the exit measure $Y_{\mathbb{B}}$ is supported on $\partial\mathbb{B}$ and it is analytically characterised as follows.

Proposition 1.5 ([20])**.** *Let $\nu \in \mathcal{M}_F(\bar{\mathbb{B}})$. For any positive, bounded, measurable function f on $\partial\mathbb{B}$,*

$$\mathbf{E}_\nu[e^{-\langle f, Y_{\mathbb{B}}\rangle}] = e^{-\langle v_f, \nu\rangle}.$$

The Laplace functional v_f is the unique non-negative solution to

$$v_f(x) = \mathbb{E}_x[f(\xi_{T_{\mathbb{B}}})] - \mathbb{E}_x\Big[\int_0^{T_{\mathbb{B}}} \psi(v_f(\xi_t))\ \mathrm{d}t\Big],$$

where $((\xi_t, t \geq 0), \mathbb{P}_x)$ is an $\mathbb{R}^d$-Brownian motion with $\xi_0 = x$ and $T_{\mathbb{B}} := \inf\{t > 0 : \xi_t \notin \mathbb{B}\}$ denotes its first exit time from $\mathbb{B}$.

The above characterisation yields yet another connection of super-Brownian motion with differential equations in that the Laplace functional v_f is the solution to the non-linear Dirichlet problem

$$\begin{aligned} \frac{1}{2}\Delta v(x) - \psi(v(x)) &= 0 \quad \text{in } \mathbb{B}, \\ v &= f \quad \text{on } \partial\mathbb{B}. \end{aligned} \tag{1.3.1}$$

Let us go back to the discrete setting considered at the beginning of this section. For our purposes, it is convenient to assume that individuals continue to evolve after their ancestral line has exited the ball and we therefore consider a standard BBM X without spatial killing. For any realisation of X, we can then construct a family of exit measures which keeps track of the configuration of particles as they exit a sequence of increasing balls. To this end, let $D_s := \{x \in \mathbb{R}^d : ||x|| < s\}$ be the open ball of radius $s > 0$ around the origin and let X_{D_s} be the exit measure of X from D_s. Dynkin [20] shows that the sequence of exit measures $(X_{D_s}, s > 0)$ has the Markov branching property, cf. Proposition 1.4.

The analogous process in the super-Brownian motion setting is the sequence of exit measures $(Y_{D_s}, s > 0)$ where Y_{D_s} is the exit measure of Y from D_s, $s > 0$. If we construct this sequence using the approximating particle picture, then we would expect that the Markov branching property is preserved. Indeed, Dynkin and Kuznetsov [23] prove that $(Y_{D_s}, s > 0)$ possesses the Markov branching property, see Proposition 3.1 in Chapter 3 for details.

1.3.2 The total mass on the boundary of balls

In Chapter 3, we study the process $Z = (Z_s, s > 0)$ where

$$Z_s := ||Y_{D_s}||, s > 0,$$

which keeps track of the total mass that is 'frozen' when it first hits the boundary of the balls D_s, $s > 0$. The radii s of the sequence of balls is now the time-parameter of Z. Clearly, if $Z_s = 0$ for some s, then this implies that the standard super-Brownian motion Y is contained in a ball of radius s and thus that the range of Y is a bounded subset of $\mathbb{R}^d$.

Sheu [64, 65] studies the asymptotic behaviour of Z and shows that the events $\{Y \text{ has compact support}\}$ and $\{\exists s > 0 : Z_s = 0\}$ agree a.s. In [64], it is found that these events agree with the event $\{\lim_{s\to\infty} Z_s = 0\}$ if and only if the branching mechanism ψ of Y satisfies

$$\int^{\infty} \frac{1}{\sqrt{\int_{\lambda^*}^{\theta} \psi(\lambda)\, d\lambda}}\, \mathrm{d}\theta < \infty. \tag{1.3.2}$$

Otherwise, the events $\{\exists s > 0 : Z_s = 0\}$ and $\{Y \text{ has compact support}\}$ both have probability 0.

Sheu's approach studies the process Z by means of differential equations, exploiting the connection pointed out in (1.3.1). In particular, the unusual condition (1.3.2) arises as an analytic condition under which solutions to the non-linear Dirichlet problem

$$\begin{aligned} \frac{1}{2}\Delta v(x) - \psi(v(x)) &= 0 \qquad \text{in } D_s, \\ v &= \infty \qquad \text{on } \partial D_s, \end{aligned} \tag{1.3.3}$$

exist. The connection to the probability that $||Y_{D_s}|| = 0$ and equivalently $Z_s = 0$ for some s, is made as follows. For $f = \theta$ constant, the solution to the non-linear Dirichlet problem (1.3.1), with $\mathbb{B}$ replaced by D_s, is $v_\theta(x) = -\log \mathbf{E}_x(e^{-\theta||Y_{D_s}||})$. Equation (1.3.3) emerges from (1.3.1) by taking $f \equiv \theta \to \infty$. As $\theta \to \infty$, we get $\lim_{\theta\to\infty} v_\theta(x) = -\log \mathbf{P}_x(||Y_{D_s}|| = 0)$ and we expect this to be a solution to (1.3.3).

In Chapter 3, we seek for a probabilistic description of the process Z and an interpretation of Sheu's condition (1.3.2). To begin with, it is not difficult to see that $Z = (Z_s, s > 0)$ is a time-inhomogeneous continuous-state branching process, meaning that it is a $[0,\infty]$-valued Markov branching process with càdlàg paths and that its law is determined by a branching mechanism which changes as time evolves (see Theorem 3.2 for details).
As part of Theorem 3.2, we characterise the branching mechanism of Z, say $\Psi(s,\cdot)$, where s indicates the time-dependence, through the differential equation

$$\begin{aligned}\frac{\partial}{\partial s}\Psi(s,\theta) + \frac{1}{2}\frac{\partial}{\partial\theta}\Psi^2(s,\theta) + \frac{d-1}{s}\Psi(s,\theta) &= 2\psi(\theta) \qquad s>0,\ \theta\in(0,\infty)\\ \Psi(s,\lambda^*) &= 0, \qquad s>0.\end{aligned}$$

We then study its behaviour as the radius of the balls tends to infinity. In particular, in Theorem 3.4 in Chapter 3, we show that $\Psi(s,\theta)$ converges towards the branching mechanism $\Psi_\infty(\theta) := \lim_{s\to\infty}\Psi(s,\theta)$ with

$$\Psi_\infty(\theta) = 2\,\mathrm{sgn}(\psi(\theta+))\sqrt{\int_{\lambda^*}^{\theta}\psi(\lambda)\,\mathrm{d}\lambda}, \qquad \theta \geq 0,$$

of a time-homogeneous continuous-state branching process as $s \to \infty$. It is now evident that the analytic condition (1.3.2) is Grey's classical condition (1.2.2) for the limiting branching mechanism $\Psi_\infty(\theta)$. That means that the compact support condition for the super-Brownian motion Y is the extinction vs. extinguishing condition for the limiting CSBP with branching mechanism $\Psi_\infty(\theta)$.
Let us briefly comment on the relation between conditions (1.2.2) and (1.3.2). Sheu [65] observed the following implication. If ψ satisfies Sheu's condition (1.3.2) then it also satisfies Grey's condition (1.2.2). He concludes that if the range of Y is bounded then Y has to become extinct. However, the converse is not necessarily true and (1.2.2) does not imply (1.3.2). This is saying that, Y can have an unbounded range even though it becomes extinct in finite time. A branching mechanism for which this behaviour can occur is $\psi(\lambda) = \int_0^1 (e^{-\lambda x} - 1 + \lambda x)\, x^{-2}|\log x|\,\mathrm{d}x$, which is given as an example in [65].

1.4 A guide to notation

Table 1.1 contains some of the notation which is frequently used throughout the thesis. The guideline regarding our notation for the different processes and their laws is as follows.
The law of a diffusion ξ is denoted by $\mathbb{P}$, laws of branching particle diffusions X are denoted by P and laws of superdiffusions Y by $\mathbf{P}$.
Superscripts indicate the domain whose boundary is the killing barrier. If there is no superscript then there is no killing.
In Chapter 2, we consider one-dimensional processes only and write $\mathbb{P}^K$, P^K, etc. instead of $\mathbb{P}^{(0,K)}$, $P^{(0,K)}$ and so on. We further need a second superscript which refers to whether the underlying process is blue (B), red (R) , dressed (D) or coloured (C). 'Spine' processes will be denoted by $\mathbb{Q}$, Q respectively $\mathbf{Q}$ with appropriate superscripts. See Chapter 2 and the references in Table 1.1 for definitions of these processes. Note that the underlying Brownian motion of all processes in Chapter 2 has a fixed drift $-\mu$, $\mu \geq 0$. To keep notation simple, this is not indicated by the probability measures.

References in Table 1.1 point to the page where the definition can be found.

Table 1.1: Table of notation

	NOTATION USED THROUGHOUT	
$\mathbb{B}$	open ball in $\mathbb{R}^d$	
$(\xi, \mathbb{P}^{\mathbb{B}})$	Brownian motion killed upon exiting $\mathbb{B}$	
$(X, P^{\mathbb{B}})$	BBM with killing upon exiting $\mathbb{B}$	10
$(Y, \mathbf{P}^{\mathbb{B}})$	super-Brownian motion with killing upon exiting $\mathbb{B}$	13
$(\mathcal{G}_t, t \geq 0)$	natural filtration of diffusion ξ	
$(\mathcal{F}_t, t \geq 0)$	natural filtration of branching diffusion X	
$(\mathcal{H}_t, t \geq 0)$	natural filtration of superdiffusion Y	
F	branching mechanism of BBM	11
ψ	branching mechanism of super-Brownian motion	12
T_D	first exit time of a diffusion ξ from a domain $D \subset \mathbb{R}^d$	
$\mathcal{M}_a(D)$	space of finite atomic measures with support in $D \subseteq \mathbb{R}$	
$\mathcal{M}_F(D)$	space of finite, compactly supported measures on $D \subseteq \mathbb{R}$	
	CHAPTER 2	
$(\xi, \mathbb{P}^K)$	Brownian motion with drift $-\mu$ and killing upon exiting $(0, K)$	
$(\xi, \mathbb{P}^{R,K})$	'red' diffusion in $(0, K)$	49

$(\xi, \mathbb{P}^{B,K})$	'blue' diffusion in $(0, K)$	52
L	infinitesimal generator of $(\xi, \mathbb{P}^K)$	28
$L^{R,K}$	infinitesimal generator of $(\xi, \mathbb{P}^{R,K})$	47
$L^{B,K}$	infinitesimal generator of $(\xi, \mathbb{P}^{B,K})$	51
$(\xi, \mathbb{Q}^K)$	Brownian motion conditioned to stay in $(0, K)$	35
(X, P^K)	BBM with killing upon exiting $(0, K)$	
$(X, P^{R,K})$	red branching diffusion in $(0, K)$	47
$(X, P^{B,K})$	blue branching diffusion in $(0, K)$	55
$(X, P^{D,K})$	dressed blue branching diffusion in $(0, K)$	51
$(X, P^{C,K})$	two-colour branching diffusion in $(0, K)$	56
$F^{R,K}$	branching mechanism of $(X, P^{R,K})$	47
$F^{B,K}$	branching mechanism of $(X, P^{B,K})$	51
(X, Q^K)	BBM with a spine conditioned to stay in $(0, K)$	36
(X^*, Q^*)	BBM with a spine conditioned to stay in $(0, K_0)$	32
$(Y, \mathbf{P}^K)$	super-Brownian motion with killing upon exiting $(0, K)$	
$(Y, \mathbf{P}^{R,K})$	'red' superdiffusion in $(0, K)$	82
$(Y, \mathbf{P}^{D,K})$	'dressed' superdiffusion in $(0, K)$	83
$(Y, \mathbf{P}^{C,K})$	'coloured' superdiffusion in $(0, K)$	83
$(Y^*, \mathbf{P}^*)$	superdiffusion with spine conditioned to stay in $(0, K_0)$	84
	CHAPTER 3	
D_s	ball of radius s in $\mathbb{R}^d$	
$Y_{D_s}, s > 0$	exit measure of Y from D_s	90
$\mathcal{H}_{D_s}$	σ-algebra generated by $(Y_{D_r}, r \leq s)$	
$v_f(x, s)$	Laplace functional of $(Y_{D_s}, s > 0)$	90
$(Z, \bar{\mathbf{P}}_r)$	process of total mass of $(Y_{D_s}, s \geq r)$	91
$\Psi(r, \theta)$	time-dependent branching mechanism of Z	91
$u(r, s, \theta)$	Laplace functional of Z	91
$(Z^\infty, \bar{\mathbf{P}}^\infty)$	limiting CSBP as $r \to \infty$	92
$\Psi_\infty(\theta)$	branching mechanism of Z^∞	92
$(R, \mathbb{P}^R)$	d-dimensional Bessel process	

1.5 Publication details

This thesis consists of two self-contained chapters, excluding the introduction. The chapters are based on the following papers which form the references [37] and [40].

CHAPTER 2

[37] *Branching Brownian motion in a strip: Survival near criticality*, with Simon C. Harris and Andreas E. Kyprianou.
Preprint arXiv:1212.1444, *submitted.*

CHAPTER 3

[40] *The total mass of super-Brownian motion upon exiting balls and Sheu's compact support condition*, with Andreas E. Kyprianou.
Preprint arXiv:1308.1656, *submitted.*

Chapter 2

Branching Brownian motion in a strip: Survival near criticality

We consider a branching Brownian motion with linear drift in which particles are killed on exiting the interval $(0, K)$ and study the evolution of the process on the event of survival as the width of the interval shrinks to the critical value at which survival is no longer possible. We combine spine techniques and a backbone decomposition to obtain exact asymptotics for the near-critical survival probability. This allows us to deduce the existence of a quasi-stationary limit result for the process conditioned on survival which reveals that the backbone thins down to a spine as we approach criticality.
Our investigations are motivated by recent work on survival of near critical branching Brownian motion with absorption at the origin by Aïdékon and Harris in [1] as well as the work of Berestycki et al. in [4] and [3].

2.1 Introduction and main results

2.1.1 Introduction and main results

We consider a branching Brownian motion X in which each particle performs a Brownian motion with drift $-\mu$, for $\mu \geq 0$, and is killed on hitting 0 or K. All living particles undergo branching at constant rate β to be replaced by a random number of offspring particles, A, where A is an independent random variable with distribution $\{q_k; k = 0, 1, ...\}$ and finite mean $m > 1$ and such that $E(A \log^+ A) < \infty$. Once born, offspring particles move off independently from their birth position, repeating the stochastic behaviour of their parent. This is simply the one-dimensional version of the branching Brownian motion in a ball in Definition 1.1 with an additional drift, a supercritical branching mechanism and the additional moment assumption $E(A \log^+ A) < \infty$.
In other words, the motion of a single particle is governed by the infinitesimal

generator

$$L = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \mu\frac{\mathrm{d}}{\mathrm{d}x}, \quad x \in (0, K), \tag{2.1.1}$$

defined for all functions $u \in C^2(0, K)$, the space of twice continuously differentiable functions on $(0, K)$, with $\mathrm{u}(0+) = u(K-) = 0$. We further recall from Section 1.1.1 that the branching activity is characterised by the branching mechanism

$$F(s) = \beta(G(s) - s), \quad s \in [0, 1], \tag{2.1.2}$$

where $G(s) = \sum_{k=0}^{\infty} q_k s^k$ is the probability generating function of A.
We will henceforth use the notation introduced in Section 1.1.1. We remind the reader that, throughout this section, the law P^K_ν of X with initial configuration $\nu \in \mathcal{M}_a(0, K)$ refers to a BBM in $(0, K)$ with fixed *drift* $-\mu$ and we will refer to X as a P^K-branching diffusion. Similarly, $(\xi, \mathbb{P}^K_x)$ will henceforth denote a Brownian motion with drift $-\mu$ upon exiting the interval $(0, K)$ starting from $x \in (0, K)$. We will sometimes neglect the dependence on the initial configuration and write P^K and $\mathbb{P}^K$ without a subscript.
For $x \in [0, K]$ we define the survival probability $p_K(x) = P^K_x(\zeta = \infty)$ where $\zeta = \inf\{t > 0 : |N_t| = 0\}$ is the time of extinction. As a first result, we identify the critical width K_0 below which survival is no longer possible.

Proposition 2.1. *If* $\mu < \sqrt{2(m-1)\beta}$ *and* $K > K_0$ *where*

$$K_0 := \frac{\pi}{\sqrt{2(m-1)\beta - \mu^2}},$$

then $p_K(x) > 0$ *for all* $x \in (0, K)$*; otherwise* $p_K(x) = 0$ *for all* $x \in [0, K]$.

Proposition 2.1 is essentially not new as, in the case of binary branching, it is already implicit in Theorem 3 in Engländer and Kyprianou [25], see also [24], Example 14. Nevertheless, we will give a short proof of Proposition 2.1 in Section 2.3 as the techniques used therein will be important later. In particular, the proof uses a spine argument, decomposing X into a Brownian motion conditioned to stay in $(0, K)$ dressed with independent copies of (X, P^K) which 'immigrate' along its path.
Our aim is to study the evolution of the P^K-branching diffusion on the event of survival. We will therefore develop a decomposition which identifies the particles with infinite genealogical lines of descent, that is, particles which produce a family of descendants which survives forever. To illustrate this, for each realisation of X, let us colour blue all particles with an infinite line of descent and colour red all remaining particles. Thus, on the event of survival, the resulting picture consists of a blue tree 'dressed' with red trees whereas, on the event of extinction,

we see a red tree only. Recall Figure 1-2 in Chapter 1 for an illustration.
For the moment, let us consider the binary branching case only in which each particle splits into two. Intuitively, the branching rates of the *blue branching diffusion*, corresponding to the blue tree in the colouring picture, and the *red branching diffusion*, corresponding to a red tree, can be derived as follows. Suppose a particle dies and is replaced by two offspring at position y. For each of the offspring, the probability that it has an infinite genealogical line of descent is the survival probability $p_K(y)$, independent of the other offspring particle. Thus, each offspring particle is blue with probability $p_K(y)$ and hence with probability $p_K(y)^2$ both offspring particles are blue. Therefore, given the parent particle is blue, it branches into two blue particles at rate $\beta\frac{p_K(y)^2}{p_K(y)} = \beta p_K(y)$.
Likewise, the probability that both offspring are red is $(1-p_K(y))^2$ and hence, given the parent particle is red, it branches into two red particles at rate $\beta(1-p_K(y))$.
Further, with probability $2p_K(y)(1-p_K(y))$ one blue and one red particle are born. Then, given a particle is blue, it branches into one blue and one red particle at rate $2\beta(1-p_K(y))$. We call such a branching event an *immigration*.
Concerning the motion of the red and blue particles, we claim that red particles move according to a diffusion with spatial drift coefficient $-(\mu+\frac{p_K'(y)}{1-p_K(y)})$, while blue particles have spatial drift coefficient $-(\mu-\frac{p_K'(y)}{p_K(y)})$, where p_K' is the derivative of p_K. Each of these diffusions is the result of a h-transform of L using $h=1-p_K$ and $h=p_K$ respectively. In fact we will show that the laws of the red and blue branching diffusions arise from martingale changes of measure which, on the level of infinitesimal generators, correspond to the aforementioned h-transforms (see Lemma 2.7 for the relation between martingale changes of measure and h-transforms).
Continuing with the binary branching case, the following two results characterise the red and the blue branching diffusion as well as the *dressed blue branching diffusion* which corresponds to the blue tree dressed with red trees in the colouring picture, see again Figure 1-2.

Proposition 2.2 (The red branching diffusion in the binary branching case)**.** *Let $K>K_0$. In the case of binary branching, the red branching diffusion on $(0,K)$ has single particle motion according to the infinitesimal generator*

$$L^{R,K} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \left(\mu + \frac{p_K'(y)}{1-p_K(y)}\right)\frac{\mathrm{d}}{\mathrm{d}y} \quad on \quad (0,K),$$

for $u\in C^2(0,K)$ with $u(0+)=u(K-)=0$, and each particle branches into two at space-dependent rate $\beta(1-p_K(y))$, $y\in(0,K)$.

Theorem 2.3 (The dressed blue branching diffusion in the binary branching case)**.** *Let $K>K_0$. In the case of binary branching, the dressed blue branching* *diffusion on* $(0, K)$ *starting from an initial particle at* $x \in (0, K)$ *evolves as follows.*

(i) From x*, we run a blue branching diffusion* X^B*, that is a branching diffusion in which the single particle movement has infinitesimal generator*

$$L^{B,K} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \left(\mu - \frac{p'_K(y)}{p_K(y)}\right)\frac{\mathrm{d}}{\mathrm{d}y} \quad on \quad (0, K),$$

defined for all $u \in C^2(0, K)$*, and each particle branches into two at space-dependent rate* $\beta p_K(y)$*,* $y \in (0, K)$*.*

(ii) Conditionally on X^B*, along the trajectory of each particle in* X^B*, an immigrant occurs at space-dependent rate* $2\beta(1 - p_K(y))$*,* $y \in (0, K)$*. Each immigrant initiates a red branching diffusion from the space-time position of its birth.*

It follows from the colouring picture that conditioning X on the event of survival is the same as conditioning the initial particle of the coloured tree on being blue. Likewise, conditioning the initial particle on being red corresponds to conditioning X on becoming extinct. Indeed, we will show that the red branching diffusion in Proposition 2.2 and the dressed blue branching diffusion in Theorem 2.3 arise naturally from changes of measure which condition (X, P^K) on the event of survival and the event of extinction respectively. In view of Theorem 2.3, we will sometimes refer to the blue branching diffusion X^B as the *backbone* and the theorem itself together with Proposition 2.2 as the *backbone decomposition.*
The corresponding results in the case of a general branching mechanism F are given as Proposition 2.13 and Theorem 2.14 in Section 2.4. In particular, we will see that a general branching mechanism induces a second type of immigration at the branching times of the backbone.
A significant convenience of these results is that the law of the P^K-branching diffusion conditioned on survival is the same as the law of the dressed blue branching diffusion. Thus, instead of studying the quasi-stationary limit $\lim_{K\downarrow K_0} P_x^K(\cdot|\zeta = \infty)$ it suffices to study the evolution of the dressed blue branching diffusion as $K \downarrow K_0$.
In a first step to understand the evolution of the dressed blue branching diffusion near criticality, we study the asymptotics of the survival probability p_K as $K \downarrow K_0$. For a first asymptotic result note that $u = 1 - p_K$ solves the differential equation $Lu + F(u) = 0$ on $(0, K)$ with boundary condition $u(0) = u(K) = 1$ (cf. Proposition 2.12). Near criticality we may assume that $p_K(x)$ is very small for a fixed x and neglecting all terms of order $(p_K(x))^2$ and higher we obtain the linearisation $Lp_K + (m - 1)\beta p_K = 0$. This suggests $p_K(x) \sim C_K \sin(\pi x/K_0)e^{\mu x}$. In fact, for a general branching mechanism F of the form in (2.1.2), we have the following result.

Theorem 2.4. *Define*

$$C_K := (K - K_0)\frac{(K_0^2\mu^2 + \pi^2)(K_0^2\mu^2 + 9\pi^2)}{12(m-1)\beta\pi K_0^3(e^{\mu K_0} + 1)}.$$

Then, as $K \downarrow K_0$*, we have* $C_K \downarrow 0$ *and*

$$p_K(x) \sim C_K \sin(\pi x/K_0)e^{\mu x}, \tag{2.1.3}$$

uniformly for all $x \in (0, K_0)$*. That is* $p_K(x)/(C_K \sin(\pi x/K_0)e^{\mu x})$ *converges to* 1 *uniformly for all* $x \in (0, K_0)$*, as* $K \downarrow K_0$*.*

In Section 2.5.1 we will prove a first part of Theorem 2.4, that is equation (2.1.3) without identifying C_K, in the fashion of [1] using spine techniques. Note that, in the sketch of the analytic argument above, we used that p_K asymptotically solves the linearisation $Lp_K + (m-1)\beta p_K = 0$. However, so does any multiple of p_K. Therefore, it is not possible to find the exact expression for C_K by studying this linearisation only. On the probabilistic side, the spine approach can be seen as an approximation of a branching diffusion by a single line of descent, the spine. It seems that the amount of information about the branching diffusion which can be gained from a probabilistic spine decomposition is similar to that gained when working with an analytic linearisation, in that both methods give the asymptotic shape of p_K in (2.1.3) but fail to identify the constant C_K.
The decomposition in Theorem 2.3 suggests using the backbone X^B as a more accurate approximation of (X, P^K) on survival. It turns out that the backbone indeed captures enough information about the evolution of (X, P^K) on survival to derive the explicit expression for C_K. A heuristic argument and an outline of the proof using large deviation theory is given in Section 2.5.2, together with a rigorous proof based on computations of the growth rate of the expected number of particles in the backbone.
We remark that in the setting of branching Brownian motion with absorption at the origin in [3, 1] it has not been possible so far to identify the analogue to our constant C_K explicitly.

With Theorem 2.3 and 2.4 in hand we look for a quasi-stationary limit result for the law of the dressed blue branching diffusion, which agrees with the law of (X, P^K) conditioned on survival, as we approach criticality.
The heuristics we used earlier in the binary case suggest the following. If we pick a particle which is currently located at position y then the probability that it has an infinite line of descent is $p_K(y)$. Thus, given the particle positions $x_u(t)$ for all particles $u \in N_t$, the number of particles in the backbone at time t is the number of successes in a sequence of independent Bernoulli trials each with probability of success $p_K(x_u(t))$ (We will make this argument rigorous in Corollary 2.17). Now, as $K \downarrow K_0$, the probabilities $p_K(x_u(t))$ tend to 0 uniformly by Theorem 2.4 and thus the blue tree becomes increasingly thinner on $(0, K_0)$. It cannot vanish

completely though since the genealogical line of the initial blue particle cannot become extinct and thus one may believe that, over a fixed time interval $[0,T]$, the blue tree thins down to a single genealogical line at criticality.
In the case of a binary branching mechanism as considered in Theorem 2.3, this conjecture can readily by confirmed by looking at the branching rates. The blue branching rate βp_K drops down to 0 as $K \downarrow K_0$, at the same time the red branching rate $\beta(1-p_K)$ increases to β and the rate of immigration $2\beta(1-p_K)$ rises to 2β at criticality.
Let us formalise this idea by defining what we expect to be the limiting branching diffusion, now already for the case of a general branching mechanism, before giving the quasi-stationary limit result below.

Definition 2.5. *Let $x \in (0,K_0)$. Let $X^* = (X^*_t, t \geq 0)$ be a $\mathcal{M}_a(0,K_0)$-valued process which is constructed as follows.*
X^* is initiated from a single particle at x performing a Brownian motion conditioned to stay in $(0,K_0)$, that is, a strong Markov process with infinitesimal generator

$$L^{K_0,*} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{\pi/K_0}{\tan(\pi y/K_0)}\frac{\mathrm{d}}{\mathrm{d}y}, \quad \text{on} \;\; (0,K_0), \tag{2.1.4}$$

defined for all $u \in C^2(0,K_0)$. Along its path we immigrate $\tilde{A}$ independent copies of (X, P^{K_0}) at rate $m\beta$ where $\tilde{A}$ has the size-biased offspring distribution $(\tilde{q}_k, k = 0,1,...)$ with

$$\tilde{q}_k = q_{k+1}\frac{k+1}{m}, \quad k \geq 0.$$

Denote the law of X^* by Q^*_x.

Theorem 2.6. *Let $x \in (0,K_0)$. Then, for any fixed time $T > 0$, the law of $(X_t, 0 \leq t \leq T)$ under the measure $\lim_{K\downarrow K_0} P^K_x(\cdot|\zeta = \infty)$ is equal to $(X^*_t, 0 \leq t \leq T)$ under Q^*_x, where the limit is understood in the weak sense.*

Theorem 2.6 can be seen as an extension of the spine decomposition we mentioned in the discussion following Proposition 2.1 to the critical width K_0. We emphasize however that the result, as stated, only holds over finite time horizons $[0,T]$.
In Section 2.7, we demonstrate the robustness of our approach by applying the results for the P^K-branching diffusion to study the evolution of a supercritical super-Brownian motion with absorption at 0 and K near criticality. We outline a backbone decomposition analogous to Theorem 2.3 in which we will see that the backbone of the super-Brownian motion with absorption at 0 and K is the same as the backbone of an associated P^K-branching diffusion. This connection allows

us to deduce asymptotic results for the survival rate of the super-Brownian motion with absorption on $(0, K)$ directly from the results on the survival probability of the associated P^K-branching diffusion. Further, we can find a quasi-stationary limit result for the super-Brownian motion equivalent to Theorem 2.6. This section is intended to highlight the applicability of the backbone approach and we will only sketch the proofs of the results therein.
The remainder of this chapter is organised as follows. In Section 2.2 we introduce some useful spine techniques which are employed in the proof of Proposition 2.1 in Section 2.3. In Section 2.4 we establish the results corresponding to Proposition 2.2 and Theorem 2.3 for the case of a general branching mechanism, see Proposition 2.13 and Theorem 2.14. In doing so, we show that the red branching diffusion and the dressed blue branching diffusion arise from martingale changes of measure which condition (X, P^K) on extinction respectively survival. We prove the asymptotic results for the survival probability given in Theorem 2.4 in Section 2.5. The proof of the quasi-stationary limit result in Theorem 2.6 follows in Section 2.6. Section 2.7 sketches the analogous results for the super-Brownian motion on $(0, K)$.

2.1.2 Literature overview

Branching Brownian motion with an absorbing barrier at the origin was studied by Kesten [45]. Our investigations are particularly motivated by recent results on the asymptotics of the survival probability of branching Brownian motion with absorption found in Berestycki et al. [3] as well as Aïdékon and Harris [1]. Some of these result were already discussed in Section 1.2.2. A discussion of branching Brownian motion in the critical width strip can be found in Berestycki et al. [4]. Spine techniques of the type used in the proof of Proposition 2.1 were developed in Chauvin and Rouault [11], Lyons [55] and Lyons et al. [56] and are now a standard approach in the theory of branching processes. See, for example, Harris et al. [36] and Kyprianou [48] for related applications in the setting of branching Brownian motion with absorption at 0 respectively absorption at a space-time barrier.

A backbone decomposition as in Theorem 2.3 for supercritical superprocesses is presented in Berestycki et al. [5]. It extends the earlier work of Evans and O'Connell [31], Fleischmann and Swart [33] and Engländer and Pinsky [26] as well as the corresponding decomposition for continuous-state branching processes in Duquesne and Winkel [14].
The results for superprocesses are complemented by the decomposition in Etheridge and Williams [28] which considers the $(1+\beta)$-superprocess conditioned on survival. This work is of particular interest in the current context since it also presents the equivalent result for the approximating branching particle system. However we should point out that in their case the immigrants are conditioned

to become extinct up to a fixed time T whereas, in our setting, we condition on extinction in the strip $(0, K)$. Thus the underlying transformations in [28] are time-dependent in contrast to the space-dependent h-transforms we see in our setting.
We also point out that our derivation of the backbone decomposition differs from the previously mentioned articles in that we show that the backbone arises from combining changes of measure which condition (X, P^K) on either the event of survival or the event of extinction.

The equivalent result to Theorem 2.4 in the setting of branching Brownian motion with absorption at the origin was shown in Berestycki et al. [3] and Aïdékon and Harris [1]. However, it has not been possible so far to give such an explicit expression for the constant analogous to C_K.

A similarly fashioned result to Theorem 2.6 was obtained in the aforementioned work by Etheridge and Williams [28]. Their result extends the Evans immortal particle representation for superprocesses in [30] which is the equivalent of the spine representation for branching processes. Again we point out that, in contrast to our setting, extinction is a time-dependent phenomenon in [28]. Further, our martingale change of measure approach to the backbone decomposition allows us to give a very simple proof of the quasi-stationary limit result.

2.2 Changes of measure and spine techniques

Let us begin this section by stating a general result on how martingale changes of measure affect the drift of a Brownian motion. Recall that we denote by $(\xi, \mathbb{P}_x^K)$ a Brownian motion with drift $-\mu$ initiated from $x \in (0, K)$ which is killed upon exiting $(0, K)$ and set $\mathcal{G}_t = \sigma(\xi_s : s \leq t)$.
We remind the reader of the following classical result, which is adapted from Revuz and Yor [59], VIII Proposition 3.4 and the discussion preceding it, since we will make use of it several times.

Lemma 2.7. *Let $x \in (0, K)$. Let $h \in C^2(0, K)$ and suppose that*

$$\frac{h(\xi_t)}{h(x)} \exp\Big\{ - \int_0^t \frac{Lh(\xi_s)}{h(\xi_s)} \, \mathrm{d}s \Big\}, \ t \geq 0, \tag{2.2.1}$$

is a $\mathbb{P}_x^K$-martingale. Define $\hat{\mathbb{P}}_x^K$ to be the probability measure which has martingale density (2.2.1) with respect to $\mathbb{P}_x^K$ on $\mathcal{G}_t$.
Under $\hat{\mathbb{P}}_x^K$, ξ has infinitesimal generator $L + \frac{h'(y)}{h(y)}\mathrm{d}y$ for all functions $u \in C^2(0, K)$ with $u(0+) = u(K-) = 0$.

In this regard, a change of measure with a martingale of the form (2.2.1) is equivalent to a h-transform of the infinitesimal generator L.

Recall that we characterised the Brownian motion conditioned to stay in $(0, K)$ via its infinitesimal generator $L^{K,*}$ given in (2.1.4) in Definition 2.5 (where K_0 can be replaced by a general $K > 0$). In view of Lemma 2.7, it is not difficult to see that its law can be obtained from the law of $(\xi, \mathbb{P}^K)$ by a martingale change of measure. In fact, note that the process

$$\Upsilon^K(t) = \sin(\pi\xi_t/K)e^{\mu\xi_t+(\mu^2/2+\pi^2/2K^2)t}, \quad t \geq 0, \tag{2.2.2}$$

is a $\mathbb{P}^K$-martingale. Define $\mathbb{Q}_x^K$ to be the probability measure which has martingale density $\Upsilon^K(t)$ with respect to $\mathbb{P}_x^K$ on $\mathcal{G}_t$, that is

$$\left.\frac{\mathrm{d}\mathbb{Q}_x^K}{\mathrm{d}\mathbb{P}_x^K}\right|_{\mathcal{G}_t} = \frac{\Upsilon^K(t)}{\Upsilon^K(0)}, \quad t \geq 0. \tag{2.2.3}$$

Then, under $\mathbb{Q}_x^K$, ξ is a Brownian motion conditioned to stay in $(0, K)$. By Lemma 2.7 with $h(x) = \sin(\pi x/K)e^{\mu x}$, its infinitesimal generator is indeed given by $L^{K,*}$ as in (2.1.4).
This process was first introduced in Knight [46], Theorem 3.1 and referred to as the *taboo process*. Let us note that $(\xi, \mathbb{Q}_x^K)$ is positive recurrent and has invariant density $\frac{2}{K}\sin^2(\pi x/K)$, for $x \in (0, K)$.

The proof of Proposition 2.1 in the subsequent section uses classical spine techniques developed in Chauvin and Rouault [11], Lyons et al. [56] and Lyons [55]. We will briefly recall the key steps in the spine construction here. For a comprehensive account we refer the reader to Hardy and Harris [35]; see also Harris et al. [36] and Kyprianou [48] for related applications in the setting of branching Brownian motion with absorption at 0.
The spine is a distinguished line of descent which is constructed as follows. To begin with, the spine contains the initial particle. When the initial particle dies we pick uniformly at random one of its offspring to continue the path of the spine. We proceed in this manner until the current spine particle gets killed or until it dies without having any offspring.
The probability measure under which the spine is a line of descent chosen as above, say $\tilde{P}^K$, needs to be defined on a larger filtration which allows us to distinguish it from all other lines of descent. We introduce the following filtrations encoding different amounts of information about the P^K-branching diffusion and the spine.

- $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ is the natural filtration generated by the P^K-branching diffusion. $\mathcal{F}_t$ contains all information about the branching particles up to time t but it does not know which particles make up the spine.
- $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_t, t \geq 0)$ is the filtration generated by the P^K-branching diffusion and the spine. $\tilde{\mathcal{F}}_t$ contains all information about the P^K-branching diffusion

up to time t and, in addition, it knows which particles are part of the spine up to time t.

- $\mathcal{G}^S = (\mathcal{G}^S_t, t \geq 0)$ is the natural filtration generated by the path of the spine. $\mathcal{G}^S_t$ contains the spatial information about the spine up to time t but it does not know which particles in the P^K-branching diffusion make up the spine.

- $\tilde{\mathcal{G}}^S = (\tilde{\mathcal{G}}^S_t, t \geq 0)$ contains all information about the spine. Like $\mathcal{G}^S_t$, it contains the spatial information about the spine up to time t. In addition, $\tilde{\mathcal{G}}^S_t$ knows the death times and offspring numbers of the particles which make up the spine up to time t. Apart from the birth times and family sizes along the spine, $\tilde{\mathcal{G}}^S_t$ does not contain any information about the other particles in the P^K-branching diffusion.

The probability measure $\tilde{P}^K$ described above is a measure on $\tilde{\mathcal{F}}$ and it is an extension of P^K in that $\tilde{P}^K|_{\mathcal{F}_\infty} = P^K$. Under $\tilde{P}^K$, the path of the spine is a Brownian motion with drift $-\mu$ and killing up on exiting $(0, K)$.
The next step is to construct a measure under which the spine moves like a Brownian motion conditioned to stay in $(0, K)$. Recall that the martingale Υ^K in (2.2.2) is used to condition a Brownian motion to stay in $(0, K)$. For each $u \in N_t$, write

$$\Upsilon^K_u(t) = \sin(\pi x_u(t)/K) e^{\mu x_u(t) + (\mu^2/2 + \pi^2/2K^2)t}, \ t \geq 0.$$

Set $\lambda(K) := (m-1)\beta - \mu^2/2 - \pi^2/2K^2$ and define the process $Z^K = (Z^K(t), t \geq 0)$ as

$$Z^K(t) = \sum_{u \in N_t} e^{-(m-1)\beta t} \Upsilon^K_u(t) = \sum_{u \in N_t} e^{\mu x_u(t) - \lambda(K)t} \sin(\pi x_u(t)/K), \ t \geq 0.$$

Then Z^K is a non-negative $(P^K_x, \mathcal{F})$-martingale. For $x \in (0, K)$, we define a martingale change of measure on the probability space of the P^K-branching diffusion via

$$\left.\frac{\mathrm{d}Q^K_x}{\mathrm{d}P^K_x}\right|_{\mathcal{F}_t} = \frac{Z^K(t)}{Z^K(0)}, \ t \geq 0. \tag{2.2.4}$$

Changing measure with the martingale Z^K in this way leads to the following 'spine'-construction of the path of X under Q^K_x:

- the initial particle moves according to a $\mathbb{Q}^K_x$-diffusion, that is a Brownian motion conditioned to stay in $(0, K)$;

- the initial particles dies at accelerated rate $m\beta$;

- when it dies it is replaced by $1 + \tilde{A}$ children, where $\tilde{A}$ has the size-biased offspring distribution

$$\tilde{q}_k = \frac{1+k}{m} q_{k+1}, \quad k \geq 0; \tag{2.2.5}$$

- from its $1 + \tilde{A}$ children we choose one particle uniformly at random;
- the chosen particle repeats the stochastic behaviour of the initial particle, i.e. from its birth position it moves like a $\mathbb{Q}^K$-diffusion, branches at rate $m\beta$ and has a size-biased offspring distribution;
- all other particles initiate a copy of (X, P^K) from the space-time position of their birth.

Again, the line of descent of the chosen particles in this construction is called the spine.
Alternatively, instead of picking the spine particles sequentially from the current spine particle's children, the path of X under Q^K_x can be constructed from a $\mathbb{Q}^K_x$-diffusion along which copies of (X, P^K) immigrate. More precisely,

- from the initial position x, we run a $\mathbb{Q}^K_x$-diffusion, this is the spine;
- at times of a Poisson process with rate $m\beta$ we immigrate $\tilde{A}$ independent copies of (X, P^K) which are rooted at the spatial position of the spine at this time;
- the number of immigrants $\tilde{A}$ has the size-biased offspring distribution given in (2.2.5).

Note that Q^K_x is a probability measure on the filtration $\mathcal{F}$. Therefore we do not know which line of descent is the spine under Q^K_x.
As before, we can construct a probability measure $\tilde{Q}^K$ on the enhanced filtration $\tilde{\mathcal{F}}$ as an extension of Q^K, in that $\tilde{Q}^K|_{\mathcal{F}_\infty} = Q^K$, such that the spine is a distinguished line of descent. The path of the spine is a $\mathbb{Q}^K_x$-diffusion under the measure $\tilde{Q}^K_x$.
Let us introduce some more notation. Denote by $\tilde{\xi} = (\tilde{\xi}_t, t \geq 0)$ the path of the spine. When it does not lead to ambiguity, we will also use $\tilde{\xi}_t$ to refer to the particle that makes up the spine at time t. In particular, we denote by $\{u < \tilde{\xi}_t\}$ the set of ancestors of the current particle in the spine at time t. We let σ_u be the death time of a spine particle u and A_u is the random number of immigrant children that are born at time σ_u. To keep notation simple we use $\mathbb{Q}^K$, Q^K and $\tilde{Q}^K$ to denote the probability measures as well as their corresponding expectation operators.
We conclude this section by stating the so-called spine decomposition which will turn out to be a useful tool in the analysis of the martingale Z^K.

Lemma 2.8 (Spine decomposition, cf. [35], Theorem 7.1)**.** *For any $t \geq 0$, under $\tilde{Q}_x^K$,*

$$Z^K(t) = e^{\mu\tilde{\xi}_t - \lambda(K)t} \sin(\pi\tilde{\xi}_t/K) + \sum_{u<\tilde{\xi}_t} e^{-\lambda(K)\sigma_u} \sum_{j=1}^{A_u} Z_{uj}^K(t-\sigma_u), \qquad (2.2.6)$$

where, conditionally on $\tilde{\mathcal{G}}_\infty^S$, $(Z_{uj}^K, \tilde{Q}_x^K)$ is an independent copy of the martingale Z^K under $P_{\xi_{\sigma_u}}^K$. In particular,

$$\tilde{Q}_x(Z^K(t)|\tilde{\mathcal{G}}_\infty^S) \;=\; e^{\mu\tilde{\xi}_t - \lambda(K)t} \sin(\pi\tilde{\xi}_t/K) + \sum_{u<\tilde{\xi}_t} A_u e^{\mu\tilde{\xi}_{\sigma_u} - \lambda(K)\sigma_u} \sin(\pi\tilde{\xi}_{\sigma_u}/K). \qquad (2.2.7)$$

2.3 Proof of Proposition 2.1 and relation to a differential equation

2.3.1 Proof of Proposition 2.1

From the description of the evolution of X under the measure Q^K in the previous section, it is clear that the process (X, Q^K) survives a.s. since the spine survives. In light of the change of measure (2.2.4), survival of X under Q^K implies a positive probability of survival of X under P^K if the martingale Z^K is uniformly integrable. For this reason, we will now the study the large time behaviour of Z^K.
Since we assumed $E(A\log^+ A) < \infty$, the following proposition gives a necessary and sufficient condition for the $L^1(P_x^K)$-convergence of Z^K. Recall that $\lambda(K) = (m-1)\beta - \mu^2/2 - \pi^2/2K^2$.

Proposition 2.9. *Let $0 < x < K$.*
(i) If $\lambda(K) > 0$ then the martingale Z^K is $L^1(P_x^K)$-convergent and in particular uniformly integrable.
(ii) If $\lambda(K) \leq 0$ then $\lim_{t\to\infty} Z^K(t) = 0$ P_x^K-a.s.

The proof is similar in nature to the proof of Theorem 1 in [48] which presents the L^1-convergence result in the case of a branching Brownian motion with absorption at a space-time barrier, see also the proof of Theorem 1 therein, as well as the proof in [55] and the proof of Theorem A in [56].

Proof of Proposition 2.9. The main argument relies on the following measure-theoretic result (see Durrett [15], p. 242). Let $Z^K(\infty) = \lim_{t\to\infty} Z^K(t)$ and

$\bar{Z}^K(\infty) = \limsup_{t\to\infty} Z^K(t)$. Then:

$$Q^K(\bar{Z}^K(\infty) < \infty) = 1 \quad \Leftrightarrow \quad P^K(Z^K(\infty)) = Z^K(0), \tag{2.3.1}$$
$$Q^K(\bar{Z}^K(\infty) = \infty) = 1 \quad \Leftrightarrow \quad P^K(Z^K(\infty) = 0) = 1. \tag{2.3.2}$$

Note that Z^K is a non-negative P^K-martingale. Therefore, the limit $Z^K(\infty)$ exists a.s. under P^K but it might not exist under Q^K. We proof part (i) and (ii) separately.
(i) Suppose $\lambda(K) > 0$. To begin with, we study the effect of the $A\log^+ A$-condition. Recall that $\tilde{A}$ has the size-biased offspring distribution (2.2.5) under $\tilde{Q}^K$. We readily compute

$$\tilde{Q}^K(\log^+ \tilde{A}) = \sum_{k=1}^{\infty} \log^+ k \frac{1+k}{m} q_{k+1} = \frac{P^K(A\log^+(A-1))}{m}$$

to see that $\tilde{Q}^K(\log^+ \tilde{A})$ is finite if and only if $P^K(A\log^+ A)$ is finite. Further, for any $c > 0$, the sum $\sum_{k\geq 0} \tilde{Q}^K(\log^+ \tilde{A} > ck)$ is finite if and only if $\tilde{Q}^K(\log^+ \tilde{A})$ is finite. Thus, for any $c > 0$, we have

$$\sum_{k\geq 0} \tilde{Q}^K(\log^+ \tilde{A} > ck) < \infty \quad \Leftrightarrow \quad P^K(A\log^+ A) < \infty.$$

Under $\tilde{Q}^K$, the sequence of the number of immigrants, say $\{A_k, k \geq 1\}$, is an i.i.d. sequence of copies of $\tilde{A}$. Therefore, we can apply the Borel-Cantelli Lemma to get

$$\tilde{Q}^K(\limsup_{k\to\infty} k^{-1}\log^+ A_k > c) = 0 \qquad \text{if and only if} \qquad P^K(A\log^+ A) < \infty$$

Since this holds for any $c > 0$ we conclude that

$$\limsup_{k\to\infty} k^{-1}\log^+ A_k = 0 \quad \tilde{Q}^K\text{-a.s.} \qquad \text{if and only if} \qquad P^K(A\log^+ A) < \infty. \tag{2.3.3}$$

Thus the $A\log^+ A$-condition ensures that, under $\tilde{Q}^K$, the extrema of the sequence $(A_k, k \geq 1)$ have sub-exponential behaviour.
Next, consider the spine decomposition in (2.2.7). $\tilde{Q}^K_x$-a.s., the first term on the right-hand side of (2.2.7) converges to zero as $t \to \infty$ since the term $\mu\tilde{\xi}_t - \lambda(K)t$ tends to $-\infty$. The second term on the right-hand side of (2.2.7) is $\tilde{Q}^K_x$-a.s. finite if the sum

$$\sum_{k\geq 1} A_k e^{-\lambda(K)\sigma_k} \tag{2.3.4}$$

is finite a.s. under $\tilde{Q}^K_x$. Here $(\sigma_k, k \geq 1)$ denotes the sequence of immigration times along the spine. Since immigration events along the spine occur at rate $m\beta$, the kth immigration time σ_k is the sum of k i.i.d exponential random variables with parameter $m\beta$. For any $\eta \in (0,1)$,

$$\begin{aligned}\tilde{Q}^K_x(\sum_{k\geq 1} A_k e^{-\lambda(K)\sigma_k} = \infty) &\leq \tilde{Q}^K_x(A_k e^{-\lambda(K)\sigma_k} > \eta^k \text{ for infinitely many } k) \\ &= \tilde{Q}^K_x(k^{-1}\log A_k > \log\eta + k^{-1}\lambda(K)\sigma_k \\ &\qquad\qquad \text{for infinitely many } k).\end{aligned}$$

By the strong law of large numbers, $k^{-1}\lambda(K)\sigma_k$ converges to $\lambda(K)/m\beta$, $\tilde{Q}^K_x$-a.s. Choosing $\eta = \exp\{-\lambda(K)/m\beta\}$, we see that

$$\tilde{Q}^K_x(\sum_{k\geq 1} A_k e^{-\lambda(K)\sigma_k} = \infty) \leq \tilde{Q}^K_x(\limsup_{k\to\infty} k^{-1}\log A_k > 0).$$

By (2.3.3) the latter probability is zero. In conclusion, the sum in (2.3.4), and hence the second term on the right-hand side of (2.2.7), is $\tilde{Q}^K_x$-a.s. finite and we obtain from (2.2.7) that

$$\limsup_{t\to\infty} \tilde{Q}^K_x(Z^K(t)|\tilde{\mathcal{G}}^S_\infty) < \infty.$$

By Fatou's lemma, we then get

$$\tilde{Q}^K_x(\liminf_{t\to\infty} Z^K(t)|\tilde{\mathcal{G}}^S_\infty) \leq \liminf_{t\to\infty} \tilde{Q}^K_x(Z^K(t)|\tilde{\mathcal{G}}^S_\infty) \leq \limsup_{t\to\infty} \tilde{Q}^K_x(Z^K(t)|\tilde{\mathcal{G}}^S_\infty) < \infty$$

and deduce that $\liminf_{t\to\infty} Z^K(t) < \infty$, $\tilde{Q}^K_x$-a.s. Since $Z^K(t)$ is $\mathcal{F}_t$-measurable and $\tilde{Q}^K_x\big|_{\mathcal{F}_\infty} = Q^K_x$, this gives

$$Q^K_x(\liminf_{t\to\infty} Z^K(t) < \infty) = 1.$$

According to [38], $(1/Z^K(t), t \geq 0)$ is a Q^K_x- supermartingale which has a Q^K_x-a.s. limit. This ensures that $\liminf_{t\to\infty} Z^K(t)$ agrees with $\limsup_{t\to\infty} Z^K(t)$, Q^K-a.s., which leads us to conclude that

$$Q^K_x(\bar{Z}^K(\infty) < \infty) = 1.$$

By (2.3.1), we get $P^K_x(\bar{Z}^K(\infty)) = Z^K(0)$ which, when combined with Scheffé's Lemma ([69], Theorem 5.10), yields the $L^1(P^K_x)$-convergence.
(ii) To begin with, suppose that $\lambda(K) < 0$. We use the first term on the right-hand side of (2.2.6) in Lemma 2.8 as a lower bound for $Z^K(t)$ under $\tilde{Q}^K_x$. That

is, under $\tilde{Q}_x^K$,

$$Z^K(t) \geq e^{\mu\tilde{\xi}_t - \lambda(K)t} \sin(\pi\tilde{\xi}_t/K), \quad t \geq 0.$$

Since the spine $\tilde{\xi}$ is a Brownian motion conditioned to stay in $(0, K)$ under $\tilde{Q}_x^K$ and since $\lambda(K)$ is assumed to be strictly negative, the right-hand side above tends to infinity $\tilde{Q}_x^K$-a.s and thus, so does $Z^K(t)$. An application of (2.3.2) gives $Z^K(\infty) = 0$, P_x^K-a.s.
To complete the proof, we consider now the critical case $\lambda(K) = 0$. Note again that the process $(1/Z^K(t), t \geq 0)$ is a non-negative Q_x^K-supermartingale, see [38]. It thus converges Q_x^K-a.s. which implies in turn that $\lim_{t\to\infty} Z^K(t) \in (0, \infty]$ under Q_x^K. Let σ_k, $k \geq 1$, be the k-th birth time along the spine and let A_k be the number of immigrant children of the spine born at this time, as defined in part (i) of this proof. Then, using the decomposition of Z^K under $\tilde{Q}_x^K$ in (2.2.6), we have

$$Z^K(\sigma_k) = e^{\mu\tilde{\xi}_{\sigma_k}} \sin(\pi\tilde{\xi}_{\sigma_k}/K) + \sum_{i=1}^{k} \sum_{j=1}^{A_i} Z_{ij}^K(\sigma_k - \sigma_i).$$

Similarly, we can decompose Z^K at time σ_k-. The terms $Z^K(\sigma_k)$ and $Z^K(\sigma_k-)$ only differ by the contribution of the immigrant children of the spine born at time σ_k, that is, under $\tilde{Q}_x^K$,

$$Z^K(\sigma_k) - Z^K(\sigma_k-) = \sum_{j=1}^{A_k} e^{\mu\xi_{\sigma_k}} \sin(\pi\xi_{\sigma_k}/K). \tag{2.3.5}$$

Here we used that $\tilde{\xi}_{\sigma_k} = \tilde{\xi}_{\sigma_k-}$ and $Z_{ij}^K(\sigma_k - \sigma_i) = Z_{ij}^K((\sigma_k-) - \sigma_i)$, for $i < k$, $\tilde{Q}_x^K$-a.s. The latter holds since the probability that a birth in the subtree initiated from the jth child of the spine born at time σ_i occurs at time σ_k is zero.
Recall that the A_k, $k \geq 1$, are i.i.d random variables with a size-biased distribution under $\tilde{Q}_x^K$ and that $\tilde{\xi}$ is ergodic under $\tilde{Q}_x^K$. Therefore, we get

$$\limsup_{k\to\infty} A_k e^{\mu\xi_{\sigma_k}} \sin(\pi\xi_{\sigma_k}/K) > c > 0, \quad \tilde{Q}_x^K\text{-a.s. for some constant } c.$$

Thus the right-hand side of (2.3.5), and hence the size of the jumps of Z^K, is infinitely often larger than some strictly positive constant which implies that Z^K cannot converge to a finite limit under $\tilde{Q}_x^K$. We conclude that $Q_x^K(Z^K(\infty) = \infty) = \tilde{Q}_x^K(Z^K(\infty) = \infty) = 1$ and complete the proof with an application of (2.3.2). □

If the P^K-branching diffusion becomes extinct, then the martingale Z^K has a zero limit since the sum over all particles in N_t will eventually be empty. We will now show that the converse is also true, that is, (X, P^K) becomes extinct if the martingale limit $Z^K(\infty)$ is zero.

Proposition 2.10. *For $x \in (0, K)$, the events $\{Z^K(\infty) = 0\}$ and $\{\zeta < \infty\}$ agree P^K_x-a.s.*

Proof. Clearly $\{\zeta < \infty\} \subset \{Z^K(\infty) = 0\}$ and it remains to show that $\{\zeta = \infty\} \cap \{Z^K(\infty) = 0\}$ has zero probability. We consider the cases $\lambda(K) \leq 0$ and $\lambda(K) > 0$ separately.

Assume $\lambda(K) \leq 0$. Proposition 2.9 gives $Z^K(\infty) = 0$, P^K-a.s. As $Z^K(t)$ is the sum of the non-negative terms $e^{-\lambda(K)t} \sin(\pi x_u(t)/K) e^{\mu x_u(t)}$, Z^K vanishes in the limit if and only if all its terms do. On extinction, this is certainly the case. On the event of survival, these terms can only vanish if all particles move arbitrarily close to the killing boundary as $\sin(\pi x/K)e^{\mu x} \approx 0$ for x close to 0 and K only. Let us show that this particle behaviour cannot occur.
We suppose for a contradiction that $Z^K(\infty) = 0$ on the event of survival. This assumption demands that, for any $\epsilon > 0$, all particles leave the interval $(\epsilon, K - \epsilon)$ eventually, and thus we may assume without loss of generality that the process survives in the small strip $(0, \epsilon)$. We will now lead this argument to a contradiction by showing that, for ϵ small enough, the P^ϵ_x-branching diffusion, $x \in (0, \epsilon)$, will become extinct a.s.
Denote by $P^{(-\delta,\epsilon+\delta)}_x$ the law under which X is our usual branching Brownian motion but with killing upon exiting the interval $(-\delta, \epsilon + \delta)$, $\delta > 0$. For any $\delta > 0$, we can embed the P^ϵ-branching diffusion in a $P^{(-\delta,\epsilon+\delta)}$-branching diffusion according to the following procedure. Let us write $v \leq u$ if v is an ancestor of u (u is considered to be an ancestor of itself), in accordance with the classical Ulam-Harris notation (see for instance [35], p.290). Under $P^{(-\delta,\epsilon+\delta)}_x$, we define

$$N_t|_{(0,\epsilon)} = \{u \in N_t : \forall s \leq t \ \forall v \in N_s \text{ s.t. } v \leq u \text{ we have } x_v(s) \in (0, \epsilon)\},$$

which is the set of particles $u \in N_t$ whose ancestors (not forgetting u itself) have not exited $(0, \epsilon)$ up to time t. Now we can define the restriction of X to $(0, \epsilon)$ under $P^{(-\delta,\epsilon+\delta)}_x$ by

$$X_t|_{(0,\epsilon)} = \sum_{u \in N_t|_{(0,\epsilon)}} \delta_{x_u(t)}, \quad t \geq 0.$$

Then we conclude immediately that, for an initial position in $(0, \epsilon)$, the restricted process $X|_{(0,\epsilon)} = \left(X_t|_{(0,\epsilon)}, t \geq 0\right)$ under $P^{(-\delta,\epsilon+\delta)}_x$ has the same law as (X, P^ϵ_x).
Now we choose δ and ϵ small enough such that $\lambda(\epsilon + 2\delta) := (m - 1)\beta - \mu^2/2 -$

$\pi^2/2(\epsilon+2\delta)^2 < 0$. Then, under $P^{(-\delta,\epsilon+\delta)}$, the process

$$\begin{aligned} &Z^{(-\delta,\epsilon+\delta)}(t) \\ &:= \sum_{u\in N_t} \left\{ e^{\mu(x_u(t)+\delta)-\lambda(\epsilon+2\delta)t} \sin(\pi(x_u(t)+\delta)/(\epsilon+2\delta)) \right\}, \quad t \geq 0, \end{aligned}$$

is a martingale of the form in Proposition 2.9. Considering now the contribution coming from the particles in the set $N_t|_{(0,\epsilon)}$ only, we first note that our assumption of survival of the P^ϵ-branching diffusion ensures that this set is non-empty for any time t. Further, for particles $u \in N_t|_{(0,\epsilon)}$, the terms $e^{\mu(x_u(t)+\delta)} \sin(\pi(x_u(t)+\delta)/(\epsilon+2\delta))$ are uniformly bounded from below by a constant $c > 0$ and hence, under $P_x^{(-\delta,\epsilon+\delta)}$, we get

$$Z^{(-\delta,\epsilon+\delta)}(t) \geq cN_t|_{(0,\epsilon)} e^{-\lambda(\epsilon+2\delta)t}.$$

Since we have chosen δ and ϵ such that $\lambda(\epsilon+2\delta) < 0$, we now conclude that $Z^{(-\delta,\epsilon+\delta)}(\infty) = \infty$, $P_x^{(-\delta,\epsilon+\delta)}$-a.s. This is a contradiction since $Z^{(-\delta,\epsilon+\delta)}$ is a non-negative martingale and therefore has a finite limit a.s. Hence, for $\lambda(K) \leq 0$, the event $\{\zeta = \infty\} \cap \{Z^K(\infty) = 0\}$ has zero probability.

Consider the case $\lambda(K) > 0$. Suppose for a contradiction that $\{\zeta = \infty\} \cap \{Z^K(\infty) = 0\}$ has positive probability. Let $z_K(x) = P_x^K(Z^K(\infty) = 0)$, for $x \in (0, K)$. Define $M_\infty := \mathbf{1}_{\{Z^K(\infty)=0\}}$ and set

$$M_t := \prod_{u\in N_t} z_K(x_u(t)) = E_x^K(M_\infty|\mathcal{F}_t). \tag{2.3.6}$$

The equality above is a consequence of the Markov branching property and can be seen as follows. By the branching property, $\{Z^K(\infty) = 0\} = \{Z_u^K(\infty) = 0 \text{ for all } u \in N_t\}$, for any $t \geq 0$, where given $\mathcal{F}_t$, the Z_u^K are independent copies of Z^K under $P_{x_u(t)}^K$. Hence, we get

$$\begin{aligned} z_K(x) = P_x^K(Z^K(\infty) = 0) &= P_x^K\big(\prod_{u\in N_t} P_{x_u(t)}^K(Z_u^K(\infty) = 0)\big) \\ &= P_x^K\big(\prod_{u\in N_t} z_K(x_u(t))\big). \end{aligned}$$

In combination with the Markov property we obtain

$$P_x^K\Big(\prod_{u\in N_{t+s}} z_K(x_u(t+s))\Big|\mathcal{F}_t\Big) = \prod_{u\in N_t} P_{x_u(t)}^K\Big(\prod_{v\in N_s} z_K(x_v(s))\Big) = \prod_{u\in N_t} z_K(x_u(t)).$$

Thus the process $(M_t, t \geq 0)$ defined in (2.3.6) is a uniformly integrable P_x^K-martingale with limit $M_\infty = \mathbf{1}_{\{Z^K(\infty)=0\}}$ and, in particular, the equality in (2.3.6) holds. On the event $\{\zeta = \infty\} \cap \{Z^K(\infty) = 0\}$, we clearly have $M_\infty = 1$, P^K_x-a.s. This requires in turn that all particles $x_u(t), u \in N_t$ move towards 0 and K as $t \to \infty$, since we know from Proposition 2.9 (i) that $z_K(x) < 1$ for x within $(0, K)$. The previous part of this proof already showed that this leads to a contradiction. Thus, for $\lambda(K) > 0$, the martingale limit cannot be zero on survival. This completes the proof. □

Proof of Proposition 2.1. Note that $\lambda(K) \geq 0$ if and only if $\mu < \sqrt{2(m-1)\beta}$ and $K > K_0$. The result follows now immediately from Proposition 2.9 and 2.10. □

2.3.2 Solutions to $Lu + F(u) = 0$

As alluded to in the introductory Chapter 1, there are several connections between branching diffusion and differential equations. Here we investigate the differential equation

$$\begin{aligned} Lu + F(u) &= 0 \quad \text{on } (0, K) \\ u(0) = u(K) &= 1, \end{aligned} \tag{2.3.7}$$

and its relation to the branching Brownian motion in a strip. The following proposition characterises the solutions to (2.3.7) as functions which generate P^K-product martingales.

Proposition 2.11. *Let $g : (0, K) \to (0, 1)$ be a continuous function satisfying $g(0) = g(K) = 1$. If*

$$\prod_{u \in N_t} g(x_u(t)), \quad t \geq 0, \tag{2.3.8}$$

is a P^K-martingale then g solves (2.3.7).

The connection between product martingales and solutions to differential equations is not new, see for instance [57], [58], [11] or [9], and it is derived from a classical Feynman-Kac argument. We present the proof for completeness.

Proof. Assume the product in (2.3.8) is a martingale. Let us denote by $\xi = (\xi_t, 0 \leq t \leq S)$ the path of the initial particle up to its branching time S, noting that it is a Brownian motion with drift $-\mu$ and killing upon exiting $(0, K)$ under P^K and $\mathbb{P}^K$. Denote by $T_{(0,K)} = \inf\{t > 0 : \xi_t \notin (0, K)\}$ its exit time from $(0, K)$. The first branching time S is exponentially distributed with rate β and considering the event that it occurs after time $t \wedge T_{(0,K)}$ and its complement we get, for $t \geq 0$,

$$\begin{aligned} g(x) &= E_x^K\Big(\prod_{u\in N_t} g(x_u(t))\Big) \\ &= \mathbb{E}_x^K\Big(g(\xi_{t\wedge T_{(0,K)}})\ e^{-\beta(t\wedge T_{(0,K)})}\Big) \qquad (2.3.9) \\ &\quad +\mathbb{E}_x^K\Big(\int_0^{t\wedge T_{(0,K)}} E_x^K\Big(E_{\xi_s}^K\Big(\prod_{i=1}^{A}\prod_{u\in N_{t-s}^i} g(x_u(t-s))\Big)\Big)\beta e^{-\beta s}\ \mathrm{d}s\Big), \end{aligned}$$

where N_{t-s}^i denotes the set of particles alive at time t which have descended from the ith child, out of the A children born at time s, of the initial particle. Since each of the children of the initial particle gives rise to an independent BBM starting from their common birth position ξ_s at time s, we have for any $s \leq t$

$$\begin{aligned} E_x^K\Big(E_{\xi_s}^K\Big(\prod_{i=1}^{A}\prod_{u\in N_{t-s}^i} g(x_u(t-s))\Big)\Big) &= E_x^K\Big(\Big(E_{\xi_s}^K\Big(\prod_{u\in N(t-s)} g(x_u(t-s))\Big)\Big)^A\Big) \\ &= E_x^K\big(g(\xi_s)^A\big), \qquad (2.3.10) \end{aligned}$$

where, in the last step, we used that (2.3.8) is a martingale. Recall that $G(s) = E(s^A)$ denoted the generating function of A. Putting together (2.3.9) and (2.3.10), this gives

$$g(x) = \mathbb{E}_x^K\Big(g(\xi_{t\wedge T_{(0,K)}})e^{-\beta(t\wedge T_{(0,K)})}\Big) + \mathbb{E}_x^K\Big(\int_0^{t\wedge T_{(0,K)}} G(g(\xi_s))\beta e^{-\beta s}\ \mathrm{d}s\Big). \qquad (2.3.11)$$

Since the martingale in (2.3.8) is uniformly integrable we can take $t \to \infty$ and, with $g(\xi_{T_{(0,K)}}) = 1$ P^K-a.s., we get

$$g(x) = \mathbb{E}_x^K\Big(e^{-\beta T_{(0,K)}} + \int_0^{T_{(0,K)}} G(g(\xi_s))\ \beta e^{-\beta s}\ \mathrm{d}s\Big). \qquad (2.3.12)$$

Then a Feynman-Kac theorem, see e.g. Dynkin [16] Theorem 13.20, tells us that the right-hand side of (2.3.12) provides the unique solution to the equation

$$\begin{aligned} Lu - \beta u &= -\beta G(g) \quad \text{on} \quad (0,K) \\ u(0) &= 1 \qquad (2.3.13) \\ u(K) &= 1. \end{aligned}$$

By the identity (2.3.12) this implies that g itself is the unique solution to (2.3.13) and hence also a solution to (2.3.7). □

We have already seen two examples of functions that generate product martingales of the form in (2.3.8).

Proposition 2.12. *The function* $z_K = P^K_\cdot(Z^K(\infty) = 0)$ *and the extinction probability* $1 - p_K$ *solve (2.3.7).*

Proof. The function z_K generates the product martingale in (2.3.6). We can apply the same argument given there to show that

$$E^K_x(\mathbf{1}_{\{\zeta<\infty\}}|\mathcal{F}_t) = \prod_{u\in N_t} (1 - p_K(x_u(t)))\,,\ t \geq 0$$

is a uniformly integrable product martingale and thus of the form (2.3.8). The result follows by Proposition 2.11. □

2.4 The backbone decomposition via martingale changes of measure

In this section we decompose the BBM in $(0, K)$ into the blue and red branching diffusions corresponding to the blue and red trees described in our intuitive picture in Section 2.1.1. Recall that the blue tree consists of all genealogical lines of descent that will never become extinct while the red trees contain all remaining lines of descent. In Section 2.1.1, we only gave a characterisation of the red, blue and dressed blue branching diffusion in the case of a binary branching mechanism (Proposition 2.2 and Theorem 2.3). For a general branching mechanism, the results will be presented in this section as Proposition 2.13 and Theorem 2.14. Recall that in the colouring procedure in Section 2.1.1 the resulting coloured tree is either a dressed blue tree or a red tree depending on whether we are on the event of survival or the event of extinction. Let us refer to the process corresponding to the coloured tree as the *two-colour branching diffusion.* The law $P^{C,K}$ of the two-colour branching diffusion is defined by the law of X under P^K and a subsequent colouring of the particles. Let $c(u)$ denote the colour of a particle u. We say a particle u is blue if it has an infinite genealogical line of descent and we write $c(u) = b$, otherwise we say it is red and write $c(u) = r$. Let us remark that the natural filtration of $(X, P^{C,K})$ is $\sigma(\mathcal{F}_t, c(u)_{u\in N_t})$ but this filtration will not play a role in the forthcoming analysis. Given $\mathcal{F}_\infty$, the colouring is deterministic. Define $c(N_t) = \{(c_u)_{u\in N_t} : c_u \in \{b, r\}\}$ as the set of all possible colourings of N_t. Trivially, for all $t \geq 0$,

$$\left.\frac{\mathrm{d}P^{C,K}_x}{\mathrm{d}P^K_x}\right|_{\mathcal{F}_\infty} = \prod_{u\in N_t} \left(\mathbf{1}_{\{c(u)=b\}} + \mathbf{1}_{\{c(u)=r\}}\right) = 1$$

and thus

$$\begin{aligned}
\left.\frac{\mathrm{d}P_x^{C,K}}{\mathrm{d}P_x^K}\right|_{\mathcal{F}_t} &= E_x^K\Big(\prod_{u\in N_t}\Big(\mathbf{1}_{\{c(u)=b\}}+\mathbf{1}_{\{c(u)=r\}}\Big)\Big|\mathcal{F}_t\Big)\\
&= \sum_{\mathbf{c}\in c(N_t)}\prod_{u\in N_t}P_x^K\Big(c(u)=c_u\Big|\mathcal{F}_t\Big)\\
&= \sum_{\mathbf{c}\in c(N_t)}\prod_{u\in N_t,c_u=b}p_K(x_u(t))\prod_{u\in N_t,c_u=r}\Big(1-p_K(x_u(t))\Big)=1,
\end{aligned}$$

where the sum is taken over all possible colourings $\mathbf{c}=(c_u)_{u\in N_t}$ in $c(N_t)$. In particular, for $A\in\mathcal{F}_t$, we get

$$\begin{aligned}
&P_x^{C,K}(A;c(u)=c_u\ \forall u\in N_t|\mathcal{F}_t)\\
&\qquad=\mathbf{1}_A\prod_{u\in N_t,c_u=b}p_K(x_u(t))\prod_{u\in N_t,c_u=r}\Big(1-p_K(x_u(t))\Big).
\end{aligned}$$

We can now derive the change of measure for the red branching diffusion. It is sufficient to consider one initial particle and we suppose that this particle is red. Let $A\in\mathcal{F}_t$ and write $c(\emptyset)=r$ for the event that the initial particle is red. Then

$$\begin{aligned}
P_x^{R,K}(A):=P_x^{C,K}(A|c(\emptyset)=r) &= \frac{P_x^{C,K}(A;c(u)=r\ \forall u\in N_t)}{P_x^{C,K}(c(\emptyset)=r)}\\
&= \frac{E_x^K\Big(\mathbf{1}_A\prod_{u\in N_t}\left(1-p_K(x_u(t))\right)\Big)}{1-p_K(x)}.
\end{aligned}\tag{2.4.1}$$

Clearly, conditioning the initial particle to be red is the same as conditioning the process to become extinct and therefore the law of X under $P^{R,K}$ agrees with the law of X conditioned on extinction. The following proposition characterises X under $P^{R,K}$ and generalises Proposition 2.2 in Section 2.1.1.

Throughout this section we will denote branching rates by β and offspring probabilities by q with super- and subscripts indicating whether they belong to the red or blue branching diffusion or the immigration procedure.

Proposition 2.13 (The red branching diffusion). *For $\nu\in\mathcal{M}_a(0,K)$, define $P_\nu^{R,K}$ via (2.4.1). Then $(X,P_\nu^{R,K})$ is a branching process with single particle motion characterised by the infinitesimal generator*

$$L^{R,K}=\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\left(\mu+\frac{p_K'(y)}{1-p_K(y)}\right)\frac{\mathrm{d}}{\mathrm{d}y}\quad on\ \ (0,K),\tag{2.4.2}$$

for $u\in C^2(0,K)$ with $u(0+)=u(K-)=0$, and the branching activity is governed *by the space-dependent branching mechanism*

$$F^{R,K}(s,y) = \frac{1}{1-p_K(y)}(F(s(1-p_K(y))) - sF(1-p_K(y))), \qquad (2.4.3)$$

for $s \in [0,1]$ *and* $y \in (0,K)$*. In particular,* $F^{R,K}$ *is of the form*

$$F^{R,K}(s,y) = \beta^R(y)(\sum_{k\geq 0} q_k^R(y)s^k - s), \qquad (2.4.4)$$

where β^R *is a space-dependent branching rate and* $(q_k^R, k \geq 0)$ *a space-dependent offspring distribution. For a fixed* $y \in (0,K)$*, the branching rate is given as*

$$\beta^R(y) \;=\; \beta \sum_{k\geq 0} q_k(1-p_K(y))^{k-1}, \qquad (2.4.5)$$

and the offspring distribution is given as

$$q_k^R(y) \;=\; \beta(\beta^R(y))^{-1} q_k(1-p_K(y))^{k-1}, \quad k \geq 0. \qquad (2.4.6)$$

Proof. The change of measure in (2.4.1) preserves the branching property in the following sense. Let $\nu = \sum_{i=1}^n \delta_{x_i}$ be an initial configuration at time 0 in $(0,K)$ and $A \in \mathcal{F}_t$. Then

$$\begin{aligned} P_\nu^{R,K}(A) &= E_\nu^K\Big(\mathbf{1}_A \frac{\prod_{u\in N_t}(1-p_K(x_u(t)))}{\prod_{i=1}^n(1-p_K(x_i))}\Big) \\ &= \prod_{i=1}^n E_{x_i}^K\Big(\mathbf{1}_A \frac{\prod_{u\in N_t^i}(1-p_K(x_u(t)))}{1-p_K(x_i)}\Big) \\ &= \Big(\otimes_{i=1}^n P_{x_i}^{R,K}\Big)(A), \end{aligned}$$

where N_t^i is the set of descendants at time t of the ith initial particle. The process $(X, P^{R,K})$ is therefore completely characterised by its evolution up to the first branching time S.
Let us denote by $\xi = \{\xi_t, 0 \leq t \leq S\}$ the path of the initial particle up to time S, noting that it is a Brownian motion with drift $-\mu$ and killing upon exiting $(0,K)$ under P^K and $\mathbb{P}^K$. Let H be a positive bounded measurable functional of this path. We begin with considering the case $t < S$. Using the change of measure in (2.4.1) and the fact that S is exponentially distributed with parameter β, we

have

$$\begin{aligned}E_x^{R,K}(H(\xi_s, s \le t); S > t) &= E_x^K\Big(H(\xi_s, s \le t)\frac{1-p_K(\xi_t)}{1-p_K(x)}; S > t\Big)\\ &= e^{-\beta t}\mathbb{E}_x^K\Big(H(\xi_s, s \le t)\frac{1-p_K(\xi_t)}{1-p_K(x)}\Big)\\ &= e^{-\beta t}\mathbb{E}_x^{R,K}\Big(H(\xi_s, s \le t)e^{-\int_0^t \frac{F(1-p_K(\xi_s))}{1-p_K(\xi_s)}\mathrm{d}s}\Big),\end{aligned} \tag{2.4.7}$$

where $\mathbb{P}_x^{R,K}$ is defined by the change of measure

$$\left.\frac{d\mathbb{P}_x^{R,K}}{d\mathbb{P}_x^K}\right|_{\mathcal{G}_t} = \frac{1-p_K(\xi_t)}{1-p_K(x)}e^{\int_0^t \frac{F(1-p_K(\xi_s))}{1-p_K(\xi_s)}\mathrm{d}s}, \quad t \ge 0. \tag{2.4.8}$$

It follows from Lemma 2.7 using $h = 1 - p_K$ and the fact that $L(1-p_K) + F(1-p_K) = 0$, see Proposition 2.12, that the motion under $\mathbb{P}^{R,K}$ is governed by the infinitesimal generator $L^{R,K}$ in (2.4.2). Note that $L^{R,K}$ depends on the branching mechanism F through p_K.
We can rewrite β^R defined in (2.4.5) as

$$\begin{aligned}\beta^R(y) &= \beta\sum_{k\ge 0} q_k(1-p_K(y))^{k-1}\\ &= \frac{\beta\big(\sum_{k\ge 0} q_k(1-p_K(y))^k - (1-p_K(y))\big) + \beta(1-p_K(y))}{1-p_K(y)}\\ &= \frac{F(1-p_K(y)) + \beta(1-p_K(y))}{1-p_K(y)},\end{aligned} \tag{2.4.9}$$

for $y \in (0, K)$. Using this, (2.4.7) simplifies to

$$E_x^{R,K}(H(\xi_s, s \le t); S > t) = \mathbb{E}_x^{R,K}\Big(H(\xi_s, s \le t)e^{-\int_0^t \beta^R(\xi_s)\mathrm{d}s}\Big).$$

Thus, under $P^{R,K}$, the motion of the initial particle is given by the change of measure in (2.4.8) and it branches at space-dependent rate β^R as given in (2.4.5). It remains to identify the offspring distribution and we therefore study the process at its first branching time S. Using (2.4.1) in the first step, and then (2.4.8) together with the definition of β^R in (2.4.9) in the last, we get,

$$
\begin{aligned}
& E_x^{R,K}(H(\xi_s, s \le S); N_S = k; S \in \mathrm{d}t) \\
= \; & E_x^K\Big(\frac{\big(1-p_K(\xi_S)\big)^{N_S}}{1-p_K(x)} H(\xi_s, s \le S); N_S = k; S \in \mathrm{d}t\Big) \\
= \; & \mathbb{E}_x^K\Big(\frac{\big(1-p_K(\xi_t)\big)^{k}}{1-p_K(x)} H(\xi_s, s \le t) \; q_k \; \beta e^{-\beta t}\Big)\mathrm{d}t \\
= \; & \mathbb{E}_x^K\Big(\frac{1-p_K(\xi_t)}{1-p_K(x)} e^{\int_0^t \frac{F(1-p_K(\xi_s))}{1-p_K(\xi_s)} \, \mathrm{d}s} H(\xi_s, s \le t) \\
& \qquad q_k \; \beta e^{-\beta t} e^{-\int_0^t \frac{F(1-p_K(\xi_s))}{1-p_K(\xi_s)} \, \mathrm{d}s} \; \big(1-p_K(\xi_t)\big)^{k-1}\Big)\mathrm{d}t \\
= \; & \mathbb{E}_x^{R,K}\Big(H(\xi_s, s \le t)\beta^R(\xi_t) e^{-\int_0^t \beta^R(\xi_s)\mathrm{d}s} \; \frac{\beta}{\beta^R(\xi_t)} q_k (1-p_K(\xi_t))^{k-1}\Big)\mathrm{d}t.
\end{aligned}
$$

We see that, in addition to the change in the motion and the branching rate, the offspring distribution under $P^{R,K}$ becomes $\{q_k^R, k \ge 0\}$ as in (2.4.6).
We complete the proof by showing that the branching mechanism $F^{R,K}$ in (2.4.3) is of the form in (2.4.4). We have

$$
\begin{aligned}
& F^{R,K}(s,y) \\
= \; & \frac{1}{1-p_K(y)}(F(s(1-p_K(y))) - sF(1-p_K(y))) \\
= \; & \frac{1}{1-p_K(y)} \; \beta\Big(\sum_{k \ge 0} q_k s^k (1-p_K(y))^k - s(1-p_K(y)) \\
& \qquad -s\big(\sum_{k\ge 0} q_k (1-p_K(y))^k - (1-p_K(y))\big)\Big) \\
= \; & \beta\Big(\sum_{k\ge 0} q_k s^k (1-p_K(y))^{k-1} - s\sum_{k \ge 0} q_k (1-p_K(y))^{k-1}\Big) \\
= \; & \beta \sum_{k\ge 0} q_k (1-p_K(y))^{k-1}\Big(\sum_{k \ge 0} s^k \frac{(1-p_K(y))^{k-1} q_k}{\sum_{k\ge 0} q_k(1-p_K(y))^{k-1}} - s\Big) \\
= \; & \beta^R(y)(\sum_{k\ge 0} s^k q_k^R(y) - s)
\end{aligned}
$$

where we used the expressions in (2.4.5) and (2.4.6) in the last step. □

The natural next step is to condition the initial particle to be blue and study the resulting law. Note that this will describe the evolution of a dressed blue branching diffusion, corresponding to a blue tree dressed with red trees, and from this process we will be able to recover the blue branching diffusion. We will give the change of measure for the blue branching diffusion in Proposition 2.15

following the next theorem.
Let us define the law of the dressed blue branching diffusion as follows, for any $A \in \mathcal{F}_t$,

$$\begin{aligned} P_x^{D,K}(A) &:= P_x^{C,K}(A|c(\emptyset) = b) \\ &= \frac{P_x^{C,K}(A; c(u) = b \text{ for at least one } u \in N_t)}{P_x^{C,K}(c(\emptyset) = b)} \\ &= \frac{E_x^K\Big(\mathbf{1}_A\Big(1 - \prod_{u \in N_t}\left(1 - p_K(x_u(t))\right)\Big)\Big)}{p_K(x)}. \end{aligned} \tag{2.4.10}$$

Then $(X, P^{D,K})$ is the same as (X, P^K) conditioned on survival.

Theorem 2.14 (The dressed blue branching diffusion)**.** *Let $K > K_0$ and $x \in (0, K)$. The process $(X, P_x^{D,K})$ evolves as follows.*

(i) From x, we run a branching diffusion X^B with single particle movement according to the infinitesimal generator

$$L^{B,K} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \left(\mu - \frac{p_K'(y)}{p_K(y)}\right)\frac{\mathrm{d}}{\mathrm{d}y} \quad on \quad (0, K), \tag{2.4.11}$$

defined for all $u \in C^2(0, K)$, and space-dependent branching mechanism $F^{B,K}$ of the form

$$F^{B,K}(s, y) = \beta^B(y)\Big(\sum_{k \geq 0} q_k^B(y)s^k - s\Big), \quad s \in [0, 1], \ y \in (0, K),$$

where, for a fixed $y \in (0, K)$, the branching rate $\beta^B(y)$ and the offspring distribution $(q_k^B(y), k \geq 2)$ are given by

$$\begin{aligned} \beta^B(y) &= \beta \sum_{k \geq 2} \sum_{n \geq k} q_n \binom{n}{k} p_K(y)^{k-1}(1 - p_K(y))^{n-k}, \\ q_k^B(y) &= \beta\, \beta^B(y)^{-1} \sum_{n \geq k} q_n \binom{n}{k} p_K(y)^{k-1}(1 - p_K(y))^{n-k}, \quad k \geq 2. \end{aligned}$$

In particular, $F^{B,K}(s, y)$ can be written as

$$\frac{1}{p_K(y)}\left(F(sp_K(y) + (1 - p_K(y))) - (1 - s)F(1 - p_K(y))\right).$$

(ii) Conditionally on the branching diffusion X^B in (i), we have the following.

- *(Continuous immigration) Along the trajectories of each particle in*

X^B, an immigration with $n \geq 1$ immigrants occurs at rate

$$\beta_n^{I,1}(y) = \beta q_{n+1}(n+1)(1-p_K(y))^n, \quad y \in (0,K).$$

- *(Branch point immigration) At a branch point of X^B at $y \in (0,K)$ with some fixed $k \geq 2$ offspring, the number of immigrants is distributed according to $(q_{n,k}^{I,2}(y), n \geq 0)$, in that we see an immigration of n immigrants with probability*

$$q_{n,k}^{I,2}(y) \;=\; (\kappa_k(y))^{-1} q_{n+k} \binom{n+k}{k} p_K(y)^{k-1}(1-p_K(y))^n,$$

with normalising constant $\kappa_k(y) = q_k^B(y)\beta^{-1}\beta^B(y)$.

Each immigrant initiates an independent copy of $(X, P^{R,K})$ from the space-time position of its birth.

Proof. We use the same notation as in the proof of Proposition 2.13 and in addition let $T_{(0,K)}$ denote the first time the initial particle exits $(0,K)$. Consider the change of measure in (2.4.10) and note that, for any time $t < S$ and $A \in \mathcal{F}_t$, it becomes

$$P_x^{D,K}(A; t < S) \;=\; E_x^K\Big(\mathbf{1}_A \frac{p_K(\xi_t)}{p_K(x)}; S > t, T_{(0,K)} > t\Big),$$

where the indicator $T_{(0,K)} > t$ appears since the product in the enumerator in (2.4.10) is empty if the initial particle gets killed before it reproduces. Then

$$\begin{aligned} E_x^{D,K}(H(\xi_s, s\leq t); S > t) &= E_x^K\Big(H(\xi_s, s \leq t)\,\frac{p_K(\xi_t)}{p_K(x)}, T_{(0,K)} > t; S > t\Big) \\ &= e^{-\beta t}\mathbb{E}_x^K\Big(H(\xi_s, s \leq t)\,\frac{p_K(\xi_t)}{p_K(x)}, T_{(0,K)} > t\Big) \\ &= e^{-\beta t}\mathbb{E}_x^{B,K}\Big(H(\xi_s, s \leq t)\; e^{\int_0^t \frac{F(1-p_K(\xi_s))}{p_K(\xi_s)}\,\mathrm{d}s}\Big), \end{aligned} \tag{2.4.12}$$

where $\mathbb{P}_x^{B,K}$ is defined by the change of measure, for $t \geq 0$,

$$\left.\frac{d\mathbb{P}_x^{B,K}}{d\mathbb{P}_x^K}\right|_{\mathcal{G}_t} \;=\; \frac{p_K(\xi_t)}{p_K(x)} \exp\left\{-\int_0^t \frac{F(1-p_K(\xi_s))}{p_K(\xi_s)}\,\mathrm{d}s\right\}\, \mathbf{1}_{\{T_{(0,K)} > t\}}. \tag{2.4.13}$$

By Lemma 2.7 using $h = p_K$ and $Lp_K - F(1-p_K) = 0$, see Proposition 2.12, the motion of ξ under $\mathbb{P}_x^{B,K}$ is governed by the infinitesimal generator $L^{B,K}$ as in

(2.4.11). Note that $L^{B,K}$ depends on F through p_K. Then, setting

$$\begin{aligned} \beta^D(y) &= \frac{\beta p_K(y) - F(1-p_K(y))}{p_K(y)} \\ &= \beta \frac{1-\sum_{k=0}^{\infty}(1-p_K(y))^k q_k}{p_K(y)}, \quad \text{for } y \in (0,K), \end{aligned} \tag{2.4.14}$$

we see that (2.4.12) simplifies to

$$E_x^{D,K}(H(\xi_s, s \le t), S > t) = \mathbb{E}_x^{B,K}\Big(H(\xi_s, s \le t) e^{-\int_0^t \beta^D(\xi_s)\,\mathrm{d}s}\Big).$$

We deduce from this that, under $P^{D,K}$, the motion of the initial particle is given by the change of measure in (2.4.13) and it branches at space-dependent rate β^D given in (2.4.14).
It remains to specify the offspring distribution. We begin with the expression in (2.4.10) and then use (2.4.13) and the expression for β^D in (2.4.14) to get

$$\begin{aligned} & E_x^{D,K}(H(\xi_s, s \le S); S \in \mathrm{d}t; N_S = k; T_{(0,K)} > t) \\ = \; & E_x^K\Big(H(\xi_s, s \le t)\frac{1-(1-p_K(\xi_t))^{N_S}}{p_K(x)}; S \in \mathrm{d}t; N_S = k; T_{(0,K)} > t\Big) \\ = \; & \mathbb{E}_x^K\Big(H(\xi_s, s \le t)\frac{1-(1-p_K(\xi_t))^k}{p_K(x)}\beta e^{-\beta t} q_k \,\mathrm{d}t; T_{(0,K)} > t\Big) \\ = \; & \mathbb{E}_x^K\Big(H(\xi_s, s \le t)\frac{p_K(\xi_t)}{p_K(x)} e^{-\int_0^t \frac{F(1-p_K(\xi_s))}{p_K(\xi_s)}\,\mathrm{d}s} \mathbf{1}_{\{T_{(0,K)}>t\}} \\ & \qquad e^{\int_0^t \frac{F(1-p_K(\xi_s))}{p_K(\xi_s)}\,\mathrm{d}s} \beta e^{-\beta t} \frac{1-(1-p_K(\xi_t))^k}{p_K(\xi_t)} q_k \,\mathrm{d}t\Big) \\ = \; & \mathbb{E}_x^{B,K}\Big(H(\xi_s, s \le t)\beta^D(\xi_s) e^{-\int_0^t \beta^D(\xi_s)\mathrm{d}s} \frac{\beta}{\beta^D(\xi_t)} \frac{1-(1-p_K(\xi_t))^k}{p_K(\xi_t)} q_k \,\mathrm{d}t\Big). \end{aligned} \tag{2.4.15}$$

Again this reveals the evolution of the initial particle as described above and we further see that the offspring distribution of the initial particle under $P^{D,K}$ is given by $\{q_k^D, k \ge 0\}$ where

$$q_k^D(y) \propto q_k \frac{1-(1-p_K(y))^k}{p_K(y)}, \quad \text{for } y \in (0,K),$$

up to the normalising constant $\beta(\beta^D(y))^{-1}$. We note that $q_0^D(y) = 0$ for all $y \in (0,K)$ which we expected to see since $(X, P^{D,K})$ is equal in law to (X, P^K) conditioned on survival. However, we have so far neglected the fact that the initial particle can give birth to particles of the same type, i.e. blue particles (referred to as branching), and red particles which evolve as under $P^{R,K}$ (referred to as immigration). We will split up the rate β^D and the offspring distribution q_k^D into terms corresponding to branching respectively immigration. We start with a little computation. With the help of the binomial theorem we get

$$\begin{aligned}
&\beta \sum_{k\geq 1}\sum_{n\geq k} q_n \binom{n}{k} p_K(y)^{k-1}(1-p_K(y))^{n-k} \\
&= \frac{\beta}{p_K(y)}\Big(\sum_{k\geq 0}\sum_{n\geq k} q_n \binom{n}{k} p_K(y)^{k}(1-p_K(y))^{n-k} - \sum_{n\geq 0} q_n (1-p_K(y))^n\Big) \\
&= \frac{\beta}{p_K(y)}\Big(1-\sum_{k=0}^{\infty}(1-p_K(y))^k q_k\Big).
\end{aligned}$$

This is the rate β^D as given in (2.4.14). Thus we can decompose β^D into

$$\begin{aligned}
&\beta^D(y) \\
&= \beta \sum_{k\geq 2}\sum_{n\geq k} q_n \binom{n}{k} p_K(y)^{k-1}(1-p_K(y))^{n-k} + \beta \sum_{n\geq 1} q_n n (1-p_K(y))^{n-1} \\
&=: \beta^B(y) + \sum_{n\geq 0}\beta_n^{I,1}(y). \qquad (2.4.16)
\end{aligned}$$

Then $\beta_n^{I,1}$ is the rate at which the initial particle gives birth to one blue particle and n (red) immigrants while β^B is the rate at which the initial particle gives birth to at least two particles of the blue type and a random number of (red) immigrants. These rates agree with the rates β^B and $\beta_n^{I,1}$ as stated in (i) and (ii) respectively. We can now rewrite the offspring distribution q_k^D. For each $k \geq 1$, it is again an application of the binomial theorem that gives

$$\begin{aligned}
q_k^D(y) &\propto q_k \frac{1-(1-p_K(y))^k}{p_K(y)} \\
&= q_k \sum_{i=0}^{k}\binom{k}{i} p_K(y)^{i-1}(1-p_K(y))^{k-i} - q_k\, p_K(y)^{-1}(1-p_K(y))^k \\
&= q_k \sum_{i=2}^{k}\binom{k}{i} p_K(y)^{i-1}(1-p_K(y))^{k-i} \qquad (2.4.17) \\
&\qquad + q_k\, k(1-p_K(y))^{k-1}. \qquad (2.4.18)
\end{aligned}$$

Then the term in (2.4.17) gives, up to normalisation, the sum of the probabilities that the initial particle branches into i blue particles and, at the same branching time, $k-i$ red particles immigrate. From this we can deduce the immigrant distribution at branching points, $(q_{n,k}^{I,2}(y), k \geq 2)$, as given in (ii) as well as the

offspring distribution $(q_k^B(y), k \geq 2)$ of the blue branching diffusion in (i). The term in (2.4.18) is the probability that $k-1$ immigrants occur, again up to a normalising constant.
Using the decompositions of β^D and $(q_k^D, k \geq 0)$ just derived, we can carry out a computation similar to (2.4.15) where we now consider a time $t > S$ (instead of $t < S$). From this it can be deduced that the rates β^B and $\beta_n^{I,1}$ and the offspring distributions $(q_k^B(y), k \geq 2)$ and $(q_{n,k}^{I,2}(y), k \geq 2)$ do indeed characterise the birth rate and offspring distribution of blue particles and red immigrants respectively. We refrain from given the explicit computation as it is straightforward but cumbersome.
Note that $(X, P^{D,K})$ inherits the branching Markov property from (X, P^K) by (2.4.10) in a similar spirit to the case of $(X, P^{R,K})$ (cf. the proof of Proposition 2.13). Thus the description of the initial particle also characterises the evolution of all particles of the blue type and together with the characterisation of the immigrating $P^{R,K}$-branching diffusions in Proposition 2.13 we have completely characterised the evolution of X under $P^{D,K}$. □

In light of Theorem 2.14, we call the blue branching diffusion X^B in step (i) the *backbone*. Let us give the change of measure under which X evolves like X^B. Using the classical Ulam-Harris notation (see for instance [35], p.290), we denote by τ_v and σ_v the birth respectively death time of a particle v, by $T^v_{(0,K)}$ its first exit time from $(0, K)$ and by A_v the random number of its offspring. Denote by $\mathcal{T}$ the set of all particles in a realisation of X. Let $\mathcal{T}_t$ be the set of all $v \in \mathcal{T}$ with $\tau_v < t$ and v is in $\mathcal{T}_{t-}$ if, in addition, $\sigma_v < t$.

Proposition 2.15 (The backbone). *For $\nu \in \mathcal{M}_a(0, K)$ such that $\nu = \sum_{i=1}^n \delta_{x_i}$ with $x_i \in (0, K)$, $n \geq 1$, we define the measure $P_\nu^{B,K}$ via the following change of measure. For $t \geq 0$,*

$$
\begin{aligned}
\left.\frac{\mathrm{d}P_\nu^{B,K}}{\mathrm{d}P_\nu^K}\right|_{\mathcal{F}_t} \quad = \quad & \prod_{v \in \mathcal{T}_t} \frac{p_K(x_v(\sigma_v \wedge t))}{p_K(x_v(\tau_v))} \mathbf{1}_{\{t < T^v_{(0,K)}\}} \\
& \times \exp\left\{ \int_{\tau_v}^{\sigma_v \wedge t} F'(1 - p_K(x_v(s))) + \beta \ \mathrm{d}s \right\} \\
& \times \prod_{v \in \mathcal{T}_{t-}} \frac{q^B_{A_v}(x_v(\sigma_v))}{q_{A_v}\beta(\beta^B(x_v(\sigma_v)))^{-1}}.
\end{aligned}
$$

The branching diffusion $(X, P_\nu^{B,K})$ has single particle movement according to the infinitesimal generator $L^{B,K}$ and branching mechanism $F^{B,K}$ as given in step (i) of Theorem 2.14.

Proof. We use (2.4.14) and (2.4.16) to get

$$
\begin{aligned}
-\frac{F(1-p_K(y))}{p_K(y)} - \beta^B(y) &= \beta^D(y) - \beta - \beta^B(y) \\
&= \sum_{n\geq 0} \beta_n^{I,1}(y) - \beta \\
&= \beta \sum_{n\geq 1} q_n n (1-p_K(y))^{n-1} - \beta \\
&= F'(1-p_K(y)).
\end{aligned}
$$

The result then follows from rewriting the change of measure up to the first branching time S as

$$
\begin{aligned}
\left.\frac{dP_x^{B,K}}{dP_x^K}\right|_{\mathcal{F}_S} &= \frac{p_K(\xi_S)}{p_K(x)} \exp\left\{-\int_0^S \frac{F(1-p_K(\xi_s))}{p_K(\xi_s)}\,\mathrm{d}s\right\} \mathbf{1}_{\{S<T_{(0,K)}\}} \\
&\quad \times \frac{1}{\beta}\beta^B(\xi_S) \exp\left\{-\int_0^S \beta^B(\xi_s) - \beta\,\mathrm{d}s\right\} \times \frac{q^B_{N_S}(\xi_S)}{q_{N_S}},
\end{aligned}
$$

noting that the first line on the right-hand side accounts for the change of motion, the first term in the second line for the change in the branching rate and the last term in the second line for the change in the offspring distribution. □

Corollary 2.16 (The backbone decomposition)**.** *Let $K > K_0$ and $\nu \in \mathcal{M}_a(0,K)$ such that $\nu = \sum_{i=1}^n \delta_{x_i}$ with $x_i \in (0,K)$, $n \geq 1$. Then $(X, P_\nu^{C,K})$ has the same law as the process*

$$
\sum_{i=1}^n \left(Y_i X_t^{D,i} + (1-Y_i) X_t^{R,i}\right), \quad t \geq 0.
$$

where $X^{R,i} = (X_t^{R,i}, t \geq 0)$ are independent copies of $(X, P_{x_i}^{R,K})$, $X^{D,i} = (X_t^{D,i}, t \geq 0)$ are independent copies of $(X, P_{x_i}^{D,K})$ and the Y_i are independent Bernoulli random variables with respective parameters $p_K(x_i)$.

Intuitively speaking, we can describe the evolution under $P_\nu^{C,K}$ and thus also under P_ν^K as follows. Independently for each initial particle x_i, we flip a coin with probability $p_K(x_i)$ of 'heads'. If it lands 'heads', we initiate a copy of $(X, P_{x_i}^{D,K})$ and otherwise we initiate a copy of $(X, P_{x_i}^{R,K})$.

Corollary 2.17. *Given the number of particles of (X, P_ν^K) and their positions, say $x_1, ..., x_n$ for some $n \in \mathbb{N}$, at a fixed time t, then the number of particles of X_t^B is the number of successes in a sequence of n independent Bernoulli trials each with success probability $p_K(x_1), ..., p_K(x_n)$.*

Remark 2.18. *With Theorem 2.14 it can be shown that, if the differential equation in (2.3.7) has a non-trivial,* $(0,1)$*-valued solution, then it is unique. We sketch the argument here.*
Assume that $g_K(x)$ *is a non-trivial,* $(0,1)$*-valued solution to (2.3.7). By a Feynman-Kac argument (cf. Champneys et al. [9], Theorem (1.36)), it follows that*

$$M^K(t) = \prod_{u \in N_t} g_K(x_u(t)), \ t \geq 0,$$

is a P_x^K*-product martingale. Since* M^K *is uniformly integrable, its limit* $M^K(\infty)$ *exits* P_x^K*-a.s. On the event of extinction,* $M^K(\infty) = 1$*. On the event of survival, it follows from Theorem 2.14 that*

$$M^K(t) = \prod_{u \in N_t} g_K(x_u(t)) \leq \prod_{u \in N_t^B} g_K(x_u^B(t)), \tag{2.4.19}$$

where N_t^B *is the set of particles in* X_t^B*.*
Clearly, $|N_t^B| \to \infty$ *as* $t \to \infty$ *since each particle in* X^B *is replaced by at least two offspring and there is no killing. Denote by* $\xi^B = (\xi_t^B, t \geq 0)$ *the path of an arbitrary line of descent of particles in* X^B*. Then* ξ^B *performs an ergodic motion in* $(0, K)$ *according to the infinitesimal generator* $L^{B,K}$ *in (2.4.2). By ergodicity,* P^K*-a.s., we have* $\liminf_{t\to\infty} \xi_t^B = 0$ *and* $\limsup_{t\to\infty} \xi_t^B = K$ *which implies*

$$\liminf_{t\to\infty} g_K(\xi_t^B) = \inf_{y \in (0,K)} g_K(y) < 1, \tag{2.4.20}$$

since g_K *is non-trivial and* $(0,1)$*-valued. At any time* $t \geq 0$*, we can choose* $|N_t^B|$ *lines of descent, each of them containing the path of one of the particles in* N_t^B*, and (2.4.20) holds true along these lines of descent. Loosely speaking, the right-hand side of (2.4.19) then tends to an infinite product of terms with* lim inf *strictly smaller than* 1 *and therefore it must converge to* 0*, that is,*

$$\liminf_{t\to\infty} M^K(t) \leq \liminf_{t\to\infty} \prod_{u \in N_t^B} g_K(x_u^B(t)) = 0, \quad P^K\text{-a.s.}$$

This argument is not quite rigorous since we are taking the lim inf *along an infinite number of lines of descent. A rigorous proof can be carried out as follows. Let* $\epsilon > 0$ *and consider the event of survival in the interval* $(\epsilon, K-\epsilon)$*, say* $\{\zeta^{(\epsilon,K-\epsilon)} = \infty\}$*. On this event, the number of particles whose genealogical lines have not exited* $(\epsilon, K-\epsilon)$ *up to time* t*, namely* $|N_t|_{(\epsilon,K-\epsilon)}|$*, tends to infinity under* P^K*, as* $t \to \infty$*. Note that* $g \leq 1-\delta$ *within* $(\epsilon, K-\epsilon)$ *for some* $\delta > 0$*. Then, on* *the event* $\{\zeta^{(\epsilon,K-\epsilon)} = \infty\}$*, we get,* P^K*-a.s.,*

$$\begin{aligned}\liminf_{t\to\infty} M^K(t) &\leq \liminf_{t\to\infty} \prod_{u\in N_t|_{(\epsilon,K-\epsilon)}} g_K(x_u(t)) \\ &\leq \liminf_{t\to\infty} (1-\delta)^{\left|N_t|_{(\epsilon,K-\epsilon)}\right|} \\ &= 0.\end{aligned}$$

Thus the martingale limit $M^K(\infty)$ *is zero on the event of survival in* $(\epsilon, K-\epsilon)$*, for any* $\epsilon > 0$*. Finally, we use the fact that, if* $\epsilon > 0$ *is small enough such that* $\lambda(\epsilon) \leq 0$ *then the process cannot survive in the interval* $(0,\epsilon)$*, to convince ourselves that* $\{\zeta = \infty\} = \bigcup_{\epsilon>0}\{\zeta^{(\epsilon,K-\epsilon)} = \infty\}$*,* P^K*-a.s. We can then conclude that* $M^K(\infty) = 0$*, on the event of survival.*
Recalling that $M^K(\infty) = 1$ *on the event of extinction, we get* $M^K(\infty) = \mathbf{1}_{\{\zeta<\infty\}}$*. Taking expectations gives*

$$g_K(x) = E_x^K(M^K(\infty)) = P_x^K(\zeta < \infty), \quad x \in (0,K).$$

As this is true for any non-trivial, $(0,1)$*-valued solution to (2.3.7) we have established uniqueness of these solutions.*

By Proposition 2.12, the function $z_K(x) = P_x^K(Z^K(\infty) = 0)$ *solves (2.3.7). With Proposition 2.9, this yields that (2.3.7) has a non-trivial solution if and only if* $\mu < \sqrt{2(m-1)\beta}$ *and* $K > K_0$*.*

Again by Proposition 2.12, $1 - p_K(x)$ *is also a solution to (2.3.7). Thus we may conclude again that the events* $\{Z^K(\infty) = 0\}$ *and* $\{\zeta < \infty\}$ *agree* P_x^K*-a.s., cf. Proposition 2.10.*

2.5 Proof of Theorem 2.4

We break up Theorem 2.4 into two parts which will be proved in the subsequent sections.

Proposition 2.19. *Uniformly for all* $x \in (0, K_0)$*,*

$$p_K(x) \sim c_K \sin(\pi x/K_0)e^{\mu x}, \qquad \text{as } K \downarrow K_0,$$

where c_K *is independent of* x *and* $c_K \downarrow 0$ *as* $K \downarrow K_0$*.*

Proposition 2.20. *The constant* c_K *in Proposition 2.19 satisfies*

$$c_K \sim (K-K_0)\frac{(K_0^2\mu^2+\pi^2)(K_0^2\mu^2+9\pi^2)}{12(m-1)\beta\pi K_0^3(e^{\mu K_0}+1)} \quad \text{as } K \downarrow K_0. \tag{2.5.1}$$

Theorem 2.4 then follows by defining C_K to be the expression on the right-hand side in (2.5.1).
We will provide probabilistic proofs of the results above. B. Derrida remarked that it is also possible to recover the asymptotics of p_K and the explicit constant C_K in an analytic approach. This analytic approach is based on an asymptotic expansion of the non-linear ODE $Lu+F(u)=0$ with boundary conditions $u(0)=u(K)=1$, which we considered in Proposition 2.11. It seems however that it would take some effort to make this argument rigorous and it does not appear to be more efficient than the probabilistic proof we present here.

2.5.1 Proof of Proposition 2.19

The following proof of Proposition 2.19 is guided by the ideas in Aïdékon and Harris [1]. We begin with a preliminary result which ensures that the survival probability p_K is right-continuous at K_0.

Lemma 2.21. *Let $x \in (0, K_0)$. Then* $\lim_{K\downarrow K_0} p_K(x) = 0$.

Proof. We fix $x \in (0, K_0)$ throughout the proof and consider $p_K(x)$ as a function in K. If we write P for the law of the branching Brownian motion without spatial killing, then we recover the law of the P^K-branching diffusion by restricting the process to those particles whose ancestral line has not exited the interval $(0, K)$ (see the proof of Proposition 2.10 for a formal construction). Under the measure P_x, we have

$$\begin{aligned} & \{\text{survival of the branching diffusion within } [0, K_0]\} \\ = & \bigcap_{K>K_0} \{\text{survival of the branching diffusion within } (0, K)\} \end{aligned}$$

By monotonicity of measures, we thus have

$$\begin{aligned} \lim_{K\downarrow K_0} p_K(x) &= \lim_{K\downarrow K_0} P_x(\text{survival of the branching diffusion in } (0, K)) \\ &= P_x\Big(\bigcap_{K>K_0} \{\text{survival of the branching diffusion in } (0, K)\}\Big) \\ &= P_x(\text{survival of the branching diffusion in } [0, K_0]) \\ &= p_{K_0}(x, t), \end{aligned}$$

where the last equality holds true as any particle that hits 0 or K will immediately pass below 0 respectively above K.
By Proposition 2.1, $p_{K_0}(x) = 0$ and so we have $\lim_{K\downarrow K_0} p_K(x) = 0$.

□

Recall that we denoted by $\mathcal{T}$ the set of all particles in a realisation of X and $v < u$ means that v is a strict ancestor of u. For $y \in (0, K_0)$, let $\mathcal{L}_{(0,y)}$ be the set containing all particles which are the first ones in their genealogical line to exit the strip $(0, y)$, i.e.

$$\mathcal{L}_{(0,y)} = \{u \in \mathcal{T} : \exists s \in [\tau_u, \sigma_u] \text{ s.t. } x_u(s) \notin (0, y) \text{ and } x_v(r) \in (0, y) \text{ for all } v < u, r \in [\tau_v, \sigma_v]\}. \tag{2.5.2}$$

The random set $\mathcal{L}_{(0,y)}$ is a stopping line in the sense of Biggins and Kyprianou [8] (see also Chauvin [10] which uses a slightly different definition though).
Let $|\mathcal{L}_{(0,y)}|$ be the number of particles which are the first ones in their line of descent to hit y (we do not count the ones exiting at 0), which can be written as

$$|\mathcal{L}_{(0,y)}| = \sum_{u \in \mathcal{L}_y} \mathbf{1}_{\left\{x_u(T^u_{(0,y)})=y\right\}}, \tag{2.5.3}$$

recalling that we denoted by $T^u_{(0,y)}$ the first exit time of a particle u from $(0, y)$. Likewise we can define the stopping line $\mathcal{L}_{(y,K_0)}$ as the set containing all particles which are the first ones in their genealogical line to exit the strip (y, K_0) and $|\mathcal{L}_{(y,K_0)}|$ as the number of particles in $\mathcal{L}_{(y,K_0)}$ which have exited at y.
The quantity $|\mathcal{L}_{(0,y)}|$ will turn out to be the essential ingredient in the proof of Proposition 2.19. To begin with, let us show that $E^{K_0}_x(|\mathcal{L}_{(0,y)}|)$ is finite. In fact, we can compute this expectation explicitly.

Lemma 2.22. *Let $x, y \in (0, K_0)$ with $x \leq y$. We have*

$$E^{K_0}_x(|\mathcal{L}_{(0,y)}|) = \frac{\sin(\pi x/K_0)}{\sin(\pi y/K_0)} e^{\mu(x-y)}, \tag{2.5.4}$$

where $|\mathcal{L}_{(0,y)}|$ is defined in (2.5.3). For $x, y \in (0, K_0)$ with $x \geq y$, (2.5.4) holds true with $|\mathcal{L}_{(0,y)}|$ replaced by $|\mathcal{L}_{(y,K_0)}|$.

Proof. To begin with, we note that a stopping line is called dissecting if there exists a P^K-a.s. finite time such that each particle alive at this time has descended from a particle in the stopping line, cf. [47]. Since we choose $y \in (0, K_0)$, the width of the strip $(0, y)$ is subcritical and hence, for any initial position $x \in (0, y)$, all particles will exit it eventually. This ensures that the stopping line $\mathcal{L}_{(0,y)}$ defined in (2.5.2) is a dissecting stopping line. Since $\mathcal{L}_{(0,y)}$ is dissecting it follows from Theorem 6 in [47] that we can apply the Many-to-one Lemma (see e.g. [35] Theorem 8.5) for the stopping line $\mathcal{L}_{(0,y)}$. Let $T_{(0,y)}$ again be the first time ξ exists $(0, y)$ and recall the definition of $\mathbb{Q}^{K_0}_x$ via the martingale change of measure in

(2.2.3). Then we get

$$\begin{aligned} &E_x^{K_0}(|\mathcal{L}_{(0,y)}|) \\ &= \mathbb{E}_x^{K_0}\left(e^{(m-1)\beta T_{(0,y)}}\mathbf{1}_{(\xi_{T_{(0,y)}}=y)}\right) \\ &= \mathbb{Q}_x^{K_0}\left(e^{(m-1)\beta T_{(0,y)}}\frac{\sin(\pi x/K_0)e^{\mu(x-\xi_{T_{(0,y)}})}}{\sin(\pi\xi_{T_{(0,y)}}/K_0)e^{(\mu^2/2+\pi^2/2K_0^2)T_{(0,y)}}}\mathbf{1}_{(\xi_{T_{(0,y)}}=y)}\right) \\ &= \frac{\sin(\pi x/K_0)}{\sin(\pi y/K_0)}e^{\mu(x-y)}\mathbb{Q}_x^{K_0}(\xi_{T_{(0,y)}}=y), \end{aligned}$$

where we have used that $(m-1)\beta-\mu^2/2-\pi^2/2K_0^2=0$ (and $\mathbb{Q}_x^{K_0}$ is used as an expectation operator). Under $\mathbb{Q}_x^{K_0}$, ξ will never hit 0 since it is conditioned to stay in $(0,K_0)$. However as ξ is positive recurrent it will eventually cross y and therefore $\mathbb{Q}_x^{K_0}(\xi_{T_{(0,y)}}=y)=1$. This gives (2.5.4). The case $x\geq y$ follows in the same way. □

The following lemma is the essential part in the proof of Proposition 2.19.

Lemma 2.23. *Let $x,y\in(0,K_0)$ with $x\leq y$. Let $|\mathcal{L}_{(0,y)}|$ be as defined in (2.5.3). Then, we have*

$$\lim_{K\downarrow K_0}\frac{p_K(x)}{p_K(y)}=E_x^{K_0}(|\mathcal{L}_{(0,y)}|). \tag{2.5.5}$$

For $x,y\in(0,K_0)$ with $x\geq y$, (2.5.5) holds true with $|\mathcal{L}_{(0,y)}|$ replaced by $|\mathcal{L}_{(y,K_0)}|$.

Proof. Fix $y\in(0,K_0)$. We begin with the case $0<x\leq y$.
We recall from Proposition 2.12 that $(\prod_{u\in N_t}(1-p_K(x_u(t))),t\geq 0)$ is a P_x^K-martingale. Since $\mathcal{L}_{(0,y)}$ is dissecting, as noted in the proof of Lemma 2.22, it follows from [10] that we can stop the martingale at $\mathcal{L}_{(0,y)}$ and obtain, for $x\in(0,y)$,

$$1-p_K(x)=E_x^K\left(\prod_{u\in\mathcal{L}_{(0,y)}}1-p_K(x_u(T^u_{(0,y)}))\right)=E_x^K((1-p_K(y))^{|\mathcal{L}_{(0,y)}|}), \tag{2.5.6}$$

where we have used that the process started at zero becomes extinct immediately, i.e. $p_K(0)=0$. Further $|\mathcal{L}_{(0,y)}|$ has the same distribution under P_x^K and $P_x^{K_0}$ since we consider particles stopped at level y below K_0 and thus we can replace E_x^K by $E_x^{K_0}$ on the right-hand side above. Now, using first (2.5.6) and then the geometric sum $\sum_{j=0}^{n-1} a^j = \frac{1-a^n}{1-a}$, we get

$$\frac{p_K(x)}{p_K(y)} = E_x^{K_0}\left(\frac{1-(1-p_K(y))^{|\mathcal{L}_{(0,y)}|}}{1-(1-p_K(y))}\right) = E_x^{K_0}\left(\sum_{j=0}^{|\mathcal{L}_{(0,y)}|-1}(1-p_K(y))^j\right). \tag{2.5.7}$$

The sum on the right-hand side is dominated by $|\mathcal{L}_{(0,y)}|$ which does not depend on K and has finite expectation, see Lemma 2.22. We can therefore apply the Dominated convergence theorem to the right-hand side in (2.5.7) and we conclude that

$$\begin{aligned}\lim_{K\downarrow K_0} & E_x^{K_0}\left(\sum_{j=0}^{|\mathcal{L}_{(0,y)}|-1}(1-p_K(y))^j\right)\\ &= E_x^{K_0}\left(\sum_{j=0}^{|\mathcal{L}_{(0,y)}|-1}\lim_{K\downarrow K_0}(1-p_K(y))^j\right) = E_x^{K_0}(|\mathcal{L}_{(0,y)}|),\end{aligned} \tag{2.5.8}$$

where the convergence holds point-wise in $x \in (0,y)$. Combining (2.5.7) and (2.5.8) we get (2.5.5) for $x \in (0,y)$.
It remains to show that (2.5.5) also holds for $x \in (y, K_0)$. Instead of approaching criticality by taking the limit in K we can now fix a $K > K_0$ and consider a (supercritical) strip (z,K) and let $z \uparrow z_0$ where $z_0 := K - K_0$. Denote by $p_{(z,K)}(x+z)$ the probability of survival in the strip (z,K) when starting from $x+z$. We then have

$$\lim_{K\downarrow K_0}\frac{p_K(x)}{p_K(y)} = \lim_{z\uparrow z_0}\frac{p_{(z,K)}(x+z)}{p_{(z,K)}(y+z)}.$$

Hence (2.5.5) is equivalent to showing that

$$\lim_{z\uparrow z_0}\frac{p_{(z,K)}(x+z)}{p_{(z,K)}(y+z)} = E_{x+z_0}^{K}(|\mathcal{L}_{(y+z_0,K)}|) = E_x^{K_0}(|\mathcal{L}_{(y,K_0)}|).$$

Here $|\mathcal{L}_{(y+z_0,K)}|$ denotes the number of particles which are the first in their genealogical line to exit the strip $(y+z_0, K)$ at $y+z_0$. Noting that this has the same law under $P_{x+z}^{z,K}$ and $P_{x+z}^{z_0,K}$, we can then repeat the argument in the first part. □

The next step is to show that the convergence in Lemma 2.23 holds uniformly in x on $(0, K_0)$.

Lemma 2.24. *Let $y \in (0, K_0)$. Then we have*

$$\lim_{K \downarrow K_0} \frac{p_K(x)}{p_K(y)} = \frac{\sin(\pi x/K_0)}{\sin(\pi y/K_0)} e^{\mu(x-y)}, \tag{2.5.9}$$

uniformly for all $x \in (0, K_0)$.

Proof. With Lemma 2.23 and 2.22, it remains to show that, for fixed $y \in (0, K_0)$, the convergence in equation (2.5.5) of Lemma 2.23 holds uniformly for all $x \in (0, K_0)$. Taking a look back at the proof of Lemma 2.23, we see that it suffices to show that the convergence in (2.5.8) holds uniformly for all $x \in (0, K_0)$.
Let us fix a $y \in (0, K_0)$ and let $x \in (0, y)$. We set

$$\varphi(x, K) = E_x^{K_0} \left(\sum_{j=0}^{|\mathcal{L}_{(0,y)}|-1} (1 - p_K(y))^j \right), \quad \text{for } x \in [0, y],$$

(with the convention that the P^K-branching diffusion becomes extinct immediately for the initial position $x = 0$ respectively stopped for $x = y$) and denote by $\varphi(x) = E_x^{K_0}(|\mathcal{L}_{(0,y)}|)$ its point-wise limit. Since $1 - p_K(y) \leq 1 - p_{K'}(y)$, for $K \geq K'$, we have $\varphi(x, K) \leq \varphi(x, K')$ and thus, for any $x \in [0, y]$, the sequence $\varphi(x, K)$ is monotone increasing as $K \downarrow K_0$. Moreover the functions $\varphi(x, K)$ and $\varphi(x)$ are continuous in x, for any K. In conclusion, we have an increasing sequence of continuous functions on a compact set with a continuous point-wise limit and therefore the convergence in (2.5.8) also holds uniformly in $x \in [0, y]$ (see e.g. [62], Theorem 7.13). This implies now that, for fixed $y \in (0, K_0)$, (2.5.5) and thus (2.5.9) holds uniformly in $x \in (0, y)$.
As outlined in the proof of Lemma 2.23, we can adapt the argument to the case $x \in (y, K_0)$ to complete the proof. □

Proof of Proposition 2.19. Choose a $y \in (0, K_0)$. Then an application of Lemma 2.24 gives, as $K \downarrow K_0$,

$$p_K(x) = p_K(y) \frac{p_K(x)}{p_K(y)} \sim p_K(y) \frac{\sin(\pi x/K_0)}{\sin(\pi y/K_0)} e^{\mu(x-y)} = c_K \sin(\pi x/K_0) e^{\mu x},$$

uniformly for all $x \in (0, K_0)$, where $c_K := \frac{p_K(y)}{\sin(\pi y/K_0)} e^{-\mu y}$. By Proposition 2.21, $c_K \downarrow 0$ as $K \downarrow K_0$ which completes the proof. □

2.5.2 Proof of Proposition 2.20

In this section we will present the proof of Proposition 2.20 which gives an explicit asymptotic expression for the constant c_K appearing in the asymptotics for the survival probability in Proposition 2.19 and Theorem 2.4 .

Heuristic argument

The starting point for the proof of Proposition 2.20 is the following idea: By Corollary 2.16, at time t, given the spatial positions $x_u(t)$ of all particles $u \in N_t$, the number of backbone particles is the number of successes in a sequence of Bernoulli trials with success probabilities $p_K(x_u(t))$. As this holds at any time t, we would expect that the proportion of backbone particles remains constant over time. This suggests that the backbone grows at the same rate as the whole process on survival. Further, the immigrating red trees are conditioned to become extinct which suggests that they do not contribute to the survival of the process. Loosely speaking, we do not expect to lose too much information about the evolution of (X, P^K) on survival if we simply study the growth of the backbone and ignore the contribution of the immigrating red trees.
We break up the heuristic argument into four steps.

Step (i) (The growth rate of the backbone) In this step, we derive an expression for the expected growth rate of the number of blue particles. The argument is based on heuristics from large deviation theory for ergodic processes, see for instance Chapter 5 in Stroock [68]. For a rigorous account of this step, the reader is referred to the outline of the large deviation proof at the end of this section.
Consider a process $\xi^B = (\xi^B_t, t \geq 0)$ performing the single particle motion of the backbone, that is according to the infinitesimal generator $L^{B,K}$ which is given in (2.4.11) in Theorem 2.14 as

$$L^{B,K} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \left(\mu - \frac{p_K'}{p_K}\right)\frac{\mathrm{d}}{\mathrm{d}y} \quad \text{on} \quad (0, K),$$

with domain $C^2(0, K)$. Let $\Pi^{B,K}$ be the invariant density for $L^{B,K}$, i.e. the positive solution of $\tilde{L}^{B,K}\Pi^{B,K} = 0$ where $\tilde{L}^{B,K}$ is the formal adjoint of $L^{B,K}$. Then we find

$$\Pi^{B,K}(y) \propto p_K(y)^2 e^{-2\mu y}, \quad y \in (0, K).$$

For $t \geq 0$ and a set $A \subset [0, K]$, we define

$$\Gamma(t, A) = \int_0^t \mathbf{1}_{\{\xi^B_s \in A\}} \,\mathrm{d}s$$

to be the occupation time up to time t of ξ^B in the set A. Then large deviation theory suggests that the probability that the measure $t^{-1}\Gamma(t, \cdot)$ is 'close' to the measure $\int_0^K \mathbf{1}_{\{\cdot\}}(y) f^2(y)\Pi^{B,K}(y)\,\mathrm{d}y$ is roughly

$$\exp\left\{t \int_0^K \{L^{B,K} f(y)\}\; f(y)\Pi^{B,K}(y)\;\mathrm{d}y\right\}. \tag{2.5.10}$$

Recall that each particle in the backbone moves according to $L^{B,K}$ and that the branching mechanism of the backbone is $F^{B,K}$ as defined in Theorem 2.14. For $y \in (0, K)$, set

$$F^{B,K\prime}(1, y) := \frac{\mathrm{d}}{\mathrm{ds}} F^{B,K}(s, y)|_{s=1} = (m-1)\beta + \frac{F(1-p_K(y))}{p_K(y)} \qquad (2.5.11)$$

to denote the mean number of offspring at a branch point located at y. Then the expected number of particles alive at time t, given the occupation density of the backbone particles up to time t is like $f^2\Pi^{B,K}$, should be

$$\exp\left\{t \int_0^K F^{B,K\prime}(1, y)\ (f(y))^2 \Pi^{B,K}(y)\ \mathrm{d}y\right\}.$$

Together with (2.5.10), we thus guess that the expected number of particles at time t with occupation density like $f^2\Pi^{B,K}$ is very roughly

$$\exp\left\{t \int_0^K \{[L^{B,K} + F^{B,K\prime}(1, y)]f(y)\}\ f(y)\Pi^{B,K}(y)\ \mathrm{d}y\right\}.$$

The expected growth rate of the backbone should then be obtained by maximising the expression above, i.e.

$$\sup_f \left\{ \int_0^K \{[L^{B,K} + F^{B,K\prime}(1, y)]f(y)\}\ f(y)\Pi^{B,K}(y)\ \mathrm{d}y\right\}, \qquad (2.5.12)$$

with the normalisation $\int_0^K f^2(y)\Pi^{B,K}(y)\mathrm{d}y = 1$. We assume henceforth that the supremum in (2.5.12) is taken over all functions f which satisfy in addition the boundary condition

$$\lim_{y\downarrow 0} f(y)f'(y)\Pi^{B,K}(y) = \lim_{y\uparrow K} f(y)f'(y)\Pi^{B,K}(y) = 0. \qquad (2.5.13)$$

Then an integration by parts shows that

$$\int_0^K \{L^{B,K} f(y)\}\ f(y)\ \Pi^{B,K}(y)\ \mathrm{d}y \quad = \quad -\frac{1}{2}\int_0^K (f'(y))^2\ \Pi^{B,K}(y)\ \mathrm{d}y. \qquad (2.5.14)$$

Thus, with (2.5.14), the variational problem (2.5.12) can be written as

$$\sup_f \left\{ \int_0^K \{ -\frac{1}{2}(f'(y))^2 + F^{B,K\prime}(1, y)f(y)^2\}\ \Pi^{B,K}(y)\ \mathrm{d}y\right\}. \quad (2.5.15)$$

Now set $h(y) = p_K(y)e^{-\mu y}f(y)$. Then h satisfies the normalisation $\int_0^K h(y)^2 \, \mathrm{d}y = 1$ and $h(0) = 0 = h(K)$. A lengthy but elementary computation (substitute h into (2.5.12) and use the definition of $F^{B,K\prime}(1,y)$ in (2.5.11) together with $Lp_K = F(1-p_K)$) shows that, instead of (2.5.15), we can consider the equivalent problem

$$\sup_h \Big\{ \int_0^K \{-\frac{1}{2}h'(y)^2 + \big((m-1)\beta - \frac{\mu^2}{2}\big)h(y)^2\} \, \mathrm{d}y \Big\}. \tag{2.5.16}$$

Equivalence means that the optimal solutions f^* and h^* of (2.5.15) and (2.5.16), respectively, satisfy $h^*(y) = p_K(y)e^{-\mu y}f^*(y)$. If we take the supremum in (2.5.16) over all functions $h \in L^2[0,K]$ with $h(0) = 0 = h(K)$ and $\int_0^K h(y)^2 \, \mathrm{d}y = 1$ then (2.5.16) is a classical Sturm-Liouville eigenvalue problem. For this case, it is well known that the optimal solution is $h^*(y) \propto \sin(\pi y/K)$, $y \in (0,K)$. Moreover, we then get

$$f^*(y) = \frac{h^*(y)}{p_K(y)}e^{\mu y} \propto \frac{\sin(\pi y/K)}{p_K(y)}e^{\mu y}, \quad y \in (0,K), \tag{2.5.17}$$

up to a normalising constant. Since (2.5.16) and (2.5.15) are equivalent, it follows that

$$\begin{aligned} &\int_0^K \big\{ -\frac{1}{2}(f^*(y)')^2 + F^{B,K\prime}(1,y)f^*(y)^2 \big\} \, \Pi^{B,K}(y) \, \mathrm{d}y \\ &= \sup_f \Big\{ \int_0^K \big\{ -\frac{1}{2}(f'(y))^2 + F^{B,K\prime}(1,y)f(y)^2 \big\} \, \Pi^{B,K}(y) \, \mathrm{d}y \Big\}, \end{aligned} \tag{2.5.18}$$

where we take the supremum over all f of the form $f(y) \propto e^{\mu y}h(y)(p_K(y))^{-1}$ with $h \in L^2(0,K)$, $h(0) = h(K) = 0$ and normalisation $\int_0^K f^2(y)\Pi^{B,K}(y)\mathrm{d}y = 1$. Further, it can be checked that f^* as in (2.5.17) solves the differential equation

$$[L^{B,K} + F^{B,K\prime}(1,y)]f^*(y) = \lambda(K)f^*(y) \quad \text{in } (0,K). \tag{2.5.19}$$

In conclusion, under the assumption that f^* satisfies (2.5.13), we get with (2.5.19), (2.5.14) and then (2.5.18)

$$\begin{aligned}\lambda(K) &= \int_0^K \{[L^{B,K} + F^{B,K\prime}(1,y)]f^*(y)\}\ f^*(y)\Pi^{B,K}(y)\ \mathrm{d}y \\ &= \int_0^K \{-\frac{1}{2}(f^*(y)')^2 + F^{B,K\prime}(1,y)f^*(y)^2\}\ \Pi^{B,K}(y)\ \mathrm{d}y \\ &= \sup_f \Big\{ \int_0^K \{-\frac{1}{2}(f'(y))^2 + F^{B,K\prime}(1,y)f(y)^2\}\ \Pi^{B,K}(y)\ \mathrm{d}y \Big\},\end{aligned} \tag{2.5.20}$$

where the supremum is taken as in (2.5.18). Heuristically, this indicates that $\lambda(K)$ is the expected growth rate of the backbone.

Step (ii) (Lower bound on $\lambda(K)$) Since f^* maximizes the expression in (2.5.20), we get a lower bound on $\lambda(K)$ by taking $f = 1$, noting that 1 is contained in the supremum set. Thus

$$\int_0^K F^{B,K\prime}(1,y)\ \Pi^{B,K}(y)\ \mathrm{d}y \ \leq\ \lambda(K).$$

Step (iii) (Upper bound on $\lambda(K)$) Let us define the 'optimal' occupation density as

$$\Pi_*^{B,K}(y) := (f^*(y))^2\Pi^{B,K}(y) = \frac{2}{K}\sin^2(\pi y/K), \quad y \in (0,K).$$

Omitting the non-positive term $-\frac{1}{2}(f'(y))^2$ in (2.5.20) will give the upper bound

$$\lambda(K) \ \leq\ \int_0^K F^{B,K\prime}(1,y)\ \Pi_*^{B,K}(y)\ \mathrm{d}y.$$

Step (iv) (Asymptotics) By Theorem 2.4, we have the asymptotics $p_K(y) \sim c_K \sin(\pi y/K_0)e^{\mu y}$, as $K \downarrow K_0$, and we can easily deduce that

$$\Pi^{B,K}(y) \sim \Pi_*^{B,K_0}(y), \quad \text{as } K \downarrow K_0.$$

We will make rigorous later that $F^{B,K\prime}(1,y) \sim (m-1)\beta c_K \sin(\pi y/K_0)e^{\mu y}$ as $K \downarrow K_0$. Our conjecture is therefore that

$$\lambda(K) \sim c_K \frac{2(m-1)\beta}{K_0} \int_0^{K_0} \sin^3(\pi y/K_0)e^{\mu y}\ \mathrm{d}y, \qquad \text{as } K \downarrow K_0.$$

Since we can calculate the integral explicitly this gives an exact asymptotic for the constant c_K.

Let us briefly comment on the heuristics above. All we need to do to deduce (2.5.20) in step (i) rigorously is to show that (2.5.13) is satisfied for f^* as defined in (2.5.17). The bounds in step (ii) and (iii) then follow immediately from (2.5.20) and it remains to complete step (iv). In particular, it is thus not necessary to show that $\lambda(K)$ is the expected growth rate of the backbone. Showing (2.5.13) for f^* is the essence of the proof of Lemma 2.29 in the following section.
However, it may seem rather unsatisfying to omit a rigorous proof of the fact that $\lambda(K)$ is indeed the expected growth rate of the backbone which was the starting point of step (i). This follows from a straightforward computation, see Proposition 2.26. Since the argument is rather simple we also show how the lower bound in step (ii) is obtained from an expected growth rate computation, see Lemma 2.27. The proof of the upper bound (iii) once again boils down to showing that f^* satisfies (2.5.13) and thus (2.5.14) which is done in Lemma 2.28 and Lemma 2.29. Thus the approach presented in Section 2.5.2 gives a self-contained proof of the explicit asymptotic form for the constant c_K without directly appealing to the variational problem considered in the heuristics.

Proof of Proposition 2.20

We briefly recall the key quantities needed in the following proofs. Recall from equation (2.4.11) that the motion of the backbone particles is given by

$$L^{B,K} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \left(\mu - \frac{p_K'}{p_K}\right)\frac{\mathrm{d}}{\mathrm{d}y} \quad \text{on} \quad (0,K),$$

which has invariant density $\Pi^{B,K}$ satisfiying

$$\Pi^{B,K}(y) = \frac{p_K(y)^2 e^{-2\mu y}}{\int_0^K p_K(z)^2 e^{-2\mu z}\mathrm{d}z}, \quad y \in (0,K).$$

We define the mean number of offspring at a branch point of the backbone by

$$F^{B,K\prime}(1,y) := \frac{\mathrm{d}}{\mathrm{d}s}F^{B,K}(s,y)|_{s=1} = (m-1)\beta + \frac{F(1-p_K(y))}{p_K(y)}, \quad y \in (0,K),$$

(cf. equation (2.5.11)). Throughout this section, let f^* be such that

$$f^*(y) \propto \frac{\sin(\pi y/K)}{p_K(y)}e^{\mu y}, \quad y \in (0,K), \tag{2.5.21}$$

which we assume to be normalised such that $\int_0^K f^*(y)^2\Pi^{B,K}(y)\mathrm{d}y = 1$. Finally we set $\Pi_*^{B,K}(y) := (f^*(y))^2\Pi^{B,K}(y)$, which using the normalisation of f^*, gives

$$\Pi_*^{B,K}(y) = \frac{2}{K}\sin^2(\pi y/K), \quad y \in (0,K).$$

Let us also restate equation (2.5.19) as it will be important in this section.

Lemma 2.25. *With the definitions given above,*

$$[L^{B,K} + F^{B,K\prime}(1,y)]f^*(y) = \lambda(K)f^*(y) \quad in\ (0,K).$$

Proof. The equation follows from a direct computation using that p_K solves (2.3.7). □

Let us now come to the proof of Proposition 2.20. First, we want to confirm the conjecture that the expected number of particles of $(X, P^{B,K})$ grows at rate $\lambda(K)$, which motivated the heuristic step (i).

Proposition 2.26. *For $x \in (0,K)$, we have*

$$\lim_{t\to\infty}\frac{1}{t}\log E_x^{B,K}(|N_t|) = \lambda(K).$$

Proof. Let $x \in (0,K)$ and $t \geq 0$. We apply the Many-to-one Lemma (see e.g. [36]), then the change of measure in (2.4.13) together with

$$F^{B,K\prime}(1,y) - \frac{F(1-p_K(y))}{p_K(y)} = (m-1)\beta,$$

and finally the change of measure in (2.2.3), to get

$$\begin{aligned} E_x^{B,K}(|N_t|) &= \mathbb{E}_x^{B,K}\left(e^{\int_0^t F^{B,K\prime}(1,\xi_s)\ \mathrm{d}s}\right) \\ &= e^{(m-1)\beta t}\mathbb{E}_x\left(\frac{p_K(\xi_t)}{p_K(x)}\mathbf{1}_{\{t<T_{(0,K)}\}}\right) \\ &= e^{\lambda(K)t}\mathbb{Q}_x^K\left(\frac{p_K(\xi_t)}{\sin(\pi\xi_t/K)}e^{-\mu\xi_t}\right)\frac{\sin(\pi x/K)}{p_K(x)}e^{\mu x}. \end{aligned}$$

Note that $\mathbb{Q}_x^K\left(\frac{p_K(\xi_t)}{\sin(\pi\xi_t/K)}e^{-\mu\xi_t}\right)$ is bounded from above by $\sin(\pi x/K)^{-1}$ which can be seen by rewriting this term as an expectation of a drift-less Brownian motion using a h-transform with $h(x) = \sin(\pi x/K)$. Then, as we take $t \to \infty$, the term $\mathbb{Q}_x^K\left(\frac{p_K(\xi_t)}{\sin(\pi\xi_t/K)}e^{-\mu\xi_t}\right)$ tends towards a positive constant since $(\xi, \mathbb{Q}_x^K)$ is an ergodic diffusion with invariant distribution $\frac{2}{K}\sin^2(\pi x/K)\mathrm{d}x$. Thus, after taking logarithms, dividing by t and taking $t \to \infty$, the result follows. □

We give a short proof of the inequality in step (ii) of the heuristic by using a lower bound on the growth rate of the expected number of backbone particles.

Lemma 2.27. *For $K > K_0$, we have*

$$\lambda(K) \geq \int_0^K F^{B,K\prime}(1,y)\Pi^{B,K}(y)\ \mathrm{d}y.$$

Proof. Using the Many-to-one Lemma (cf. e.g. [35]), and Jensen's inequality, we get for $x \in (0,K)$, $t \geq 0$,

$$\begin{aligned} E_x^{B,K}(|N_t|) &= \mathbb{E}_x^{B,K}(e^{\int_0^t F^{B,K\prime}(1,\xi_s)\ \mathrm{d}s}) \\ &\geq e^{\mathbb{E}_x^{B,K}(\int_0^t F^{B,K\prime}(1,\xi_s)\ \mathrm{d}s)}. \end{aligned}$$

Under $\mathbb{P}_x^{B,K}$, ξ has invariant distribution $\Pi^{B,K}(y)\mathrm{d}y$. Therefore we can apply an ergodic theorem for diffusions (see e.g. Rogers & Williams [61], V.53 Theorem (53.1) and Exercise (53.6)) which gives

$$\lim_{t\to\infty} \frac{1}{t}\Big(\int_0^t F^{B,K\prime}(1,\xi_s)\ \mathrm{d}s\Big) = \int_0^K F_K^{B\prime}(1,y)\Pi^{B,K}(y)\ \mathrm{d}y, \quad \mathbb{P}_x^{B,K} - \text{a.s.}$$

Since $F^{B,K\prime}(1,y)$ is bounded for $y \in (0,K)$ (cf. the argument following (2.5.28) in the proof of Proposition 2.20), the bounded convergence theorem gives

$$\lim_{t\to\infty} \mathbb{E}_x^{B,K}\Big(\frac{1}{t}\int_0^t F^{B,K\prime}(1,\xi_s)\ \mathrm{d}s\Big) = \int_0^K F_K^{B\prime}(1,y)\Pi^{B,K}(y)\ \mathrm{d}y,$$

which, together with Proposition 2.26, gives the desired lower bound on $\lambda(K)$. □

In step (iii) of the heuristic we claimed that we can replace $(L^{B,K}f^*)f^*$ by the non-positive term $-(f^*(y)')^2$ to get an upper bound on $\lambda(K)$. This is essentially what we will do rigorously in the next lemma. Recall the definitions of $L^{B,K}$, $F^{B,K\prime}(1,\cdot)$, f^*, $\Pi^{B,K}$ and $\Pi_*^{B,K}$ given at the beginning of this section.

Lemma 2.28. *For $K > K_0$, we have*

$$\lambda(K) \leq \int_0^K F^{B,K\prime}(1,y)\ \Pi_*^{B,K}(y)\ \mathrm{d}y.$$

Proof. In Lemma 2.25, we multiply both sides of the equation by f^* and integrate from 0 to K. With the normalisation $\int_0^K f^*(y)^2\Pi^{B,K}(y)\,\mathrm{d}y = 1$, we get

$$\begin{aligned}
\lambda(K) &= \int_0^K \left\{[L^{B,K} + F^{B,K\prime}(1,y)]f^*(y)\right\} f^*(y)\Pi^{B,K}(y)\,\mathrm{d}y \\
&= \int_0^K \left\{\frac{1}{2}(f^*(y))'' - (\mu - \frac{p_K(y)'}{p_K(y)})(f^*(y))'\right\} f^*(y)\Pi^{B,K}(y)\,\mathrm{d}y \\
&\qquad + \int_0^K F^{B,K\prime}(1,y)(f^*(y))^2\Pi^{B,K}(y)\,\mathrm{d}y. \qquad (2.5.22)
\end{aligned}$$

Recalling that $\Pi_*^{B,K}(y) = (f^*(y))^2\Pi^{B,K}(y)$, the result then follows if we can show that the first integral on the right-hand side in (2.5.22) is non-positive.
We show in Lemma 2.29 below that f^* satisfies the boundary condition (2.5.13). Further, we have $(\Pi^{B,K}(y))' = -2(\mu - \frac{p_K'(y)}{p_K(y)})\Pi^{B,K}(y)$. An integration by parts thus gives

$$\begin{aligned}
&\int_0^K \left\{\frac{1}{2}(f^*(y))'' - \Big(\mu - \frac{p_K'(y)}{p_K(y)}\Big)(f^*(y))'\right\} f^*(y)\Pi^{B,K}(y)\,\mathrm{d}y \\
= &-\frac{1}{2}\int_0^K \left\{((f^*(y))')^2 - 2\Big(\mu - \frac{p_K'(y)}{p_K(y)}\Big)(f^*(y))'f^*(y)\right\}\Pi^{B,K}(y)\,\mathrm{d}y \\
&\qquad - \int_0^K \Big(\mu - \frac{p_K'(y)}{p_K(y)}\Big)(f^*(y))'f^*(y)\Pi^{B,K}(y)\,\mathrm{d}y \\
= &-\frac{1}{2}\int ((f^*(y))')^2\Pi^{B,K}(y)\,\mathrm{d}y,
\end{aligned}$$

which is less than or equal to zero and the proof is complete. □

Lemma 2.29. *The function f^* satisfies the following boundary condition:*

$$\lim_{y\downarrow 0}(f^*(y))'f^*(y)\Pi^{B,K}(y) = \lim_{y\uparrow K}(f^*(y))'f^*(y)\Pi^{B,K}(y) = 0. \qquad (2.5.23)$$

Proof. We begin by showing that f^* is uniformly bounded in $(0,K)$. Recall from (2.5.21) that

$$f^*(y) \propto \frac{\sin(\pi y/K)}{p_K(y)}e^{\mu y}, \quad y \in (0,K).$$

Since f^* is continuous in $(0,K)$ it is sufficient to show that $\limsup_{x\downarrow 0} f^*(x)$ and $\limsup_{x\uparrow K} f^*(x)$ are bounded.
An application of L'Hôpital's rule gives

$$\lim_{x\downarrow 0}\frac{\sin(\pi x/K)e^{\mu x}}{\frac{\pi}{2K\mu}(1-e^{-2\mu x})} = 1. \qquad (2.5.24)$$

To conclude that $\limsup_{x\downarrow 0} f^*(x) < \infty$, it therefore suffices to show that there exists a constant $c > 0$ such that

$$c\,(1-e^{-2\mu x}) \leq \; p_K(x), \qquad \text{for all } x \text{ sufficiently close to zero.}$$

By Proposition 2.12, $(\prod_{u\in N_t}(1-p_K(x_u(t))), t \geq 0)$ is a P_x^K-martingale and it follows then by a standard Feynman-Kac argument that $1-p_K(x)$ satisfies

$$1-p_K(x) = 1 + \mathbb{E}_x^K \int_0^{T_{(0,K)}} F(1-p_K(\xi_s))\,\mathrm{d}s, \quad x \in (0,K),$$

where $T_{(0,K)}$ is the first time ξ exists the interval $(0,K)$. We can compute the expectation above using the potential density of ξ, which is given for instance in Theorem 8.7 in [50] (where we take $q = 0$ and use that $W(x) = \frac{1}{\mu}(1-e^{-2\mu x})$ is the scale function for a Brownian motion with drift $-\mu$). Then we get

$$\begin{aligned} -p_K(x) &= \mathbb{E}_x^K \int_0^{T_{(0,K)}} F(1-p_K(\xi_s))\,\mathrm{d}s \\ &= \frac{1}{\mu}(1-e^{-2\mu x}) \int_0^K F(1-p_K(y)) \frac{(1-e^{-2\mu(K-y)})}{(1-e^{-2\mu K})}\mathrm{d}y \\ &\quad -\frac{1}{\mu}\int_0^K F(1-p_K(y))(1-e^{-2\mu(x-y)})\,\mathrm{d}y. \end{aligned} \tag{2.5.25}$$

Since $F(s) < 0$ for $0 < s < 1$, the first integral in the last equality on the right-hand side of (2.5.25) is strictly negative and bounded. Hence we can set

$$c := -\frac{1}{\mu}\int_0^K F(1-p_K(y))\frac{(1-e^{-2\mu(K-y)})}{(1-e^{-2\mu K})}\mathrm{d}y > 0.$$

The second integral on the right-hand side of (2.5.25) is non-negative, for x close to 0, since the term $1-e^{-2\mu(x-y)}$ is non-positive for $x \leq y$. Therefore, we get

$$p_K(x) \geq c(1-e^{-2\mu x}) \qquad \text{for all } x \text{ sufficiently close to zero.}$$

which, together with (2.5.24), gives the desired result.
To establish boundedness as x approaches K, we observe that $p_K(x) = \bar{p}_K(K-x)$, where $\bar{p}_K$ denotes the survival probability for a branching diffusion which evolves as under P_x^K but with positive drift μ. Similar to the previous argument we can then show that there exists a constant $c > 0$ such that $c\bar{p}_K(K-x) \geq \sin(\pi x/K)e^{\mu x}$, for x sufficiently close to K.
We can now show (2.5.23). Since f^* takes a finite value at 0 and K it suffices to show that $(f^*(y))'\Pi^{B,K}(y)$ evaluated at 0 and K is zero. Differentiating f^* and

recalling that $\Pi^{B,K}(y) \propto p_K(y)^2 e^{-2\mu y}$ gives

$$\begin{aligned}
&(f^*(y))'\Pi^{B,K}(y) \\
&\propto \ e^{-\mu y}\Big((\mu \sin(\pi y/K) + \frac{\pi}{K}\cos(\pi y/K))p_K(y) - \sin(\pi y/K)p_K'(y)\Big).
\end{aligned}$$

Differentiating both sides of equation (2.5.25) with respect to x, it is easily seen that $p_K'(x)$ is bounded for all $x \in [0, K]$. Therefore $(f^*(y))'\Pi^{B,K}(y)$ is equal to 0 at 0 and K which completes the proof. □

We complete the proof of Proposition 2.20 by making step (iv) rigorous.

Proof of Proposition 2.20. By Lemma 2.27 and 2.28, we get the following bounds on $\lambda(K)$:

$$\int_0^K F^{B,K\prime}(1,y)\Pi^{B,K}(y)\ \mathrm{d}y \leq \lambda(K) \leq \int_0^K F^{B,K\prime}(1,y)\Pi_*^{B,K}(y)\ \mathrm{d}y. \quad (2.5.26)$$

By Proposition 2.19, we get, as $K \downarrow K_0$,

$$\Pi^{B,K}(y) = \frac{p_K(y)^2 e^{-2\mu y}}{\int_0^K p_K(z)^2 e^{-2\mu z}\mathrm{d}z} \sim \frac{2}{K_0}\sin^2(\pi y/K) = \Pi_*^{B,K_0}(y), \tag{2.5.27}$$

where we have used that the asymptotics in Proposition 2.19 hold uniformly to deal with the integral in the denominator. The uniformity in Proposition 2.19 also ensures that (2.5.27) holds uniformly for all $y \in (0, K_0)$. Further, we have

$$\lim_{s\uparrow 1}\frac{F(s)}{s(s-1)} = \lim_{s\uparrow 1}\frac{F'(s)}{2s-1} = (m-1)\beta,$$

where we applied L'Hôpital's rule in the first equality above. We apply this for $s = 1 - p_K(y)$ and $K \downarrow K_0$. Then, together with the definition of $F^{B,K\prime}(1,y)$ in (2.5.11) and the asymptotics in Proposition 2.19, we obtain, as $K \downarrow K_0$,

$$\begin{aligned}
F^{B,K\prime}(1,y) &= (m-1)\beta + \frac{F(1-p_K(y))}{p_K(y)} \\
&\sim (m-1)\beta - (m-1)\beta(1-p_K(y)) \\
&\sim (m-1)\beta c_K \sin(\pi y/K_0)e^{\mu y}.
\end{aligned} \tag{2.5.28}$$

Moreover, we note that for $y \in (0, K)$

$$\Big|\frac{F(1-p_K(y))}{p_K(y)}\Big| = \Big|\frac{F(1) - F(1-p_K(y))}{1-(1-p_K(y))}\Big| \leq \max_{s\in[0,1]} F'(s).$$

Convexity of F yields that the maximum above is attained at either 0 or 1 and we know that $F'(0)$ and $F'(1)$ are both finite. Hence, by (2.5.28), $|F^{B,K'}(1,y)|$ is bounded in $(0,K)$ and we can therefore appeal to bounded convergence as we take the limit in (2.5.26). With (2.5.27) and (2.5.28) we get

$$\lambda(K) \quad \sim \quad c_K \frac{2(m-1)\beta}{K_0} \int_0^{K_0} \sin^3(\pi y/K_0) e^{\mu y} \, \mathrm{d}y, \quad \text{as } K \downarrow K_0.$$

Evaluating the integral gives

$$\lambda(K) \sim c_K \frac{12\,(m-1)\beta\,\pi^3\,(e^{\mu K_0}+1)}{(K_0^2\mu^2+\pi^2)(K_0^2\mu^2+9\pi^2)}, \quad \text{as } K \downarrow K_0.$$

Finally, $\lambda(K) \sim \pi^2(K-K_0)K_0^{-3}$ as $K \downarrow K_0$ which follows from the linearisation

$$\begin{aligned}
\lambda(K) &= (m-1)\beta - \frac{\mu^2}{2} - \frac{\pi^2}{2K^2} \\
&= \underbrace{(m-1)\beta - \frac{\mu^2}{2} - \frac{\pi^2}{2K_0^2}}_{=0} + \frac{\pi^2}{2K_0^2} - \frac{\pi^2}{2K^2} \\
&= \frac{\pi^2 K^2}{2K_0^2 K^2} - \frac{\pi^2}{2K^2} \\
&= \frac{\pi^2[(K-K_0)^2 + 2(K-K_0)K_0 + K_0^2]}{2K_0^2 K^2} - \frac{\pi^2}{2K^2} \\
&= \frac{\pi^2 (K-K_0)^2}{2K_0^2 K^2} + \frac{\pi^2 (K-K_0)}{K_0 K^2}
\end{aligned}$$

and noting that the second term in the last line is the leading order term as $K \downarrow K_0$. This completes the proof. □

Large deviations revisited

Previously in Proposition 2.26, we showed that $\lambda(K)$ is the expected growth rate of the backbone. As an alternative to the proof given there, one can make the large deviation argument in step (i) of the heuristics rigorous. We will give a brief outline of the large deviation argument here.
Denote by $\mathcal{P}^{B,K} = (\mathcal{P}_t^{B,K}, t \geq 0)$ the diffusion semi-group associated with the infinitesimal generator $L^{B,K}$. As an operator on $C^1[0,K]$, the space of bounded continuous functions on $[0,K]$ with bounded continuous derivatives, the semi-group $\mathcal{P}^{B,K}$ is symmetric with respect to the invariant measure $\mathbf{\Pi}^{B,K}(dy) =$

$\Pi^{B,K}(y)dy$. That is saying that, for $f, g \in C^1[0,K]$ and $t \geq 0$, we have

$$\int_0^K f(y)\ \mathcal{P}_t^{B,K}[g](y)\ \mathbf{\Pi}^{B,K}(\mathrm{d}y) = \int_0^K g(y)\ \mathcal{P}_t^{B,K}[f](y)\ \mathbf{\Pi}^{B,K}(\mathrm{d}y),$$

which results from an integration by parts. This puts us in the setting of symmetric operators as studied in Chapter 4 in Deuschel and Stroock [13], whose arguments we will now follow. See also Section 6.2 in [13] as well as Chapter 5 and 7 in Stroock [68].
We can define the expected growth rate of the number of particles in the backbone as

$$\begin{aligned}\lambda^K &:= \lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in(0,K)} E_x^{B,K}(N_t) \\ &= \lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in(0,K)} \mathbb{E}_x^{B,K}\big(\exp \int_0^t F^{B,K\prime}(1,\xi_s)\ \mathrm{d}s\big). \qquad (2.5.29)\end{aligned}$$

Here we used the Many-to-one Lemma, cf. [35]. Define $\mathcal{Q}^{B,K} = (\mathcal{Q}_t^{B,K}, t \geq 0)$ to be the semi-group on $C^1[0,K]$ which is such that, for $f \in C^1[0,K]$ and $x \in [0,K]$,

$$\mathcal{Q}_t^{B,K}[f](x) = \mathbb{E}_x^{B,K}\Big(\exp\Big\{\int_0^t F^{B,K\prime}(1,\xi_s)\ \mathrm{d}s\Big\}\ f(\xi_t)\Big).$$

Let $||\cdot||_{op(C^1[0,K])}$ denote the operator norm on $C^1[0,K]$. Then we can write the growth rate λ^K in (2.5.29) as

$$\lambda^K = \lim_{t\to\infty} \frac{1}{t} \log ||\mathcal{Q}_t^{B,K}||_{op(C^1[0,K])},$$

cf. equation (4.2.21) and (4.2.28) in [13]. Following the arguments in Theorem 4.2.58 in [13], we can extend $\mathcal{Q}^{B,K}$ to a strongly continuous semi-group $\bar{\mathcal{Q}}^{B,K} = (\bar{\mathcal{Q}}_t^{B,K}, t \geq 0)$ on $L^2(\mathbf{\Pi}^{B,K})$. From [13], Lemma 4.2.50 and the proof of Lemma 5.3.2 it can then be deduced that

$$\lambda^K = \lim_{t\to\infty} \frac{1}{t} \log ||\mathcal{Q}_t^{B,K}||_{op(C^1[0,K])} = \lim_{t\to\infty} \frac{1}{t} \log ||\bar{\mathcal{Q}}_t^{B,K}||_{op(L^2(\mathbf{\Pi}^{B,K}))},$$

where $||\cdot||_{op(L^2(\mathbf{\Pi}^{B,K}))}$ is the operator norm on $L^2(\mathbf{\Pi}^{B,K})$. Consequently, Lemma 4.2.50 in [13] then yields that

$$\begin{aligned}\lambda^K \ = \ \sup_f \Big\{ \int_0^K F^{B,K\prime}(1,y) f(y)^2\ \mathbf{\Pi}^{B,K}(\mathrm{d}y) - \mathcal{E}(f,f) :& \\ ||f||_{L^2(\mathbf{\Pi}^{B,K})} = 1\Big\},& \qquad (2.5.30)\end{aligned}$$

where $\mathcal{E}(\cdot,\cdot)$ denotes the Dirichlet form associated with $\bar{\mathcal{P}}^{B,K}$. Here $\bar{\mathcal{P}}^{B,K} := (\bar{\mathcal{P}}^{B,K}_t, t \geq 0)$ is the unique extension of $\mathcal{P}^{B,K}$ to a strongly continuous semigroup on $L^2(\mathbf{\Pi}^{B,K})$.
By formula (6.2.11) in Theorem 6.2.9 in [13], the Dirichlet form is given by

$$\mathcal{E}(f,f) = \begin{cases} \frac{1}{2}\int_0^K (f'(y))^2 \; \mathbf{\Pi}^{B,K}(\mathrm{d}y) & \text{if } f, f' \in L^2(\mathbf{\Pi}^{B,K}), \\ \infty & \text{otherwise.} \end{cases}$$

Thus, to find the supremum in (2.5.30), we only have to consider $L^2(\mathbf{\Pi}^{B,K})$-functions with derivative also in $L^2(\mathbf{\Pi}^{B,K})$ and we get

$$\lambda^K = \sup_f \Big\{ \int_0^K F^{B,K\prime}(1,y) f(y)^2 - \frac{1}{2}(f'(y))^2 \; \mathbf{\Pi}^{B,K}(\mathrm{d}y) \Big\}, \qquad (2.5.31)$$

where the supremum is taken over all functions f with $f, f' \in L^2(\mathbf{\Pi}^{B,K})$ and normalisation $||f||_{L^2(\mathbf{\Pi}^{B,K})} = 1$.
For this class of functions f, we set $h(y) = p_K(y)e^{-\mu y}f(y)$. This implies that $h \in L^2[0,K]$ with $h(0) = h(K) = 0$ and normalisation $||h||_{L^2[0,K]} = 1$. A computation (the same computation that turns (2.5.15) into (2.5.16)) then shows that solving the variational problem in (2.5.31) is equivalent to solving

$$\sup_h \Big\{ \int_0^K \{ -\frac{1}{2}h'(y)^2 + \big((m-1)\beta - \frac{\mu^2}{2}\big)h(y)^2 \} \; \mathrm{d}y \Big\}, \qquad (2.5.32)$$

where the supremum is taken over all h of the form $h(y) = p_K(y)e^{-\mu y}f(y)$ and f as above. This is the variational problem in (2.5.16) in step (i) of the heuristics. We can thus proceed from here in the same way as we did in the heuristics. First, if we allow any $h \in L^2[0,K]$ with $h(0) = h(K) = 0$ and $||h||_{L^2[0,K]} = 1$ in the supremum, then (2.5.32) becomes a classical Sturm-Liouville problem (cf. (2.5.16)). We can then solve (2.5.32) and with it (2.5.31) to find again the optimal functions h^* and f^* given in (2.5.17).
Under the assumptions that $f^*, (f^*)' \in L^2(\mathbf{\Pi}^{B,K})$ and that f^* satisfies (2.5.13), then, together with (2.5.14) and (2.5.19), this gives

$$\lambda^K = \sup_f \Big\{ \int_0^K F^{B,K\prime}(1,y) f^*(y)^2 - \frac{1}{2}((f^*(y))')^2 \; \mathbf{\Pi}^{B,K}(\mathrm{d}y) \Big\} = \lambda(K).$$

The assumptions $f^*, (f^*)' \in L^2(\mathbf{\Pi}^{B,K})$ and (2.5.13) hold indeed, see Lemma 2.29 and the computations within its proof. This completes the argument.

The a.s. growth rate of the backbone

The starting point of our heuristics was the idea that the growth rate of the number of backbone particles agrees with the overall growth rate of particles in (X, P^K). By Example 1 in [39], the a.s. growth rate of $|N_t|$ under P^K is $\lambda(K)$. We will now show the a.s. growth rate of $|N_t|$ under $P^{B,K}$ is also $\lambda(K)$. Thus the backbone grows a.s. at the same rate as the whole process on survival.

Proposition 2.30. *For $x \in (0, K)$,*

$$\lim_{t\to\infty} \frac{1}{t} \log |N_t| = \lambda(K), \qquad P_x^{B,K}\text{-a.s.}$$

Proof. Let us begin with an upper bound on the growth rate. It follows from Theorem 1 and Example 1 in [39] that, on survival, the a.s. growth rate of the number of particles under P^K is $\lambda(K)$. By Corollary 2.17, given (N_t, P^K), the number of particles in the backbone at time t is the number of success in a sequence of N_t independent Bernoulli trials. Therefore the number of backbone particles cannot grow faster than the overall number of particles. From this we conclude that the growth rate of N_t under $P^{B,K}$ is bounded from above by $\lambda(K)$.
For the lower bound, we begin with constructing a $P^{B,K}$-martingale similar in fashion to the P^K-martingale Z^K of Section 2.2. Since f^* satisfies the equation in Lemma 2.25, it follows by an application of Itô's formula that

$$f^*(\xi_t)\; e^{\int_0^t F^{B,K\prime}(1,\xi_s)\mathrm{d}s - \lambda(K)t}, \quad t \geq 0$$

is a $\mathbb{P}^{B,K}$-martingale. Following [35], we can then construct a $P^{B,K}$-martingale $M_{f^*} = (M_{f^*}(t), t \geq 0)$ by setting

$$M_{f^*}(t) = \sum_{u\in N_t} f^*(x_u(t))e^{-\lambda(K)t}, \quad t \geq 0.$$

The proof of $L^1(P_x^{B,K})$-convergence of M_{f^*} follows by a classical spine decomposition argument in the same fashion as the proof of Proposition 2.9 and is therefore omitted. The $L^1(P_x^{B,K})$-convergence implies that $P_x^{B,K}(M_{f^*}(\infty) > 0) > 0$. In fact, we will now show that $M_{f^*}(\infty) > 0$, $P_x^{B,K}$-a.s.
To this end, set $g(x) := P_x^{B,K}(M_{f^*}(\infty) = 0)$, for $x \in (0, K)$. Then the product

$$\pi^g(t) = \prod_{u\in N_t} g(x_u(t)), \quad t \geq 0,$$

is a $P_x^{B,K}$-martingale with almost sure limit $\mathbf{1}_{\{M_{f^*}(\infty)=0\}}$, which can be shown in the same way as for the martingale M in the proof of Proposition 2.1. Since g is $[0,1]$-valued, the product $\pi^g(t)$ is stochastically bounded by $g(\xi_t)$, for any $t \geq 0$.

Together with the martingale property of π^g we get

$$g(\xi_s) = E^{B,K}_{\xi_s}(\pi^g(t)) \leq \mathbb{E}^{B,K}_{\xi_s}(g(\xi_t)), \quad \text{for all } x \in (0, K)$$

which shows that the process $(g(\xi_t), t \geq 0)$ is a $[0,1]$-valued $\mathbb{P}^{B,K}_x$-submartingale. The submartingale $(g(\xi_t), t \geq 0)$ converges $\mathbb{P}^{B,K}_x$-a.s. to a limit, say g_∞. Since ξ is positive recurrent under $\mathbb{P}^{B,K}_x$, the process $g(\xi_t)$ can only converge if it is constant, that is $g(x) = g_\infty \in [0,1]$ for all $x \in (0, K)$. Hence, we can write π^g as

$$\pi^g(t) = g_\infty^{|N_t|}, \quad t \geq 0.$$

Assume now that $g_\infty \in [0,1)$. Under $P^{B,K}_x$, $|N_t|$ tends to infinity as $t \to \infty$ since each particle in $(X, P^{B,K}_x)$ is replaced by at least two offspring when it dies and there is no killing. With $g_\infty < 1$, we thus get $\pi^g(t) \to 0$, $P^{B,K}_x$-a.s. and, using uniform integrability of π^g, this gives

$$g(x) = E^{B,K}_x(\pi^g(\infty)) = 0.$$

In conclusion, $g \equiv g_\infty$ is either identical to 0 or identical to 1. We already know that the martingale limit $M_{f^*}(\infty)$ is strictly positive with positive probability and consequently,

$$0 < P^{B,K}_x(M_{f^*}(\infty) > 0) = 1 - g(x).$$

Hence, we conclude $g(x) = 0$, for all $x \in (0, K)$ which is saying that $M_{f^*}(\infty)$ is strictly positive, $P^{B,K}$-a.s. This allows us to deduce that

$$\liminf_{t\to\infty} \log M_{f^*}(t)/\lambda(K)t \geq 0, \qquad P^{B,K}_x\text{-a.s.} \tag{2.5.33}$$

By Lemma 2.29, f^* is bounded from above by a constant $c > 0$ in $(0, K)$. Thus, under $P^{B,K}_x$, for $t \geq 0$, we get

$$M_{f^*}(t) \leq c\,|N_t|\, e^{-\lambda(K)t}.$$

Together with (2.5.33) we see that, $P^{B,K}_x$-a.s.,

$$\liminf_{t\to\infty} \frac{\log |N_t|}{\lambda(K)t} \geq \liminf_{t\to\infty} \frac{\log M_{f^*}(t) - \log c + \lambda(K)t}{\lambda(K)t} \geq 1,$$

which completes the proof. □

2.6 Proof of Theorem 2.6

Proof of Theorem 2.6. Recall that $(X, P^{D,K})$ was defined as the process (X, P^K) conditioned on the event of survival and characterised via the change of measure in (2.4.10) and Theorem 2.14.
Fix a $K' > K_0$ and further denote by $N_t|_{(0,K)}$ the set of particles whose ancestors (including themselves) have not exited $(0, K)$ up to time t. Then, for $0 \leq K \leq K'$, and for $x \in (0, K_0)$ and $A \in \mathcal{F}_t$, we can write

$$\lim_{K \downarrow K_0} P_x^{D,K}(A) = \lim_{K \downarrow K_0} E_x^{K'} \left(\mathbf{1}_A \frac{1 - \prod_{u \in N_t|_{(0,K)}} (1 - p_K(x_u(t)))}{p_K(x)} \right),$$

since $N_t|_{(0,K)}$ has the same law under P^K and $P^{K'}$. Suppose the particles in $N_t|_{(0,K)}$ are ordered, for instance according to their spatial positions, and we write $u_1, ..., u_{N_t|_{(0,K)}}$. We can now expand the term within the expectation on the right-hand side as

$$\begin{aligned} &\frac{1 - \prod_{u \in N_t|_{(0,K)}} (1 - p_K(x_u(t)))}{p_K(x)} \\ &\quad = \sum_{i=1}^{|N_t|_{(0,K)}|} \frac{p_K(x_{u_i}(t))}{p_K(x)} \prod_{j<i} (1 - p(x_{u_j}(t))). \end{aligned} \tag{2.6.1}$$

By Lemma 2.24, for each u_i, we have

$$\lim_{K \downarrow K_0} \frac{p_K(x_{u_i}(t))}{p_K(x)} = \frac{\sin(\pi x_{u_i}(t)/K_0)}{\sin(\pi x/K_0)} e^{\mu(x_{u_i}(t) - x)} \mathbf{1}_{\{x_{u_i}(t) \in (0,K_0)\}}.$$

Further, $|\ N_t|_{(0,K)}\ |$ has finite expectation. Therefore, we can apply the Dominated convergence theorem twice to get

$$\begin{aligned} \lim_{K \downarrow K_0} P_x^{D,K}(A) &= E_x^{K'} \Big(\mathbf{1}_A \lim_{K \downarrow K_0} \sum_{i=1}^{|N_t|_{(0,K)}|} \frac{p_K(x_{u_i}(t))}{p_K(x)} \prod_{j<i} (1 - p(x_{u_j}(t))) \Big) \\ &= E_x^{K_0} \Big(\mathbf{1}_A \sum_{i=1}^{|N_t|_{(0,K_0)}|} \frac{\sin(\pi x_{u_i}(t))/K_0) e^{\mu x_{u_i}(t)}}{\sin(\pi x/K_0) e^{\mu x}} \Big) \\ &= E_x^{K_0} \left(\mathbf{1}_A \frac{Z^{K_0}(t)}{Z^{K_0}(0)} \right), \end{aligned}$$

where Z^{K_0} is the martingale used in the change of measure in (2.2.4) in Section 2.2. The evolution under this change of measure is described in the paragraph following (2.2.4) and agrees with that of (X^*, Q_x^*) as defined in Definition 2.5. □

2.7 Super-Brownian motion in a strip

This section is intended to show how the results for branching Brownian motion in a strip can be transferred into the setting of super-Brownian motion in a strip, thus highlighting the robustness of our method. The super-Brownian motion in a strip is simply the one-dimensional version of the super-Brownian motion in a ball in Definition 1.3.

Recall from (2.1.1) that the infinitesimal generator L of the single particle motion is defined for all functions $u \in C^2(0, K)$ with u(0+) $= u(K-) = 0$. Change the domain to $u \in C^2(0, K)$ with $u''(0+) = u''(K-) = 0$, then L corresponds to Brownian motion with absorption (instead of killing) at 0 and K. As pointed out in Section 1.1.2, for technical reason, we prefer to consider the absorption case from now on. The results for branching Brownian motion with killing at 0 and K also hold in the absorption setting if we restrict the process with absorption to particles within $(0, K)$, in particular when defining N_t as the number of particles alive at time t who have not been absorbed.
Suppose $Y = (Y_t, t \geq 0)$ is the super-Brownian motion in $(0, K)$, meaning that the underlying single particle motion is a Brownian motion with drift $-\mu, \mu \geq 0$ and with absorption upon exiting $(0, K)$ and the branching mechanism ψ is of the form

$$\psi(\lambda) = -a\lambda + b\lambda^2 + \int_0^\infty (e^{-\lambda y} - 1 + \lambda y)\, \Pi(\mathrm{d}y), \quad \lambda \geq 0,$$

where $a = -\psi'(0+) \in (0, \infty)$, $b \geq 0$ and Π is a measure concentrated on $(0, \infty)$ satisfying $\int_{(0,\infty)} (x \wedge x^2)\, \Pi(\mathrm{d}x) < \infty$. Recall that for an initial configuration $\eta \in \mathcal{M}_F(0, K)$ we denote the law of Y by $\mathbf{P}^K_\eta$.

Since we assume $a = -\psi'(0+) > 0$, the function ψ is the branching mechanism of a supercritical continuous-state branching process (CSBP), say $\hat{Z}$, as explained in Section 1.2.1. We assume henceforth that ψ satisfies the non-explosion condition $\int_{0+} |\psi(s)|^{-1}\, \mathrm{d}s = \infty$, see (1.2.1), and further that $\psi(\infty) = \infty$. As pointed out in Section 1.2.1, it follows then that the probability of the event of becoming extinguished, namely $\{\lim_{t\to\infty} \hat{Z}_t = 0\}$, given $\hat{Z}_0 = x$ is equal to $e^{-\lambda^* x}$, where λ^* is the largest root of the branching mechanism ψ. The root λ^* is strictly positive since ψ is supercritical and we assumed $\psi(\infty) = \infty$.
We further assume from now on that $\int^{+\infty} (\psi(s))^{-1}\, \mathrm{d}s < \infty$, and remind the reader that this condition guarantees that the event of becoming extinguished agrees with the event of extinction, that is $\{\exists t > 0 : \hat{Z}_t = 0\}$, a.s., see (1.2.2) in Section 1.2.1. This implies in turn that, for the super-Brownian motion Y, the event of becoming extinguished and the event of extinction agree $\mathbf{P}^K$-a.s. We denote the event of extinction of Y by $\mathcal{E} = \{\exists t > 0 : Y_t(0, K) = 0\}$, where $Y_t(0, K)$ is the total mass within $(0, K)$ at time t.
We define the survival rate w_K of the $\mathbf{P}^K$-superdiffusion Y as the function satis-

fying

$$-\log \mathbf{P}^K_\eta(\mathcal{E}) = \langle w_K, \eta\rangle, \quad \text{for } \eta \in \mathcal{M}_F[0,K].$$

It can be shown, see e.g. [22], that w_K is a solution to

$$Lu - \psi(u) = 0 \quad \text{with} \quad u(0) = u(K) = 0. \tag{2.7.1}$$

Analogous to Proposition 2.1, and assuming henceforth in addition that the condition $\int_1^\infty x \log x \Pi(\mathrm{d}x) < \infty$ is satisfied, it is possible to give a necessary and sufficient condition for a positive survival rate. This follows from a spine change of measure argument in the spirit of Section 2.2 and 2.3, now using the $\mathbf{P}^K_x$-martingale

$$\tilde{Z}^K(t) = \int_0^K \sin(\pi x/K) e^{\mu x - \lambda(K)t} Y_t(\mathrm{d}x), \quad t \geq 0, \tag{2.7.2}$$

where here $\lambda(K) = -\psi'(0+) - \mu^2/2 - \pi^2/2K^2$. One can then show, in the fashion of Kyprianou et al. [51], that w_K is positive if $\tilde{Z}^K$ is an $L^1(\mathbf{P}^K_x)$-martingale and the latter holds if and only if $\lambda(K) > 0$.
In Section 1.1.2 we constructed the super-Brownian motion in a ball through an approximation by branching Brownian motions in a ball. Here we present yet another connection between the $\mathbf{P}^K$-superdiffusion and a P^K-branching diffusion via the following relations. Set

$$F(s) = \frac{1}{\lambda^*}\psi(\lambda^*(1-s)), \quad s \in (0,1), \tag{2.7.3}$$

$$\bar{w}_K(x) = \lambda^* p_K(x), \qquad x \in (0,K), \tag{2.7.4}$$

where p_K is the survival probability of the P^K-branching diffusion with branching mechanism F of (2.7.3). Bertoin et al. [7] show that (2.7.3) is the branching mechanism of a Galton-Watson process which they identify as the backbone of the CSBP with branching mechanism ψ. Asymptotic results for the survival rate w_K of Y follow immediately from those for the survival probability p_K if we can show that $\bar{w}_K$ as defined via (2.7.4) agrees with w_K.

Theorem 2.31. *(i) If $\mu < \sqrt{-2\psi'(0+)}$ and $K > K_0$, where*

$$K_0 := \frac{\pi}{\sqrt{-2\psi'(0+) - \mu^2}},$$

then $w_K(x) > 0$ for all $x \in (0,K)$; otherwise $w_K(x) = 0$ for all $x \in [0,K]$.

(ii) Uniformly for $x \in (0, K_0)$, as $K \downarrow K_0$,

$$w_K(x) \sim \lambda^*(K - K_0)\frac{(K_0^2\mu^2 + \pi^2)(K_0^2\mu^2 + 9\pi^2)}{12\psi'(0+)\pi K_0^3(e^{\mu K_0} + 1)} \sin(\pi x/K_0)e^{\mu x}.$$

Proof of Theorem 2.31. The relation in (2.7.3) gives $(m-1)\beta = -\psi'(0+)$. Hence, the K_0 and the $\lambda(K)$ defined in this section are the same as the ones in Proposition 2.1 and 2.9.
Suppose $\mu < \sqrt{-2\psi'(0+)}$ and $K > K_0$. By Proposition 2.12 and Remark 2.18, p_K is the unique non-trivial solution to $Lu - F(1-u) = 0$ on $(0, K)$ with $u(0) = u(K) = 0$. Using (2.7.3) it follows then that $\bar{w}_K$ given by (2.7.4) solves (2.7.1). We can further deduce from this transformation that (2.7.1) has a unique non-trivial solution. On the other hand, we know that w_K solves (2.7.1) and, by the spine argument we mentioned after (2.7.2), we know that w_K is positive within $(0, K)$. By uniqueness, we thus have $\bar{w}_K = w_K$.
Suppose $\mu \geq \sqrt{-2\psi'(0+)}$ or $K \leq K_0$. Then p_K is identically zero and (2.3.7) does not have a non-trivial solution. By the transformation in (2.7.3), the same holds true for (2.7.1) and since w_K is always a solution to (2.7.1) it must be equal to zero. Thus $w_K = \bar{w}_K$ holds true again.
The result is now a consequence of Proposition 2.1 and Theorem 2.4. □

Let us now outline the backbone decomposition for the $\mathbf{P}_\eta^K$-superdiffusion. We begin by studying $(Y, \mathbf{P}^K)$ conditioned on becoming extinct.

Proposition 2.32. *For $\eta \in \mathcal{M}_F[0, K]$ and $t \geq 0$, we define*

$$\left.\frac{\mathrm{d}\mathbf{P}_\eta^{R,K}}{\mathrm{d}\mathbf{P}_\eta^K}\right|_{\mathcal{H}_t} = \frac{e^{-\langle w_K, Y_t\rangle}}{e^{-\langle w_K, \eta\rangle}}, \tag{2.7.5}$$

where $(\mathcal{H}_t, t \geq 0)$ is the natural filtration generated by $(Y, \mathbf{P}_\eta^K)$. Then $(Y, \mathbf{P}_\eta^{R,K})$ is equal in law to $(Y, \mathbf{P}_\eta^K(\cdot|\mathcal{E}))$. Further $(Y, \mathbf{P}_\eta^{R,K})$ has spatially dependent branching mechanism

$$\psi^{R,K}(s, x) = \psi(s + w_K(x)) - \psi(w_K(x)), \quad s \geq 0 \text{ and } x \in [0, K]$$

and the underlying motion is a Brownian motion with absorption upon exiting $(0, K)$.

The proof of Proposition 2.32 is just a straightforward adaptation of the proof of Lemma 2 in [5] and thus omitted. We point out that the motion of the $\mathbf{P}^{R,K}$-superdiffusion remains unchanged and it is therefore different from the motion of the $P^{R,K}$-branching diffusion in Proposition 2.13.
Let us introduce some notation before we proceed with the backbone decomposition. Associated to the laws $\{\mathbf{P}_{\delta_x}^{R,K}, x \in [0, K]\}$ is the family of the so-called

excursion measures $\{\mathbb{N}_x^{R,K}, x \in [0,K]\}$, defined on the same measurable space, which satisfy

$$\mathbb{N}_x^{R,K}(1-\exp\{-\langle f,\ Y_t\rangle\}) = -\log \mathbf{E}_{\delta_x}^{R,K}(\exp\{-\langle f, Y_t\rangle\}),$$

for any positive, bounded, measurable f on $[0,K]$ and $t \geq 0$. These measures are formally defined and studied in Dynkin and Kuznetsov [23]. Further, we define

$$\rho_n(\mathrm{d}y, x) = \frac{b w_K(x)^2 \delta_0(dy) \mathbf{1}_{\{n=2\}} + w_K(x)^n \frac{y^n}{n!} e^{w_K(x) y} \Pi(\mathrm{d}y)}{q_n^{B,K}(x) w_K(x) \beta^{B,K}(x)},$$

for $n \geq 2$, $x \in (0,K)$.

Definition 2.33 (The dressed backbone)**.** *Let $K > K_0$ and $\nu \in \mathcal{M}_a(0,K)$. Let $X^B = (X_t^B, t \geq 0)$ be a $P_\nu^{B,K}$- branching diffusion (which is the backbone of the P^K-branching diffusion with branching mechanism F given by (2.7.3)).*
Dress the trajectories of X^B in such a way that a particle at space-time position $(x,t) \in \mathbb{R}^d \times [0,\infty)$ has an independent $\mathcal{M}_F(0,K)$-valued process grafted on with rate

$$2b \ \mathrm{d}t \times \mathrm{d}\mathbb{N}_x^R + \int_0^\infty y \exp\{-w_K(x) y\} \ \Pi(\mathrm{d}y) \times \mathrm{d}\mathbf{P}_{y\delta_x}^{R,K}.$$

Moreover, when an individual in X^B gives birth to $n \geq 2$ offspring, then an additional independent copy of $(Y, \mathbf{P}^{R,K})$ with initial mass $y \geq 0$ is grafted on to the space-time branch point (x,t) with probability $\rho_n(\mathrm{d}y, x)$.
For $t \geq 0$, let Y_t^D consists of the total dressed mass present at time t. We define the process $Y^D := (Y_t^D, t \geq 0)$ and denote its law by $\mathbf{P}_\nu^{D,K}$.

Theorem 2.34 (Backbone decomposition)**.** *Let $K > K_0$ and $\eta \in \mathcal{M}_F[0,K]$. Suppose that ν is a Poisson random measure on $(0,K)$ with intensity $w_K(x)\eta(dx)$. Let $Y^R = (Y_t^R, t \geq 0)$ be an independent copy of $(Y, \mathbf{P}_\eta^{R,K})$ and let $(Y^D, \mathbf{P}_\nu^{D,K})$ be the process constructed in Definition 2.33. Define the process $\tilde{Y} = (\tilde{Y}_t, t \geq 0)$ by*

$$\tilde{Y}_t = Y_t^R + Y_t^D, \quad t \geq 0,$$

and denote its law by $\mathbf{P}_\eta^{C,K}$. Then the process $(\tilde{Y}, \mathbf{P}_\eta^{C,K})$ is Markovian and equal in law to $(Y, \mathbf{P}_\eta^K)$.

The proof of Theorem 2.34 is a simple adaptation of the proofs of Theorem 1 and 2 in [5] and therefore omitted.
Conditioning $(Y, \mathbf{P}_\eta^K)$ on non-extinction is the same as conditioning the Poisson random measure ν in Theorem 2.34 on having at least one atom from which a copy of $(Y^D, \mathbf{P}^{D,K})$ is then issued. In principle it should be possible to give a proof

analogous to the ones presented in Section 2.4, using that $(Y, \mathbf{P}^K_\eta)$ conditioned on non-extinction arises from a change of measure using the martingale

$$1 - e^{-\langle w_K, Y_t \rangle}, \quad t \geq 0, \tag{2.7.6}$$

together with the martingale change of measure in (2.7.5) which conditions $(Y, \mathbf{P}^K_\eta)$ on extinction.
The analogy between the P^K-branching diffusion and the $\mathbf{P}^K$-superdiffusion indicates that there is a quasi-stationary limit result equivalent to Theorem 2.6. We begin with constructing the limiting process. To this end, define the family of excursion measures $\{\mathbb{N}^{K_0}_x, x \in [0, K_0]\}$, now associated with the laws $(\mathbf{P}^{K_0}_{\delta_x}, x \in [0, K_0])$, satisfying

$$\mathbb{N}^{K_0}_x(1 - \exp\langle f, Y_t \rangle) = -\log \mathbf{E}^{K_0}_{\delta_x}(e^{-\langle f, Y_t \rangle}), \quad t \geq 0$$

for any positive, bounded, measurable function f on $(0, K)$.

Definition 2.35. *Let $\eta \in \mathcal{M}_F[0, K]$. Suppose $\xi^* = (\xi^*_t, t \geq 0)$ is a Brownian motion conditioned to stay in $(0, K_0)$ with initial position x distributed according to*

$$\frac{\sin(\pi x/K_0)e^{\mu x}}{\int_{(0,K_0)} \sin(\pi z/K_0)e^{\mu z}\ \eta(\mathrm{d}z)}\eta(\mathrm{d}x), \quad x \in (0, K_0).$$

Along the the space-time trajectory $\{(\xi^*_s, s) : s \geq 0\}$, we immigrate $\mathcal{M}_F(0, K)$-valued processes at rate

$$2b\ \mathrm{d}s \times \mathrm{d}\mathbb{N}^{K_0}_{\xi^*_s} + \int_0^\infty y\Pi(\mathrm{d}y) \times \mathrm{d}\mathbf{P}^{K_0}_{y\delta_{\xi^*_s}}.$$

Then, let $Y^* = (Y^*_t, t \geq 0)$ be such that Y^*_t consists of the total immigrated mass present at time t together with the mass present at time t of an independent copy of $(Y, \mathbf{P}^{K_0}_\eta)$ issued at time zero. We denote the law of Y^* by $\mathbf{P}^*_\eta$

The evolution of Y^* under $\mathbf{P}^*$ can thus be seen as a path-wise description of Evans' immortal particle picture in [30] for the critical width K_0; for a similar construction of Evans' immortal particle picture see Kyprianou et al. [51].
Further, we note that $(Y^*, \mathbf{P}^{K_0}_\eta)$ has the same law as Y under the measure which has martingale density $\tilde{Z}^{K_0}(t)$, given in (2.7.2), with respect to $\mathbf{P}^{K_0}_\eta$; for similar results see for instance Engländer and Kyprianou [25], Kyprianou et al. [51] and Liu et al. [54].

Theorem 2.36. *Let $K > K_0$ and $\eta \in \mathcal{M}_F[0, K_0]$. For a fixed time $T > 0$, the law of $(Y_t, t \leq T)$ under the measure $\lim_{K \downarrow K_0} \mathbf{P}^K_\eta(\cdot | \lim_{t\to\infty} ||Y_t|| > 0)$ is equal to $(Y^*_t, t \leq T)$ under $\mathbf{P}^*_\eta$.*

To prove Theorem 2.36 it suffices to show that the $\mathbf{P}_\eta^K$-martingale in (2.7.6) converges to the martingale $\tilde{Z}^{K_0}$ in (2.7.2). This is a straightforward adaptation of the proof of Theorem 2.6 and we only sketch the proof.

Sketch of the proof of Theorem 2.36. We obtain the law of $(Y, \mathbf{P}_\eta^K)$ conditioned on non-extinction from a change of measure using the martingale

$$1 - e^{-\langle w_K, Y_t \rangle}, \quad t \geq 0,$$

as in (2.7.6). The uniform asymptotics for w_K in Theorem 2.31 let us conclude that

$$\begin{aligned}\lim_{K \downarrow K_0} \frac{1 - e^{-\langle w_K, Y_t \rangle}}{1 - e^{-\langle w_K, \eta \rangle}} &= \lim_{K \downarrow K_0} \frac{\langle w_K, Y_t \rangle}{\langle w_K, \eta \rangle} \\ &= \frac{\int_0^{K_0} \sin(\pi x / K_0) e^{\mu x} \; Y_t(\mathrm{d}x)}{\int_0^{K_0} \sin(\pi x / K_0) e^{\mu x} \; \mu(\mathrm{d}x) \rangle} = \frac{\tilde{Z}^{K_0}(t)}{\tilde{Z}^{K_0}(0)},\end{aligned}$$

where $\tilde{Z}^{K_0}$ is the martingale in (2.7.2). As mentioned before, the law of Y under a change of measure with $\tilde{Z}^{K_0}$ is equal to $(Y^*, \mathbf{P}_\eta^{K_0})$. □

2.8 Concluding remarks

In this chapter we have considered the one-dimensional setting of a branching Brownian motion in a strip. Most of the results can easily be modified to hold in higher dimensions for a branching Brownian motion in a ball in $\mathbb{R}^d$, $d \geq 1$, as in Definition 1.1.

Let us set the drift $-\mu$ to zero for simplicity. To begin with, for the spine construction in Section 2.2 we need the eigenfunction corresponding to the first positive eigenvalue of the Dirichlet problem

$$\begin{aligned} Lu &= -\lambda u \quad \text{in } D_r & (2.8.1) \\ u &= 0 \quad \text{on } \partial D_r, & (2.8.2) \end{aligned}$$

where L is now $\frac{1}{2}\Delta$, the Laplacian in $\mathbb{R}^d$, and D_r is a ball of radius r. The first positive eigenvalue $\lambda^{(r)}$ is given as

$$\lambda^{(r)} = r^{-2} \lambda^{(1)} = r^{-2} j_{d/2-1}^2,$$

where j_n denotes the first positive zero of the Bessel function J_n of order n. Let $u^{(r)}$ be the corresponding eigenfunction which can be expressed in terms of spherical Bessel functions. Then a h-transform with $u^{(r)}$ conditions the Brownian motion to stay within D_r. The spine construction outlined in Section 2.2 and the proofs in Section 2.3 then carry through. As a result, we find that the probability of survival in a ball of radius r is positive if and only if $2(m-1)\beta - \lambda^{(1)}/r^2 > 0$. This result is a special case of the condition given in Sevast'yanov [63] which deals with BBM with killing upon exiting more general domains in $\mathbb{R}^d$. The critical radius of the ball is therefore

$$r_0 = \sqrt{\frac{\lambda^{(1)}}{2(m-1)\beta}}. \tag{2.8.3}$$

Indeed, this agrees with our result in one dimension where the critical value is $r_0 = \frac{1}{2}\sqrt{\frac{\pi^2}{2(m-1)\beta}} = \frac{1}{2}K_0$ since $\lambda^{(1)}$ is equal to $\pi^2/4$ here.
Consequently, it should be possible to adapt the proof for the asymptotics of the survival probability. Denote by $p_{(r)}(x)$ the probability that the BBM in D_r survives when the initial particle is located at $x \in D_r$. Then we expect to find that $p_{(r)}$ satisfies

$$p_{(r)}(x) \sim C(r)\ u^{(r)}(x) \quad \text{as } r \downarrow r_0.$$

for some constant $C(r)$, with $C(r) \downarrow 0$, uniformly for all $x \in D_{r_0}$. Identifying the constant C_K in the one-dimensional setting relied on the backbone decomposition and having an explicit expression for the solution to the differential equation (2.5.19) and the invariant measures $\Pi^{B,K}$ and $\Pi_*^{B,K}$. While the backbone decomposition does not cause any difficulties, see the comments below, it would be most tedious to try to carry out the computations for $C(r)$ in the higher dimensional setting.

The results presented in this chapter are specific to the BBM in a strip but spine techniques and the backbone decomposition apply in a more general setting.

As pointed out above, for the spine construction in Section 2.2, all we need to do is condition a single particle to stay within D_r. Unfortunately, it is not known how to do this for general Markov processes. A class of Markov processes for which this conditioning is possible is the class of one-dimensional spectrally negative Lévy processes. Lambert [52] shows that a one-dimensional spectrally negative Lévy process can be conditioned to stay in an interval using a certain h-transform. The h-function used in [52] is the eigenfunction corresponding to the first positive eigenvalue, say $\tilde{\lambda}^{(r_0)}$, of the Dirichlet problem (2.8.1) where $\frac{1}{2}\Delta$ is replaced by the infinitesimal generator of the Lévy process. Consequently, it can then be shown by following the proofs in Section 2.2 and 2.3 that the critical radius r_0 is such that $2(m-1)\beta - \tilde{\lambda}^{(r_0)} = 0$.

The idea underlying the backbone decomposition is that every realisation of a BBM in a strip can be decomposed into a subtree containing all infinite lines of descent which is 'dressed' with finite subtrees containing all other lines of descent. This is a result of the branching structure of the BBM in a strip and does not depend on the single particle motion and does not require any particular form for the branching mechanism. In Section 2.4, we derived the backbone decomposition from a combination of changes of measures using the martingales

$$\Big(\prod_{u\in N_t} p_K(x_u(t)), t\geq 0\Big) \quad \text{and} \quad \Big(1-\prod_{u\in N_t} p_K(x_u(t)), t\geq 0\Big),$$

which condition the BBM in $(0,K)$ on extinction and survival respectively. The martingale property of these two processes is a simple consequence of the Markov branching property of the BBM in a strip, cf. Proposition 2.12, and only requires the underlying single particle motion to be Markovian. In principle, it should therefore be possible to derive the backbone decomposition for general supercritical Markov branching processes in the same way as in Section 2.4.
Particular features of the underlying Brownian motion are only used when we identify explicitly the single particle motion under $P^{R,K}$ and $P^{B,K}$ and the branching mechanism $F^{R,K}$ and $F^{B,K}$. Essentially, to derive the single particle motions, we need the processes

$$\Big((1-p_K(\xi_t))e^{\int_0^t \frac{F(1-p_K(\xi_s))}{1-p_K(\xi_s)}\,\mathrm{d}s}, t\geq 0\Big) \text{ and } \Big(p_K(\xi_t)e^{-\int_0^t \frac{F(1-p_K(\xi_s))}{p_K(\xi_s)}\,\mathrm{d}s}, t\geq 0\Big)$$

to be martingales which follows from an application of Itô's formula and a Feynman-Kac representation (for the latter recall the proof of Proposition 2.11). Once this is known, the characterisation of the red and the (dressed) blue branching processes can be carried out in exactly the same way as in Section 2.4.

As we have not been able to transfer the martingale change of measure approach into the super-Brownian motion setting, we have to derive the backbone decomposition by using the Laplace functional characterisation in Proposition 1.4 instead, cf. the proofs in [5]. In doing so, we rely on analytical tools such as comparison principles from PDE theory. Therefore we cannot simply drop the diffusion assumption even though we would expect that the backbone decomposition exists for more general measure-valued Markov branching processes.

Chapter 3

The total mass of super-Brownian motion upon exiting balls and Sheu's compact support condition

We study the total mass of a d-dimensional super-Brownian motion as it first exits an increasing sequence of balls. The process of the total mass is a time-inhomogeneous continuous-state branching process, where the increasing radii of the balls are taken as the time parameter. We are able to characterise its time-dependent branching mechanism and show that it converges, as time goes to infinity, towards the branching mechanism of the total mass of a one-dimensional super-Brownian motion as it first crosses above an increasing sequence of levels. Our results allow us to identify the compact support criterion given in Sheu [64] as a classical Grey condition [34] for the aforementioned limiting branching mechanism.

3.1 Introduction and main results

Suppose that $Y = (Y_t, t \geq 0)$ is a super-Brownian motion in $\mathbb{R}^d$, $d \geq 1$, with general branching mechanism ψ of the form

$$\psi(\lambda) = -a\lambda + b\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x)\Pi(\mathrm{d}x), \quad \lambda \geq 0, \tag{3.1.1}$$

where $a = -\psi'(0+) \in (-\infty, \infty)$, $b \geq 0$ and Π is a measure concentrated on $(0, \infty)$ which satisfies $\int_{(0,\infty)} (x \wedge x^2)\Pi(\mathrm{d}x) < \infty$. This is the super-Brownian motion of Definition 1.3 when there is no killing. Assume in addition that $\psi(\infty) = \infty$.
Recall that we denote by $\mathbf{P}_\nu$ the law of Y with initial configuration according to $\nu \in \mathcal{M}_F(\mathbb{R}^d)$. We further remind the reader that we call Y (sub)critical if $\psi'(0+) \geq 0$ and supercritical if $\psi'(0+) < 0$ as explained in Section 1.2.1. In the

supercritical case, convexity of ψ and the condition $\psi(\infty) = \infty$ ensure that there is a unique and finite $\lambda^* := \inf\{\lambda \geq 0 : \psi(\lambda) > 0\} > 0$, while $\lambda^* = 0$ in the (sub)critical case. We already noted in Section 1.2.1 that in both cases

$$\mathbf{P}_\nu(\lim_{t\to\infty} ||Y_t|| = 0) = e^{-\lambda^*||\nu||},$$

where $||\nu||$ denotes the total mass of the measure $\nu \in \mathcal{M}_F(\mathbb{R}^d)$.

The aim of this chapter is to study the total mass of the super-Brownian motion Y upon its first exit from a sequence of increasing balls. To this end, we make use of Dynkin's theory of exit measures [20] introduced in Section 1.3. Recall that the exit measure of Y from a ball $\mathbb{B} \subset \mathbb{R}^d$ is the measure $Y_{\mathbb{B}}$ which is supported on the boundary $\partial\mathbb{B}$ and, loosely speaking, consists of the accumulated mass of Y which got 'frozen' upon its first exit from $\mathbb{B}$. The measure $Y_{\mathbb{B}}$ can be characterised analytically through its Laplace functional, see Proposition 1.5. We further recall from Section 1.3 that we can describe the mass of Y as it first exits a sequence of growing balls as a sequence of random measures on $\mathbb{R}^d$, known as Markov branching exit measures. Let us fix an initial radius $r > 0$ and let $D_s = \{x \in \mathbb{R}^d : ||x|| < s\}$ be the open ball of radius $s \geq r$ around the origin. Then we denote the aforementioned sequence of Markov branching exit measures by $(Y_{D_s}, s \geq r)$, where Y_{D_s} is the exit measure of Y from D_s. Formally, $(Y_{D_s}, s \geq r)$ is characterised by the following Markov branching property, see for instance Section 1.1 in Dynkin and Kuznetsov [23]. For $z \geq r$, define $\mathcal{H}_{D_z} := \sigma(Y_{D_{z'}}, r \leq z' \leq z)$.

Proposition 3.1 ([23]). *Let $r > 0$ and $\nu \in \mathcal{M}_F(\bar{D}_r)$. For any positive, bounded, continuous function f on ∂D_s,*

$$\mathbf{E}_\nu[e^{-\langle f, Y_{D_s}\rangle}|\mathcal{H}_{D_z}] = e^{-\langle v_f(\cdot,s), Y_{D_z}\rangle}, \quad 0 < r \leq z \leq s, \tag{3.1.2}$$

where the Laplace functional v_f is the unique non-negative solution to

$$v_f(x,s) = \mathbb{E}_x[f(\xi_{T_{D_s}})] - \mathbb{E}_x\Big[\int_0^{T_{D_s}} \psi(v_f(\xi_z, s))\, \mathrm{d}z\Big], \tag{3.1.3}$$

and $((\xi_z, z \geq 0), \mathbb{P}_x)$ is an $\mathbb{R}^d$-Brownian motion with $\xi_0 = x$ and with $T_{D_s} = \inf\{z > 0 : \xi_z \notin D_s\}$ denoting its first exit time from D_s.

In (3.1.2), we used again the inner product notation $\langle f, \nu\rangle = \int_{\mathbb{R}^d} f(x)\nu(\mathrm{d}x)$. For $s \geq r$, let $Z_s := ||Y_{D_s}||$ denote the total mass that is 'frozen' when it first hits the boundary of the ball D_s. We can then define the total mass process $Z = (Z_s, s \geq r)$ which uses the radius s as its time-parameter and note that, by radial symmetry, the law of Z depends on Y_{D_s} through its total mass only. Let us write $\bar{\mathbf{P}}_r$, for the law of the process $(Z_s, s \geq r)$ starting at time $r > 0$ with unit initial mass. In case we start with non-unit initial mass $a > 0$ we shall use

the notation $\bar{\mathbf{P}}_{a,r}$ for its law.
It is not difficult to see that Z is a time-inhomogeneous continuous-state branching process and we can characterise it as follows.

Theorem 3.2. *(i) Let $r > 0$. The process $Z = (Z_s, s \geq r)$ is a time-inhomogeneous continuous-state branching process. This is to say it is a $[0,\infty]$-valued strong Markov process with càdlàg paths satisfying the branching property*

$$\bar{\mathbf{E}}_{(a+a'),r}[e^{-\theta Z_s}] \quad = \quad \bar{\mathbf{E}}_{a,r}[e^{-\theta Z_s}]\bar{\mathbf{E}}_{a',r}[e^{-\theta Z_s}], \tag{3.1.4}$$

for all $a, a' > 0$, $\theta \geq 0$ and $s \geq r$.
(ii) Let $r > 0$ and $a > 0$. Then, for $s \geq r$, we have

$$\bar{\mathbf{E}}_{a,r}[e^{-\theta Z_s}] \quad = \quad e^{-u(r,s,\theta)a}, \quad \theta \geq 0, \tag{3.1.5}$$

where the Laplace functional $u(r,s,\theta)$ satisfies

$$u(r,s,\theta) = \theta - \int_r^s \Psi(z, u(z,s,\theta))\ \mathrm{d}z, \tag{3.1.6}$$

for a family of branching mechanisms $(\Psi(r,\cdot), r > 0)$ of the form

$$\Psi(r,\theta) \quad = \quad -q_r + a_r\theta + b_r\theta^2 + \int_{(0,\infty)} (e^{-\theta x} - 1 + \theta x \mathbf{1}_{(x<1)})\Lambda_r(\mathrm{d}x), \tag{3.1.7}$$

for $\theta \geq 0$, and for each $r > 0$ we have $q_r \geq 0$, $a_r \in \mathbb{R}$, $b_r \geq 0$ and Λ_r is a measure concentrated on $(0,\infty)$ satisfying $\int_{(0,\infty)}(1 \wedge x^2)\Lambda_r(\mathrm{d}x) < \infty$.
(iii) The branching mechanism Ψ satisfies the PDE

$$\begin{aligned} \frac{\partial}{\partial r}\Psi(r,\theta) + \frac{1}{2}\frac{\partial}{\partial\theta}\Psi^2(r,\theta) + \frac{d-1}{r}\Psi(r,\theta) \quad &= \quad 2\psi(\theta) \qquad r > 0,\ \theta \in (0,\infty) \\ \Psi(r,\lambda^*) \quad &= \quad 0, \qquad r > 0. \end{aligned} \tag{3.1.8}$$

We are not aware of a result in the literature which states that the definition of the time-dependent CSBP in (i) implies the characterisation in (ii). It is therefore outlined in the proof of Theorem 3.2 (ii) in Section 3.2.1 how this implication can be derived as a generalisation of the equivalent result for standard CSBPs in Silverstein [66].
As part of Theorem 3.2 (iii), we later prove that the root λ^* of ψ is also the root for each $\Psi(r,\cdot)$, $r > 0$, cf. Lemma 3.7. This will be a key property for the forthcoming analysis of the family of branching mechanism $(\Psi(r,\cdot), r > 0)$.

Let us now describe how Ψ changes as r increases. We observe the following change in the shape of the branching mechanism, see Figure 3-1.

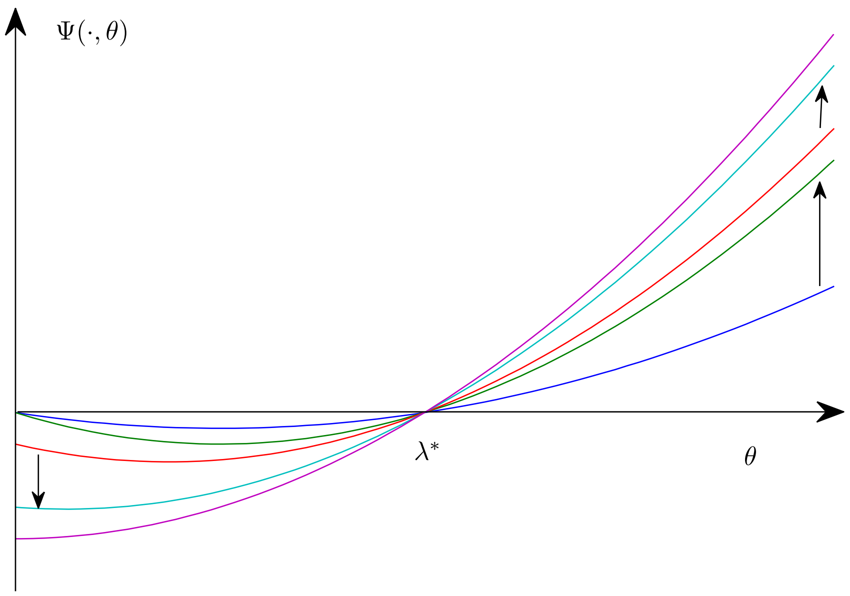


Figure 3-1: Shape of the branching mechanism $\Psi(r,\cdot)$ as $r\to\infty$ in the supercritical case

Proposition 3.3. *(i) For (sub)critical ψ, we have, for $0<r\le s$,*

$$\Psi(r,\theta)\le\Psi(s,\theta)\quad \textit{for all}\ \ \theta\ge 0.$$

(ii) For supercritical ψ, we have, for $0<r\le s$,

$$\Psi(r,\theta)\ge\Psi(s,\theta)\quad \textit{for all}\ \ \theta\le\lambda^*$$
$$\Psi(r,\theta)\le\Psi(s,\theta)\quad \textit{for all}\ \ \theta\ge\lambda^*.$$

This result suggests that there is a limiting branching mechanism $\Psi_\infty(\cdot):=\lim_{r\to\infty}\Psi(r,\cdot)$. Intuitively speaking, in the case where the initial mass is supported on a large ball, the local behaviour of the super-Brownian motion when exiting increasingly larger balls should look like a one-dimensional super-Brownian upon crossing levels. This idea is supported by the following result.

Theorem 3.4. *For each $\theta\ge 0$, the limit $\lim_{r\uparrow\infty}\Psi(r,\theta)=\Psi_\infty(\theta)$ is finite and the convergence holds uniformly in θ on any bounded, closed subset of $[0,\infty)$.*
(i) For any $\theta\ge 0$, we have

$$\Psi_\infty(\theta)=2\,\mathrm{sgn}(\psi(\theta+))\sqrt{\int_{\lambda^*}^{\theta}\psi(\lambda)\;\mathrm{d}\lambda},\tag{3.1.9}$$

with $\lambda^* = 0$ *in the (sub)critical case.*
(ii) Denote by $((Z_s^\infty, s \geq 0), \bar{\mathbf{P}}^\infty)$ *the standard CSBP associated with the limiting branching mechanism* Ψ_∞*, with unit initial mass at time* 0*.*
Then, $(Z_s^\infty, s \geq 0)$ *is the total mass of the process of Markov branching exit measures of a one-dimensional super-Brownian motion with unit initial mass at time zero as it first exits the family of intervals* $((-\infty, s), s \geq 0)$*.*
Further, for any $s > 0$*,* $\theta \geq 0$*,*

$$\lim_{r\to\infty} \bar{\mathbf{E}}_r[e^{-\theta Z_{r+s}}] = \bar{\mathbf{E}}^\infty[e^{-\theta Z_s^\infty}]. \tag{3.1.10}$$

Let us remark that, in the supercritical case, the limiting branching mechanism Ψ_∞ is critical and possesses an explosion coefficient, that is $\Psi'_\infty(0+) = 0$ and $\Psi_\infty(0) < 0$. Thanks to the uniform continuity in θ, this implies that $\Psi(t, 0) < 0$ for all sufficiently large t.
The limiting process Z^∞ in Theorem 3.4 has already been studied in Theorem 3.1 in Kyprianou et al. [51]. Note that therein the underlying Brownian motion has a positive drift which is chosen such that the resulting branching mechanism is non-explosive. The characterisation can easily be adapted to the drift-less case as stated in Theorem 3.4 (ii). Kaj and Salminen [43, 44] studied the analogous process in the setting of branching particle diffusions, that is the process of the number of particles of a one-dimensional branching Brownian motion stopped upon exiting the interval $((-\infty, s), s \geq 0)$, which was already introduced by Neveu [58] for the case of a binary branching Brownian motion with drift. Kaj and Salminen discover in the supercritical case [43] that the resulting offspring distribution is degenerate, meaning that

$$\sum_{i\geq 0} p_i < 1, \tag{3.1.11}$$

where p_i is the probability of having i offspring, $i \geq 0$. In particular, the probability of a birth event with an infinite number of offspring is strictly positive. In this view, (3.1.11) is the analogue of $\Psi_\infty(0) < 0$.

In Sheu [64, 65], asymptotics of the process Z are studied in order to obtain a compact support criterion for the super-Brownian motion Y. It is found that the event of extinction of Z, i.e. $\{\exists s > 0 : Z_s = 0\}$, and the event $\{Y \text{ has compact support}\}$ agree $\mathbf{P}_\nu$-a.s., c.f. [65], Theorem 4.1.
The following result on the asymptotic behaviour of Z is given by Sheu [64].

Theorem (Sheu [64] Theorem 1.1, Theorem 1.2, Cor 1.1)**.** *Let* $\nu \in \mathcal{M}_F(\mathbb{R}^d)$*. The event* $\{\exists s > 0 : Z_s = 0\}$ *agrees* $\mathbf{P}_\nu$*-a.s. with the event* $\{\lim_{s\to\infty} Z_s = 0\}$ *if* ψ

satisfies

$$\int^{\infty} \frac{1}{\sqrt{\int_{\lambda^*}^{\lambda} \psi(\theta)\ \mathrm{d}\theta}}\ \mathrm{d}\lambda < \infty. \tag{3.1.12}$$

Otherwise, $\{\exists s > 0 : Z_s = 0\}$ *has probability* 0.

In short, the event of extinction of Z agrees with the event of extinguishing of Z, that is $\{\lim_{s\to\infty} Z_s = 0\}$, if and only if (3.1.12) holds, and it has zero probability otherwise. We have stated the theorem slightly differently from its original version in which, in the supercritical case, condition (3.1.12) reads $\int_s^{\infty} \frac{1}{\sqrt{\int_0^{\lambda} \phi(\theta)\ \mathrm{d}\theta}}\ \mathrm{d}\lambda < \infty$, for $\phi(s) := \psi(s) - as$. The equivalence of these two conditions was already pointed out in [51].
The unusual condition (3.1.12) corresponds to Grey's condition in [34] for extinction vs. extinguishing in the following sense. Recall from Section 1.2.1 that Grey's condition says that, for a standard CSBP with branching mechanism F, the event of extinction agrees with the event of becoming extinguished if and only if $\int^{\infty} F(\theta)^{-1}\ \mathrm{d}\theta < \infty$, and has probability zero otherwise. Then the following interpretation of (3.1.12) is an immediate consequence of Theorem 3.4 (i).

Corollary 3.5. *Sheu's compact support condition (3.1.12) is Grey's condition for the limiting standard CSBP* Z^{∞} *with branching mechanism* Ψ_{∞} *in (3.1.9).*

Sheu's compact support condition (3.1.12) plays an important role when studying the radial speed of the support of supercritical Super-Brownian motion. In the one-dimensional case, assuming (3.1.12), Kyprianou et. al [51], Corollary 3.2, show that

$$\lim_{t\to\infty} \frac{\mathcal{R}_t}{t} = \sqrt{-2\psi'(0+)}, \quad \mathbf{P}_{\nu} - a.s,\ \nu \in \mathcal{M}_F(\mathbb{R}), \tag{3.1.13}$$

where $\mathcal{R}_t := \sup\{r > 0 : Y_t(r, \infty) > 0\}$ is the right-most point of the support of Y_t. A key step in the proof is to study the total mass of the process of branching exit measures of a one-dimensional super-Brownian motion with drift $c := -\sqrt{-2\psi'(0+)}$ upon exiting the increasing sequence of intervals $((-\infty, s), s \geq 0)$, which we denote here by $Z^c = (Z^c_s, s \geq 0)$. It is proved in Theorem 3.1 in [51] that Z^c is a subcritical standard CSBP. Now condition (3.1.12) comes in. Corollary 3.5 interprets (3.1.12) as Grey's condition for the standard CSBP Z^{∞}. The CSBPs Z^{∞} and Z^c only differ in that the underlying Brownian motion of the latter has drift c and it is not difficult to convince ourselves that the drift term is irrelevant when studying the extinction vs. extinguishing problem, see (29) in [51] for a rigorous argument. Therefore condition (3.1.12) is also equivalent to Grey's condition for the subcritical CSBP Z^c and hence ensures that Z^c becomes extinct $\mathbf{P}_{\nu}$-a.s. Extinction of Z^c now implies that the right-most point

of the support of Y cannot travel at a speed faster than $\sqrt{-2\psi'(0+)}$.
In order to make this last conclusion, extinguishing of Z^c is clearly not sufficient and it remains an open questions whether a strong law for $(\mathcal{R}_t, t \geq 0)$ can exist when (3.1.12) fails.
In the d-dimensional case, $d \geq 1$, and with a quadratic branching mechanism of the form $\psi(\lambda) = -a\lambda + b\lambda^2$, for $a, b \geq 0$, Kyprianou [49] shows that (3.1.13) holds, where $\mathcal{R}_t$ is now replaced by $\tilde{\mathcal{R}}_t := \sup\{r > 0 : Y_t(\mathbb{R}^d \backslash D_r) > 0\}$, the radius of the support of Y_t. It can be checked that condition (3.1.12) is satisfied for this choice of ψ. It is possible to adapt the higher-dimensional result in [49] to hold for general branching mechanisms provided (3.1.12) holds.

The remainder of this chapter is organised as follows. In Section 3.2 we prove Theorem 3.2 which is followed by the proof of Proposition 3.3 and Theorem 3.4 in Section 3.3.

3.2 Characterising the process Z - Proof of Theorem 3.2

3.2.1 Proof of Theorem 3.2 (i) and (ii)

Proof of Theorem 3.2 (i). Take a look at equation (3.1.2) which characterises the sequence of branching exit measures $(Y_{D_s}, s \geq r)$. For any measure $\nu \in \mathcal{M}_F(\partial D_r)$ with $||\nu|| = a$, we can write

$$\bar{\mathbf{P}}_{a,r}[e^{-\theta Z_s}] \;=\; \mathbf{E}_\nu[e^{-\theta||Y_{D_s}||}] = e^{-\langle v_\theta(\cdot,s),\nu\rangle} = e^{-v_\theta(x,s)a},$$

for any $x \in \partial D_r$, by radial symmetry. The branching property of Z now follows easily from the branching property of $(Y_{D_s}, s > r)$ in (3.1.2) since, for $a, a' > 0$, $0 < r \leq s$,

$$\begin{aligned}\bar{\mathbf{E}}_{(a+a'),r}[e^{-\theta Z_s}] &= \mathbf{E}_{\nu+\nu'}[e^{-\theta||Y_{D_s}||}] \\ &= e^{-v_\theta(x,s)(a+a')} \\ &= \mathbf{E}_\nu[e^{-\theta||Y_{D_s}||}]\mathbf{E}_{\nu'}[e^{-\theta||Y_{D_s}||}] = \bar{\mathbf{E}}_{a,r}[e^{-\theta Z_s}]\bar{\mathbf{E}}_{a',r}[e^{-\theta Z_s}],\end{aligned}$$

for measures ν, $\nu' \in \mathcal{M}_F(\partial D_r)$ with $||\nu|| = a$, $||\nu'|| = a'$. The Markov property is also an immediate consequence of (3.1.2). □

Proof of Theorem 3.2 (ii). First note that, by radial symmetry as seen in the proof of Theorem 3.2 (i), (3.1.5) holds with $u(r, s, \theta) = v_\theta(x, s)$ for $x \in \partial D_r$ where $r = ||x||$. It remains to show that (3.1.6) and (3.1.7) are satisfied.
The analogue result to (3.1.6) and (3.1.7) for standard (time-homogeneous) CSBPs is given in Theorem 4 in Silverstein [66] and it was already discussed at the end

of Section 4 in [66] that it is possible to allow time-dependence of the CSBP in Theorem 4, [66]. Let us briefly explain how the arguments in [66], Section 4, can be adapted to our time-inhomogeneous case. For any $0 < r \le z \le s$, $\theta \ge 0$,

$$\bar{\mathbf{E}}_r[e^{-\theta Z_s}] = \bar{\mathbf{E}}_r[\bar{\mathbf{E}}_{Z_z,z}[e^{-\theta Z_s}]] = \bar{\mathbf{E}}_r[e^{-u(z,s,\theta)Z_z}] = e^{-u(r,z,u(z,s,\theta))},$$

which shows that the Laplace functional satisfies the composition property

$$u(r,s,\theta) = u(r,z,u(z,s,\theta)) \quad \text{for } 0 < r \le z \le s,\ \theta \ge 0. \tag{3.2.1}$$

The branching property of Z implies that, for any fixed $0 < r \le s$, the law of $(Z_s, \bar{\mathbf{P}}_r)$ is an infinitely divisible distribution on $[0,\infty]$. It follows then from the Lévy-Khintchin formula that, for fixed r and s, $u(r,s,\theta)$ is a non-negative, completely concave function as considered in Section 4 in Silverstein [66]. The process Z thus has the properties of a time-dependent version of the CSBP considered in Definition 4 in Section 4 of [66]. We can then adapt the proof of Theorem 4 in [66] to show that there exists a branching mechanism Ψ of the form (3.1.7) such that

$$\frac{\partial}{\partial r}u(r,s,\theta)\big|_{r=s} = \Psi(s,\theta), \quad \text{for } s > 0,\ \theta \ge 0.$$

With the composition property (3.2.1), we then get

$$\frac{\partial}{\partial r}u(r,s,\theta) \;=\; \Psi(r,u(r,s,\theta)), \quad \text{for } 0 < r \le s,\ \theta \ge 0.$$

Together with the initial condition $u(r,r,\theta) = \theta$, we obtain equation (3.1.6). □

From (3.1.6), we get an alternative characterisation of the relation between the Laplace functional u and the branching mechanism Ψ as

$$\begin{aligned} \frac{\partial}{\partial s}u(r,s,\theta) &= -\Psi(s,\theta)\frac{\partial}{\partial \theta}u(r,s,\theta) && (3.2.2)\\ \frac{\partial}{\partial r}u(r,s,\theta) &= \Psi(r,u(r,s,\theta)) && (3.2.3)\\ u(r,r,\theta) &= \theta, \end{aligned}$$

for any $s > r > 0$ and $\theta \ge 0$. To see where equation (3.2.2) comes from, compare the derivatives of (3.1.6) in s and θ, that is

$$\begin{aligned} \frac{\partial}{\partial s}u(r,s,\theta) &= -\Psi(s,\theta) - \int_r^s \frac{\partial}{\partial u}\Psi(z,u(z,s,\theta))\frac{\partial}{\partial s}u(z,s,\theta)\,\mathrm{d}z\\ \frac{\partial}{\partial \theta}u(r,s,\theta) &= 1 - \int_r^s \frac{\partial}{\partial u}\Psi(z,u(z,s,\theta))\frac{\partial}{\partial \theta}u(z,s,\theta)\,\mathrm{d}z, \end{aligned}$$

where $\partial\Psi(\cdot,\cdot)/\partial u$ denotes the derivative in the second component of Ψ. We see that $\frac{\partial}{\partial s}u(r,s,\theta)$ and $-\Psi(s,\theta)\frac{\partial}{\partial\theta}u(r,s,\theta)$ are solutions to the same integral equation. With an application of Gronwall's inequality it can be shown that this integral equation has a unique solution.

3.2.2 Proof of Theorem 3.2 (iii)

We have already seen in the previous section that, for any measure $\nu \in \mathcal{M}_F(\partial D_r)$ with $||\nu|| = a$, we can write

$$\bar{\mathbf{E}}_{a,r}[e^{-\theta Z_s}] \;=\; \mathbf{E}_\nu[e^{-\theta||Y_{D_s}||}] = e^{-\langle v_\theta(\cdot,s),\nu\rangle} = e^{-v_\theta(x,s)a},$$

for any $x \in \partial D_r$, by radial symmetry. In particular, we saw that $u(r,s,\theta) = v_\theta(x,s)$ for any $x \in \partial D_r$. From the integral equation for v_θ in (3.1.3), we thus get a representation of u, alternative to the representation in (3.1.6), as the unique non-negative solution to

$$u(r,s,\theta) = \theta - \mathbb{E}_r^{\mathrm{R}}\Big[\int_0^{\tau_s} \psi(u(R_z,s,\theta))\,\mathrm{d}z\Big], \tag{3.2.4}$$

where $(R,\mathbb{P}_r^{\mathrm{R}})$ is a d-dimensional Bessel process and $\tau_s := \inf\{z > 0 : R_z > s\}$ its first passage time above level s.
Equation (3.2.4) tells us that the process Z can be viewed as the total mass process of the Markov branching exit measures of a d-dimensional super-Bessel process with branching mechanism ψ as it first exits the intervals $(0,s)$, $s \geq r$.
Equivalently to the characterisation of $u(r,s,\theta)$ as the unique non-negative solution to the integral equation (3.2.4), we can characterise it as the unique non-negative solution to the differential equation

$$\begin{aligned}\frac{1}{2}\frac{\partial^2}{\partial r^2}u(r,s,\theta) + \frac{d-1}{2r}\frac{\partial}{\partial r}u(r,s,\theta) &= \psi(u(r,s,\theta)), \quad 0 < r < s, \theta \geq 0,\\ u(r,r,\theta) &= \theta. \end{aligned} \tag{3.2.5}$$

We postpone the proof of this equivalence to the final Section 3.4. In the following section, we will use the differential equation (3.2.5) to prove the PDE characterisation of the branching mechanism Ψ in Theorem 3.2 (iii).
We prove Theorem 3.2 (iii) in two parts. In Lemma 3.6 we show that Ψ satisfies the PDE in (3.1.8) before we prove that $\Psi(r,\lambda^*) = 0$, for all $r > 0$, in Lemma 3.7 below.

Lemma 3.6. *The branching mechanism Ψ satisfies the PDE (3.1.8), i.e.*

$$\frac{\partial}{\partial r}\Psi(r,\theta) + \frac{1}{2}\frac{\partial}{\partial\theta}\Psi^2(r,\theta) + \frac{d-1}{r}\Psi(r,\theta) \;=\; 2\psi(\theta) \qquad r > 0,\ \theta \in (0,\infty).$$

Proof of Lemma 3.6. Using (3.2.3), the left-hand side of (3.2.5) becomes

$$\begin{aligned}
&\frac{\partial^2}{\partial r^2}u(r,s,\theta)+\frac{d-1}{r}\frac{\partial}{\partial r}u(r,s,\theta)\\
&= \frac{\partial}{\partial r}\Psi(r,u(r,s,\theta))+\frac{d-1}{r}\Psi(r,u(r,s,\theta))\\
&= \frac{\partial}{\partial y}\Psi(y,u(r,s,\theta))|_{y=r}\\
&\qquad +\frac{\partial}{\partial u}\Psi(r,u(r,s,\theta))\ \Psi(r,u(r,s,\theta))+\frac{d-1}{r}\Psi(r,u(r,s,\theta))\\
&= \frac{\partial}{\partial y}\Psi(y,u(r,s,\theta))|_{y=r}+\frac{1}{2}\frac{\partial}{\partial u}\Psi^2(r,u(r,s,\theta))+\frac{d-1}{r}\Psi(r,u(r,s,\theta)),
\end{aligned}$$

where $\partial\Psi(\cdot,\cdot)/\partial u$ denotes the derivative with respect to the second argument. Note that this equation holds for all $s>r$ and $\theta\geq 0$. Since $u(r,s,\theta)\to\theta$ as $s\downarrow r$, we see that, for fixed r, the range of $u(r,s,\theta)$ is $(0,\infty)$ as we vary $s\in(r,\infty)$ and $\theta\in[0,\infty)$. Hence, we can replace $u(r,s,\theta)$ above by an arbitrary $\theta\in(0,\infty)$ and conclude that the PDE (3.1.8) holds true. □

Recall that $\lambda^*=\inf\{\lambda\geq 0:\psi(\lambda)>0\}$ denotes the root of ψ and define $\lambda^*(r):=\inf\{\lambda\geq 0:\Psi(r,\lambda)>0\}$, for $r>0$.

Lemma 3.7. *(i) In the (sub)critical case, for all $r>0$, we have $\lambda^*(r)=0$. In particular, $\Psi(r,\theta)\geq 0$ for all $\theta\geq 0$.*
(ii) In the supercritical case, for all $r>0$, we have $\lambda^*(r)=\lambda^*$. In particular, $\Psi(r,\theta)\leq 0$ for $\theta\leq\lambda^*$, while $\Psi(r,\theta)\geq 0$ for $\theta\geq\lambda^*$.

We prove part (i) and (ii) separately.

Proof of Lemma 3.7 (i). As we are in the (sub)critical case we have $\psi(\theta)\geq 0$ for all $\theta\geq 0$. For $r<z<s$, (3.2.4) yields

$$\begin{aligned}
u(r,s,\theta) &= \theta-\mathbb{E}_r^{\mathrm{R}}\int_0^{\tau_s}\psi(u(R_v,s,\theta))\ \mathrm{d}v\\
&= \theta-\mathbb{E}_r^{\mathrm{R}}\int_0^{\tau_z}\psi(u(R_v,s,\theta))\ \mathrm{d}v-\mathbb{E}_z^{\mathrm{R}}\int_0^{\tau_s}\psi(u(R_v,s,\theta))\ \mathrm{d}v\\
&\leq \theta-\mathbb{E}_z^{\mathrm{R}}\int_0^{\tau_s}\psi(u(R_v,s,\theta))\ \mathrm{d}v\\
&= u(z,s,\theta). \qquad (3.2.6)
\end{aligned}$$

Hence, $u(r,s,\theta)$ is non-decreasing in r. With (3.2.3) we thus see that, for all $0<r<s$, $\theta\geq 0$,

$$\Psi(r,u(r,s,\theta))=\frac{\partial}{\partial r}u(r,s,\theta)\geq 0. \qquad (3.2.7)$$

As we take $s \downarrow r$, we get $u(r,s,\theta) \to \theta$ and hence $\Psi(r,\theta) \geq 0$ for all $\theta > 0$, $r > 0$. Continuity of Ψ ensures $\Psi(r,0) = 0$ and, in particular, $\lambda^*(r) = 0$ for all $r > 0$. □

The key to the proof of part (ii) of Lemma 3.7 is the following lemma.

Lemma 3.8. *Fix $r > 0$.*
(i) For any $\lambda > 0$, the process

$$M_s^\lambda \;=\; e^{-\lambda Z_s} - \int_r^s \Psi(v,\lambda) Z_v e^{-\lambda Z_v} \mathbf{1}_{\{Z_v<\infty\}} \mathrm{d}v, \quad s \geq r, \tag{3.2.8}$$

is a $\bar{\mathbf{P}}_r$-martingale.
(ii) The process $(e^{-\lambda^* Z_s}, s \geq r)$ is a $\bar{\mathbf{P}}_r$-martingale.
Here we use the convention $e^{-\lambda Z_s}\mathbf{1}_{\{Z_s=\infty\}} = 0$, for any $\lambda > 0$.

Proof Lemma 3.8 (i). Taking expectations in (3.2.8) and interchanging expectation and integral gives

$$\bar{\mathbf{E}}_r[M_s^\lambda] \;=\; e^{-u(r,s,\lambda)} - \int_r^s \Psi(v,\lambda) \frac{\partial}{\partial\lambda} u(r,v,\lambda)\, e^{-u(r,v,\lambda)} \mathrm{d}v.$$

Differentiating in s, together with (3.2.2), gives

$$\frac{\partial}{\partial s}\bar{\mathbf{E}}_r[M_s^\lambda] \;=\; \Big(-\frac{\partial}{\partial s}u(r,s,\lambda) - \Psi(s,\lambda)\frac{\partial}{\partial\lambda}u(r,s,\lambda)\Big)e^{-u(r,s,\lambda)} = 0.$$

Hence, $\bar{\mathbf{E}}_r[M_s^\lambda]$ is constant for all $s \geq r$ and in particular, taking $s = r$, equal to $e^{-\lambda}$. Note that the same computation gives that $E_{a,v}[M_s^\lambda] = e^{-\lambda a}$, for $a > 0$ and $0 < r \leq v \leq s$. An application of the Markov property then shows that $(M_s^\lambda, s \geq r)$ is a martingale for any $\lambda > 0$. □

The proof of Lemma 3.8 (ii) relies on the following idea. Since $(||Y_t||, t \geq 0)$ is a CSBP with branching mechanism ψ it is well-known that the process $(e^{-\lambda^*||Y_t||}, t \geq 0)$ is a martingale with respect to the filtration $(\sigma(||Y_u||, u \leq t), t \geq 0)$. Denoting by $\mathcal{E}(Y) := \{||Y_u|| \to 0 \text{ as } u \to \infty\}$ the event of extinguishing of Y, the martingale property follows on account of the fact that

$$\mathbf{E}_\nu[[\mathbf{1}_{\mathcal{E}(Y)}|\sigma(||Y_u||, u \leq t)] = e^{-\lambda^*||Y_t||}, \ t \geq 0,$$

by a simple application of the tower property. Now, fix $r > 0$, and consider the filtration $(\sigma(||Y_{D_v}||, r \leq v \leq s), s \geq r) = (\sigma(Z_v, r \leq v \leq s), s \geq r)$ instead. If we can show that, for $\nu \in \mathcal{M}_F(\partial D_r)$,

$$\mathbf{E}_\nu[\mathbf{1}_{\mathcal{E}(Y)}|\sigma(||Y_{D_v}||, r \leq v \leq s)] = e^{-\lambda^*||Y_{D_s}||} = e^{-\lambda^* Z_s},$$

holds, then we can deduce in the same way that the process $(e^{-\lambda^*||Y_{D_s}||}, s \geq r)$ is a martingale with respect to the filtration $(\sigma(||Y_{D_v}||, r \leq v \leq s), s \geq r)$. The proof is slightly cumbersome and therefore postponed to the end of this section.

The proof of Lemma 3.7 (ii) is now a simple consequence of Lemma 3.8.

Proof of Lemma 3.7 (ii). By Lemma 3.8, the process

$$e^{-\lambda^* Z_s} - M_s^{\lambda^*} = \int_r^s \Psi(v, \lambda^*) Z_v e^{-\lambda^* Z_v} \mathbf{1}_{\{Z_v < \infty\}} \, dv, \quad s \geq r, \tag{3.2.9}$$

must be a $\bar{\mathbf{P}}_r$-martingale. However this is only possible if the expectation of the Lebesgue-integral above is constant in s which requires $\Psi(s, \lambda^*) = 0$ on $\{0 < Z_s < \infty\}$ for all $s \geq r$. Since the event $\{0 < Z_s < \infty\}$ has positive probability under $\bar{\mathbf{P}}_r$, we reason that $\Psi(s, \lambda^*) = 0$ for all $s \geq r$. Choosing $r > 0$ arbitrarily small yields $\Psi(s, \lambda^*) = 0$ for all $s > 0$. Convexity of $\Psi(s, \theta)$ immediately implies that $\Psi(s, \theta) \geq 0$ for $\theta \geq \lambda^*$ and, further noting that $\Psi(s, 0) \leq 0$, that $\Psi(s, \theta) \leq 0$ for $\theta \leq \lambda^*$. □

Proof of Theorem 3.2 (iii). Combine Lemma 3.6 and 3.7. □

Let us now come to the proof of Lemma 3.8 (ii). For $r > 0$, $t \geq 0$, define the space-time domain D_r^t as

$$D_r^t = \{(x, u) : ||x|| < r, u < t\} \subset \mathbb{R}^d \times [0, \infty).$$

Let $(Y_{D_r^t}, t \geq 0, r > 0)$ be the system of Markov branching exit measures describing the mass of Y as it first exits the space-time domains D_r^t, see again Dynkin [20] and the introductory Section 1.3.

For the proof of Lemma 3.8 (ii), we will need the following result which seems rather obvious but nevertheless needs a careful proof.

Lemma 3.9. *Let $r > 0$. For any $\nu \in \mathcal{M}_F(D_r)$, we have $\mathbf{P}_\nu$-a.s.,*

$$\lim_{t \to \infty} ||Y_{D_r^t}|| = ||Y_{D_r}|| = Z_r.$$

Proof. For $r > 0$, $t \geq 0$, denote by ∂D_r^t the boundary of the set D_r^t, i.e.

$$\begin{aligned} \partial D_r^t &= (\ \{x : ||x|| = r\} \times [0, t)\ ) \cup (\ \{x : ||x|| < r\} \times \{t\}\ ) \\ &=: \partial D_r^{t-} \cup \partial D_{r-}^t. \end{aligned}$$

By monotonicity, we have $\lim_{t \to \infty} \big|\big| Y_{D_r^t}\big|_{\partial D_r^{t-}} \big|\big| = \big|\big| Y_{D_r} \big|\big| = Z_r$, $\mathbf{P}_\nu$-a.s. Next, define the event that Y becomes extinguished within D_r, i.e.

$$\mathcal{E}(Y, D_r) := \{\limsup_{t \to \infty} \big|\big| Y_{D_r^t}\big|_{\partial D_{r-}^t} \big|\big| = 0\}.$$

On the complement of $\mathcal{E}(Y, D_r)$, we have

$$\limsup_{t\to\infty} \big|\big|Y_{D_r^t}\big|_{\partial D_{r-}^t}\big|\big| = \infty, \quad \mathbf{P}_\nu - a.s.$$

This is to say that, on $\mathcal{E}(Y, D_r)^c$, the total mass within the open ball D_r at time t tends to infinity as t tends to infinity. This follows from Proposition 7 in [25] which says that $\limsup_{t\to\infty} ||Y_{D_r^t}|_{B\times\{t\}}|| \in \{0, \infty\}$, $\mathbf{P}_\nu$-a.s. for any non-empty open set $B \subset D_r$ (noting that Proposition 7 in [25] indeed holds for the general branching mechanism we are considering here). Hence, we have shown so far that

$$\limsup_{t\to\infty} ||Y_{D_r^t}|| \;\; = \;\; Z_r + \infty \mathbf{1}_{\mathcal{E}(Y,D_r)^c}.$$

Thus it remains to prove that, on $\mathcal{E}(Y, D_r)^c$, Z_r is also infinite. Fix a $K > 0$. Thanks to Proposition 7 of [25], on $\mathcal{E}(Y, D_r)^c$, we can define an infinite sequence of stopping times

$$\begin{aligned} T_0 &= \inf\{t > 0 : \big|\big|Y_{D_r^t}\big|_{\partial D_{r-}^t}\big|\big| \geq K\} \\ T_{i+1} &= \inf\{t > T_i + 1 : \big|\big|Y_{D_r^t}\big|_{\partial D_{r-}^t}\big|\big| \geq K\}, \;\; i = 0, 1, 2, ... \end{aligned}$$

At times T_i, $i \geq 0$, the total mass within the open ball D_r is greater than or equal to K. Fix an $M > 0$ and define the event

$$A_i = \{\big|\big|Y_{D_r^{T_i}}\big|_{[T_{i-1},T_i)\times\partial D_r}\big|\big| > M\}, \quad i = 1, 2, ...$$

which is the event that the mass that exits D_r during the time interval $[T_{i-1}, T_i)$ exceeds M. Note that there exists a strictly positive constant $\epsilon(M, K)$, such that

$$\begin{aligned} \mathbf{P}_{Y_{D_r^{T_i}}}(A_{i+1}) &\geq \mathbf{P}_{K\delta_0}(A_1) \\ &\geq \mathbf{P}_{K\delta_0}(\big|\big|Y_{D_r^1}\big|_{[0,1)\times\partial D_r}\big|\big| > M) > \epsilon(M, K). \end{aligned} \qquad (3.2.10)$$

Thus, we can partition time into infinitely many intervals $[T_i, T_{i+1})$, $i \geq 0$, of length at least 1. During each time interval the mass that exits D_r, and thus contributes to Z_r, exceeds M with positive probability. These probabilities are uniformly bounded from below by $\epsilon(M, K) > 0$ in (3.2.10). Therefore $\limsup_{t\to\infty} ||Y_{D_r^t}|| = Z_r = \infty$, $\mathbf{P}_\nu$-a.s on the event $\mathcal{E}(Y, D_r)^c$.
In conclusion we have $\limsup_{t\to\infty} ||Y_{D_r^t}|| = Z_r$, $\mathbf{P}_\nu$-a.s. Further, again using monotonicity for $\lim_{t\to\infty} \big|\big|Y_{D_r^t}\big|_{\partial D_r^{t-}}\big|\big| = Z_r$, we get

$$\liminf_{t\to\infty} \big|\big|Y_{D_r^t}\big|\big| = Z_r + \liminf_{t\to\infty} \big|\big|Y_{D_r^t}\big|_{\partial D_{r-}^t}\big|\big|.$$

Noting that

$$Z_r \leq \liminf_{t\to\infty} \left|\left|Y_{D_r^t}\right|\right| \leq \limsup_{t\to\infty} \left|\left|Y_{D_r^t}\right|\right| = Z_r, \quad \mathbf{P}_\nu - \text{a.s.}$$

completes the proof. □

Proof of Lemma 3.8 (ii). For $s > 0$, $t \geq 0$, define $\mathcal{H}_{D_s^t} = \sigma(Y_{D_{s'}^{t'}}, s' \leq s, t' \leq t)$. Fix $r > 0$. The characterising branching Markov property for exit measures, see for instance Section 1.1 in [23], yields that, for $\nu \in \mathcal{M}_F(D_r)$, $s \geq r$ and $u \geq t \geq 0$, we have

$$\mathbf{E}_\nu[e^{-\theta||Y_u||}|\mathcal{H}_{D_s^t}] = \exp\{-\langle w_\theta(u-\cdot), Y_{D_s^t}\rangle\}. \tag{3.2.11}$$

where w_θ is the Laplace functional of the standard CSBP $(||Y_u||, u \geq 0)$ with branching mechanism ψ. Taking $\theta = \lambda^*$, it is well known that $w_{\lambda^*}(t) = \lambda^*$ for all $t \geq 0$. Therefore (3.2.11), with θ replaced by λ^*, turns into

$$\mathbf{E}_\nu[e^{-\lambda^*||Y_u||}|\mathcal{H}_{D_s^t}] \;=\; \exp\{-\int w_{\lambda^*}(u-t')\, \mathrm{d}Y_{D_s^t}(x,t')\} = e^{-\lambda^*||Y_{D_s^t}||}.$$

Taking $u \to \infty$, recalling that $\mathcal{E}(Y) = \{||Y_u|| \to 0 \text{ as } u \to \infty\}$ denoted the event of extinguishing of Y, we conclude

$$\mathbf{E}_\nu[\mathbf{1}_{\mathcal{E}(Y)}|\mathcal{H}_{D_s^t}] = \lim_{u\to\infty} \mathbf{E}_\nu[e^{-\lambda^*||Y_u||}|\mathcal{H}_{D_s^t}] = e^{-\lambda^*||Y_{D_s^t}||}. \tag{3.2.12}$$

Now, we want to take the limit in t. By Lemma 3.9, we have $||Y_{D_s^t}|| \to Z_s$ as $t \to \infty$ and thus the right-hand side of (3.2.12) tends to $\exp\{-\lambda^* Z_s\}$, $\mathbf{P}_\nu$-a.s. For the left-hand side, by the strong Markov property, we can replace $\mathcal{H}_{D_s^t}$ by $\sigma(Y_{D_s^t})$. Further, note that

$$\mathbf{P}_\nu(\mathcal{E}(Y)) = e^{-\lambda^*||\nu||} \quad \text{for any } \nu \in \mathcal{M}_F(D_s),$$

with $\mathbf{P}_\nu(\mathcal{E}(Y)) = 0$ if ν has infinite mass. Thus, the event $\mathcal{E}(Y)$ only depends on the total mass of ν. Therefore we can replace $\sigma(Y_{D_s^t})$ by $\sigma(||Y_{D_s^t}||)$ on the left-hand side in (3.2.12). To sum up, we get

$$\mathbf{E}_\nu[\mathbf{1}_{\mathcal{E}(Y)}|\mathcal{H}_{D_s^t}] = \mathbf{E}_\nu[\mathbf{1}_{\mathcal{E}(Y)}|\sigma(Y_{D_s^t})] = \mathbf{E}_\nu[\mathbf{1}_{\mathcal{E}(Y)}|\sigma(||Y_{D_s^t}||)].$$

By Lemma 3.9, we have $\lim_{t\to\infty} ||Y_{D_s^t}|| = Z_s$, with the possibility of the limit being infinite. Hence,

$$\lim_{t\to\infty} \mathbf{E}_\nu[\mathbf{1}_{\mathcal{E}(Y)}|\sigma(||Y_{D_s^t}||)] = \mathbf{E}_\nu[\mathbf{1}_{\mathcal{E}(Y)}|\sigma(Z_s)].$$

Putting the pieces together, taking $t \to \infty$ in (3.2.12) gives

$$\mathbf{E}_\nu[\mathbf{1}_{\mathcal{E}(Y)}|\sigma(Z_s)] = \lim_{t\to\infty} \mathbf{E}_\nu[\mathbf{1}_{\mathcal{E}(Y)}|\mathcal{H}_{D_s^t}] = \lim_{t\to\infty} e^{-\lambda^*||Y_{D_s^t}||} = e^{-\lambda^* Z_s}.$$

Finally take $\nu \in \mathcal{M}_F(\partial D_r)$ and let $r \le s' \le s$. Then conditioning on $\sigma(Z_s)$ and using the tower property, gives

$$\begin{aligned} e^{-\lambda^* Z_{s'}} &= \mathbf{E}_\nu[\mathbf{1}_{\mathcal{E}(Y)}|\sigma(Z_{s'})] \\ &= \mathbf{E}_\nu[\mathbf{E}[\mathbf{1}_{\mathcal{E}(Y)}|\sigma(Z_s)]|\sigma(Z_{s'})] = \bar{\mathbf{E}}_r[e^{-\lambda^* Z_s}|\sigma(Z_{s'})], \end{aligned}$$

from which we conclude that $(e^{-\lambda^* Z_s}, s \ge r)$ is a $\bar{\mathbf{P}}_r$-martingale. □

3.3 The limiting branching mechanism - Proof of Proposition 3.3 and Theorem 3.4

In this section we prove Proposition 3.3 and Theorem 3.4 which describe the change in the branching mechanism and the limiting branching mechanism as $r \to \infty$.

3.3.1 Changing shape - Proof of Proposition 3.3

Proof of Proposition 3.3. (i) Fix $0 < r \le r'$, $h > 0$ and $\theta \ge 0$. The first step is to show that $u(r, r+h, \theta) \ge u(r', r'+h, \theta)$. Said another way, we want to show that

$$\bar{\mathbf{E}}_{r'}[e^{-\theta Z_{r'+h}}] \ge \bar{\mathbf{E}}_r[e^{-\theta Z_{r+h}}]. \tag{3.3.1}$$

Under $\bar{\mathbf{P}}_r$, Z_{r+h} is the total mass of the exit measure of Y from D_{r+h}, when Y is initiated from one unit of mass distributed on ∂D_r. By radial symmetry of Y, we may assume that the initial mass of Y is concentrated in a point $x_r \in \partial D_r$, i.e. $\bar{\mathbf{E}}_r[e^{-\theta Z_{r+h}}] = \mathbf{E}_{\delta_{x_r}}[e^{-\theta||Y_{D_{r+h}}||}]$.
Now we shift the point x_r to the point $x_{r'} \in \partial D_{r'}$ where $||x_{r'} - x_r|| = r' - r$. We also shift the ball D_{r+h} in the same direction and by the same distance $r' - r$ and denote its new centre by $x_{r'-r}$, see Figure 3-2. By translation invariance of Y we then have

$$\bar{\mathbf{E}}_r[e^{-\theta Z_{r+h}}] = \mathbf{E}_{\delta_{x_r}}\left[e^{-\theta||Y_{D_{r+h}}||}\right] = \mathbf{E}_{\delta_{x_{r'}}}\left[e^{-\theta||Y_{D(x_{r'-r},r+h)}||}\right],$$

where $D(x_{r'-r}, r+h)$ is the open ball centred at $x_{r'-r}$ with radius $r+h$. We can then write (3.3.1) as

$$\mathbf{E}_{\delta_{x_{r'}}}\left[e^{-\theta||Y_{D_{r'+h}}||}\right] \ \ge \ \mathbf{E}_{\delta_{x_{r'}}}\left[e^{-\theta||Y_{D(x_{r'-r},r+h)}||}\right]. \tag{3.3.2}$$

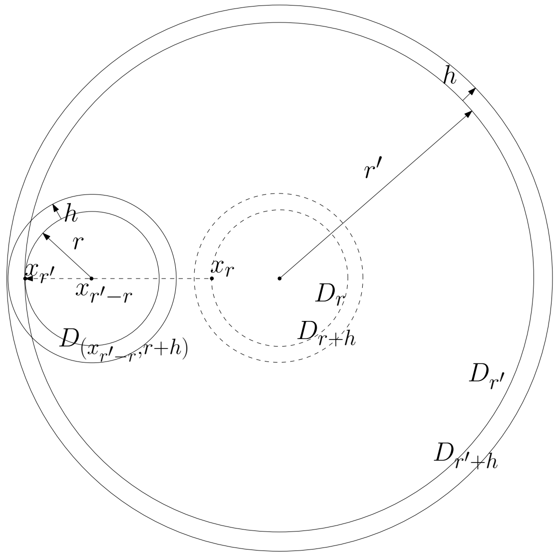


Figure 3-2: Shifting the balls D_r and D_{r+h} by a distance $r' - r$

Recall that equation (3.1.2) shows that the process of Markov branching exit measure Y_{D_s} indexed by the increasing sequence of balls $(D_s, s \geq r)$ has the strong Markov property. By Dynkin [20], the strong Markov property holds more generally for any increasing sequence of open Borel subsets of $\mathbb{R}^d$. In particular,

$$\mathbf{E}_{\delta_{x_{r'}}}\left[e^{-\theta||Y_{D_{r'+h}}||}\Big|\mathcal{H}_{D(x_{r'-r},r+h)}\right] = \mathbf{E}_{Y_{D(x_{r'-r},r+h)}}\left[e^{-\theta||Y_{D_{r'+h}}||}\right], \qquad (3.3.3)$$

where $\mathcal{H}_{D(x_{r'-r},r+h)} = \sigma(Y_{D(x_{r'-r},s)}, s \leq r+h)$. Hence, assuming that

$$\mathbf{E}_{Y_{D(x_{r'-r},r+h)}}\left[e^{-\theta||Y_{D_{r'+h}}||}\right] \geq e^{-\theta||Y_{D(x_{r'-r},r+h)}||} \qquad (3.3.4)$$

holds true, we get, together with (3.3.3), that

$$\begin{aligned}
\mathbf{E}_{\delta_{x_{r'}}}\left[e^{-\theta||Y_{D_{r'+h}}||}\right] &= \mathbf{E}_{\delta_{x_{r'}}}\left[\mathbf{E}_{\delta_{x_{r'}}}\left[e^{-\theta||Y_{D_{r'+h}}||}\big|\sigma(Y_{D(x_{r'-r},r+h)})\right]\right] \\
&= \mathbf{E}_{\delta_{x_{r'}}}\left[\mathbf{E}_{Y_{D(x_{r'-r},r+h)}}\left[e^{-\theta||Y_{D_{r'+h}}||}\right]\right] \\
&\geq \mathbf{E}_{\delta_{x_{r'}}}\left[e^{-\theta||Y_{D(x_{r'-r},r+h)}||}\right],
\end{aligned}$$

which is the desired inequality (3.3.2). Thanks to the branching Markov property for exit measures, for (3.3.4) to hold, it suffices to show that

$$\mathbf{E}_{\delta_x}\left[e^{-\theta||Y_{D_{r'+h}}||}\right] \geq e^{-\theta}, \quad \text{for any } x \in \partial D(x_{r'-r}, r+h). \qquad (3.3.5)$$

For fixed $x \in \partial D(x_{r'-r}, r+h)$, set $s = ||x||$ and note that $s \le r'+h$. By (3.2.7), $u(s, r'+h, \theta)$ is increasing in s and bounded from above by $u(r'+h, r'+h, \theta) = \theta$. Hence we obtain

$$\mathbf{E}_{\delta_x}[e^{-\theta ||Y_{D_{r'+h}}||}] = \bar{\mathbf{E}}_s[e^{-\theta Z_{r'+h}}] = e^{-u(s,r'+h,\theta)} \ge e^{-\theta},$$

which is (3.3.5). This means we have proved (3.3.1) and thus $u(r, r+h, \theta) \ge u(r', r'+h, \theta)$. The latter yields that, for all $\theta \ge 0$,

$$\begin{aligned} \frac{\partial}{\partial s} u(r,s,\theta)|_{s=r} &= \lim_{h \downarrow 0} \frac{u(r, r+h, \theta) - u(r,r,\theta)}{h} \\ &\ge \lim_{h \downarrow 0} \frac{u(r', r'+h, \theta) - u(r', r', \theta)}{h} = \frac{\partial}{\partial s} u(r', s, \theta)|_{s=r'}. \end{aligned} \tag{3.3.6}$$

Now we apply (3.2.2) to get

$$\frac{\partial}{\partial s} u(r,s,\theta)|_{s=r} = \Big(- \Psi(s,\theta) \frac{\partial}{\partial \theta} u(r,s,\theta) \Big)|_{s=r} = -\Psi(r,\theta) \cdot 1, \tag{3.3.7}$$

where we used that $\lim_{s \downarrow r} \frac{\partial}{\partial \theta} u(r,s,\theta) = 1$ which can be seen as follows. By dominated convergence, we have

$$\lim_{s \downarrow r} \frac{\partial}{\partial \theta} e^{-u(r,s,\theta)} = \lim_{s \downarrow r} \frac{\partial}{\partial \theta} \bar{\mathbf{E}}_r[e^{-\theta Z_s} \mathbf{1}_{\{Z_s < \infty\}}] = \lim_{s \downarrow r} \bar{\mathbf{E}}_r[-Z_s e^{-\theta Z_s} \mathbf{1}_{\{Z_s < \infty\}}] = -e^{-\theta}.$$

On the other hand,

$$\lim_{s \downarrow r} \frac{\partial}{\partial \theta} e^{-u(r,s,\theta)} = -\lim_{s \downarrow r} \frac{\partial}{\partial \theta} u(r,s,\theta)\ e^{-u(r,s,\theta)} = -\lim_{s \downarrow r} \frac{\partial}{\partial \theta} u(r,s,\theta)\ e^{-\theta}$$

and we may conclude that $\lim_{s \downarrow r} \frac{\partial}{\partial \theta} u(r,s,\theta) = 1$ as claimed.
Combining (3.3.6) with (3.3.7) gives $\Psi(r,\theta) \le \Psi(r',\theta)$ for $\theta \ge 0$ and $r \le r'$, which completes the proof.

(ii) Define $\Psi^*(r,\theta) := \Psi(r, \lambda^* + \theta)$ for $\theta \ge 0$. The family of branching mechanisms $(\Psi^*(r,\cdot), r > 0)$ is obtained when we run the super-Brownian motion Y with branching mechanism $\psi^*(\theta) := \psi(\theta + \lambda^*)$, $\theta \ge 0$, (instead of ψ)[1]. Clearly, the branching mechanism ψ^* is subcritical. It thus follows from part (i) that the family of branching mechanisms $(\Psi^*(r,\cdot), r > 0)$ has the property that $\Psi^*(r,\theta) \le \Psi^*(r',\theta)$, for $r \le r'$ and all $\theta \ge 0$. Clearly this gives $\Psi(r,\theta) \le \Psi(r',\theta)$ for $r \le r'$ and $\theta \ge \lambda^*$.
Let $\theta \le \lambda^*$. First, note that $u(r,s,\lambda^*) = -\log \bar{\mathbf{E}}_r[e^{-\lambda^* Z_s}] = \lambda^*$, which is a con-

[1] ψ^* is the branching mechanism of the super-Brownian motion Y with branching mechanism ψ conditioned on becoming extinguished.

sequence of Lemma 3.8 (ii). Thus, $u(r,s,\theta) \le u(r,s,\lambda^*) = \lambda^*$ for all $\theta \le \lambda^*$, $0 < r \le s$, and in particular $\psi(u(r,s,\theta)) \le 0$. We therefore get

$$\begin{aligned} u(r,s,\theta) &= \theta - \mathbb{E}_r^{\mathrm{R}} \int_0^{\tau_z} \psi(u(R_v,s,\theta))\,\mathrm{d}v - \mathbb{E}_z^{\mathrm{R}} \int_0^{\tau_s} \psi(u(R_v,s,\theta))\,\mathrm{d}v \\ &\ge \theta - \mathbb{E}_z^{\mathrm{R}} \int_0^{\tau_s} \psi(u(R_v,s,\theta))\,\mathrm{d}v \\ &= u(z,s,\theta) \end{aligned}$$

for any $0 < r \le z \le s$, $\theta \le \lambda^*$. We can then use $\frac{\partial}{\partial r} u(r,s,\theta) \le 0$ in place of the inequality (3.2.7) in the proof of part (i). Thus, following the same arguments as in the proof of part (i) with all inequalities reversed, we see that $\Psi(r,\theta) \ge \Psi(r',\theta)$ for $r \le r'$ and all $\theta \le \lambda^*$. □

3.3.2 Limiting branching mechanism - Proof of Theorem 3.4

To begin with, we show the existence and finiteness of the limiting branching mechanism Ψ_∞ and derive a PDE characterisation.

Proposition 3.10. *For each $\theta \ge 0$, the limit $\lim_{r\uparrow\infty} \Psi(r,\theta) = \Psi_\infty(\theta)$ is finite and the convergence holds uniformly in θ on any bounded, closed subset of $\mathbb{R}_+$.*
(i) In the (sub)critical case, Ψ_∞ solves the equation

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial\theta}\Psi_\infty^2(\theta) &= 2\psi(\theta), \\ \Psi_\infty(0) &= 0. \end{aligned} \tag{3.3.8}$$

(ii) In the supercritical case, Ψ_∞ solves (3.3.8) with the initial value at 0 replaced by

$$\Psi_\infty(0) = -2\sqrt{\int_0^{\lambda^*} |\psi(\theta)|\,\mathrm{d}\theta}$$

and with $\Psi_\infty(\lambda^*) = 0$.

Proof. From the monotonicity in Proposition 3.3, we conclude that the pointwise-limit $\Psi_\infty(\theta) := \lim_{r\uparrow\infty} \Psi(r,\theta)$ exists. We will have to show that $|\Psi_\infty(\theta)|$ is finite for each $\theta \ge 0$. Uniform convergence on any bounded, closed subset of $\mathbb{R}$ will then follow by convexity, see for example Theorem 10.8 in [60]. We consider the (sub)critical case and the supercritical case separately.

(i) Suppose we are in the (sub)critical case. We have $\Psi(r,0) = 0$ for all $r > 0$ and hence $\Psi_\infty(0) = 0$. For $\theta > 0$, recall the PDE (3.1.8), which can be written slightly differently as

$$\frac{\partial}{\partial r}\Psi(r,\theta)+\Psi(r,\theta)\frac{\partial}{\partial\theta}\Psi(r,\theta)+\frac{d-1}{r}\Psi(r,\theta) \;=\; 2\psi(\theta), \quad r>0,\theta>0. \tag{3.3.9}$$

By Proposition 3.3 (i), $\frac{\partial}{\partial r}\Psi(r,\theta)\geq 0$ and, by Lemma 3.7(i), $\Psi(r,\theta)\geq 0$. Thus,

$$\Psi(r,\theta)\frac{\partial}{\partial\theta}\Psi(r,\theta) \;\leq\; 2\psi(\theta), \qquad \text{for all } r>0 \text{ and } \theta\geq 0. \tag{3.3.10}$$

Fix a $\theta_0>0$. Suppose for contradiction that $\Psi(r,\theta_0)\uparrow\infty$ as $r\to\infty$. For any $K>0$, we can find an r_0 large enough such that

$$\Psi(r_0,\theta_0) \;>\; 2K\psi(\theta_0). \tag{3.3.11}$$

By (3.3.10), this implies that $\frac{\partial}{\partial\theta}\Psi(r_0,\theta_0)<\frac{1}{K}$. As Ψ is convex in θ with $\Psi(r_0,0)=0$, we get that

$$\Psi(r_0,\theta_0) \;\leq\; \frac{\theta_0}{K}.$$

Now we can choose K large enough such that $\theta_0/K<2K\psi(\theta_0)$, which then contradicts (3.3.11). Hence, $\lim_{r\to\infty}\Psi(r,\theta)=\Psi_\infty(\theta)<\infty$ for all $\theta\geq 0$.
Note that $\limsup_{r\to\infty}\frac{\partial}{\partial\theta}\Psi(r,\theta)$ is also finite for each $\theta\geq 0$. Indeed, if we supposed the contrary for some $\theta>0$, that is, $\limsup_{r\to\infty}\frac{\partial}{\partial\theta}\Psi(r,\theta)=\infty$, then (3.3.10) would imply that $\liminf_{r\to\infty}\Psi(r,\theta)=0$, which contradicts Proposition 3.3 (i). By convexity, we can pick any $\theta>0$ to get $\limsup_{r\to\infty}\frac{\partial}{\partial\theta}\Psi(r,0+)\leq\limsup_{r\to\infty}\frac{\partial}{\partial\theta}\Psi(r,\theta)<\infty$.
Next, we want to take $r\to\infty$ in (3.3.9) and we know that the limit of the left-hand side exists since the right-hand side does not depend on r. We keep $\theta_0>0$ fixed and consider each term on the left-hand side of (3.3.9) separately.
We have just seen that $\lim_{r\to\infty}\Psi(r,\theta_0)<\infty$ which implies that the third term on the left-hand side of (3.3.9), namely $\frac{d-1}{r}\Psi(r,\theta_0)$, vanishes as $r\to\infty$.
Consider the term $\Psi(r,\theta_0)\frac{\partial}{\partial\theta}\Psi(r,\theta_0)$ next. Since $\Psi(r,\cdot)$ is a sequence of continuous, convex functions, the pointwise limit Ψ_∞ is also continuous and convex in θ, cf. Theorem 10.8 in Rockafellar [60]. The convexity ensures that the set of points at which Ψ_∞ is not differentiable is at most countable. If Ψ_∞ is differentiable at θ_0, then by Theorem 25.7 in [60], it follows that $\lim_{r\to\infty}\frac{\partial}{\partial\theta}\Psi(r,\theta_0)=\frac{\partial}{\partial\theta}\Psi_\infty(\theta_0)$ and hence

$$\lim_{r\to\infty}\Psi(r,\theta_0)\frac{\partial}{\partial\theta}\Psi(r,\theta_0)=\Psi_\infty(\theta_0)\frac{\partial}{\partial\theta}\Psi_\infty(\theta_0). \tag{3.3.12}$$

So far we have seen that, for all $\theta\geq 0$ at which Ψ_∞ is differentiable, the second

and third term on the left-hand side of (3.3.9) converge to a finite limit as $r \to \infty$ which implies that the limit of the first term, that is $\lim_{r\to\infty} \frac{\partial}{\partial r}\Psi(r,\theta)$, also exists and is finite. With $\lim_{r\to\infty} \Psi(r,\theta) < \infty$ it thus follows that $\frac{\partial}{\partial r}\Psi(r,\theta)$ tends to 0 as $r \to \infty$, for all $\theta \geq 0$ at which Ψ_∞ is differentiable.
In conclusion, for any θ at which Ψ_∞ is differentiable, the first and third term on the left-hand side of (3.3.9) vanish as $r \to \infty$ and with (3.3.12) we get

$$\Psi_\infty(\theta)\frac{\partial}{\partial\theta}\Psi_\infty(\theta) = 2\psi(\theta). \tag{3.3.13}$$

For $\theta > 0$, we have $\Psi_\infty(\theta) > 0$ and we can write (3.3.13) as

$$\frac{\partial}{\partial\theta}\Psi_\infty(\theta) = 2\frac{\psi(\theta)}{\Psi_\infty(\theta)}, \tag{3.3.14}$$

which again holds for all $\theta > 0$ at which Ψ_∞ is differentiable. By convexity, Ψ_∞ admits left and right derivatives for every $\theta > 0$. Since the right-hand side of (3.3.14) is continuous and (3.3.14) holds true for all but countably many $\theta > 0$, we conclude that the left and the right derivative of $\Psi_\infty(\theta)$ agree for every $\theta > 0$. Thus (3.3.14), and equivalently (3.3.8), holds in fact for every $\theta > 0$. By convexity, for any $\theta > 0$, we get

$$\frac{\partial}{\partial\theta}\Psi_\infty(0+) \leq \frac{\partial}{\partial\theta}\Psi_\infty(\theta) = 2\frac{\psi(\theta)}{\Psi_\infty(\theta)} < \infty,$$

which shows that (3.3.8) holds true for $\theta = 0$ with both sides being equal to 0.

(ii) We consider the supercritical case now. Again we first have to show that $\Psi_\infty(\theta)$ is finite for each $\theta \geq 0$.
Let us begin with the case $\theta \in [\lambda^*, \infty)$. We can consider the (sub)critical branching mechanisms $\Psi^*(r,\lambda) := \Psi(r, \lambda + \lambda^*)$ for $\lambda \geq 0$. Then part (i) applies to the (sub)critical Ψ^* and we conclude that, for any $\theta \geq \lambda^*$,

$$\Psi_\infty(\theta) = \lim_{r\to\infty} \Psi(r,\theta) = \lim_{r\to\infty} \Psi^*(r, \theta - \lambda^*) = \Psi^*_\infty(\theta - \lambda^*) < \infty.$$

In particular, the equation (3.3.8) holds for all $\theta \geq \lambda^*$ and $\Psi_\infty(\lambda^*) = \Psi^*_\infty(0) = 0$. Further, it follows from the monotonicity in Proposition 3.3 that $\frac{\partial}{\partial\theta}\Psi^*(r, 0+) \leq \frac{\partial}{\partial\theta}\Psi^*_\infty(0+)$. The latter derivative was shown to be finite in the proof of part (i). Thus, for any $r > 0$,

$$\frac{\partial}{\partial\theta}\Psi(r,\theta)|_{\theta=\lambda^*} = \frac{\partial}{\partial\theta}\Psi^*(r, 0+) \leq \frac{\partial}{\partial\theta}\Psi^*_\infty(0+) < \infty.$$

Hence, we have a uniform upper bound for the θ-derivative of $\Psi(r,\cdot)$ at λ^*. Recalling that $\Psi(r,\lambda^*) = 0$, convexity ensures that $\Psi(r,\cdot)$ is uniformly bounded

from below by the function $\frac{\partial}{\partial\theta}\Psi^*_\infty(0+)(\cdot-\lambda^*)$ on the interval $[0,\lambda^*]$. This implies already that $\lim_{r\to\infty}|\Psi(r,\theta)|<\infty$ for all $\theta\in[0,\lambda^*]$.
To show that the equation (3.3.8) holds for all $\theta\le\lambda^*$ we can now simply repeat the argument given in the proof of part (i). Finally, with $\Psi_\infty(\lambda^*)=0$, we can derive the initial condition for $\Psi_\infty(0)$ by integrating (3.3.8) from 0 to λ^*. □

Proof of Theorem 3.4. Proposition 3.10 guarantees the existence and finiteness of Ψ_∞. If we integrate (3.3.8) from λ^* to θ, and note that $\Psi_\infty(\theta)$ and $\psi(\theta)$ are negative if and only if $\theta\le\lambda^*$, we obtain the expression in (3.1.9). It thus remains to show (ii).

It follows from an obvious adaptation of the proof of Theorem 3.1 in Kyprianou et al. [51] that Z^∞ is indeed the process of the total mass of the Markov branching exit measures of a one-dimensional super-Brownian as it first exits the family of intervals $((-\infty,s),s\ge0)$ as claimed.
Concerning the convergence in (3.1.10), we have to show that, for $s\ge0$ and $\theta\ge0$, $u^\infty(s,\theta):=\lim_{r\to\infty}u(r,s+r,\theta)$ exists and solves

$$u^\infty(s,\theta)=\theta-\int_0^s\Psi_\infty(u^\infty(s-v,\theta))\,\mathrm{d}v, \tag{3.3.15}$$

which is the characterising equation for the Laplace functional of Z^∞.
This is trivially satisfied for $s=0$. Henceforth, let $s>0$ and $\theta\ge0$ be fixed. Recall that $u(r,s+r,\theta)$ solves equation (3.1.6), which can be written as

$$u(r,s+r,\theta)=\theta-\int_0^s\Psi(v+r,u(v+r,s+r,\theta))\,\mathrm{d}v,\quad r>0. \tag{3.3.16}$$

Note that the convergence of the convex functions $\Psi(r,\cdot)$ to $\Psi_\infty(\cdot)$ in Theorem 3.4 holds uniformly in θ on each bounded closed subset of $\mathbb{R}_+$. Therefore, for fixed $\epsilon>0$, we can choose r large enough such that $|\Psi(s+r,\lambda)-\Psi_\infty(\lambda)|<\epsilon$ for all $\lambda\in\{u(v+r,s+r,\theta),0\le v\le s\}$. Thus, for large r,

$$\begin{aligned}&\Big|u(r,s+r,\theta)-\big(\theta-\int_0^s\Psi_\infty(u(v+r,s+r,\theta))\,\mathrm{d}v\big)\Big|\\ =&\ \Big|\int_0^s\Psi(v+r,u(v+r,s+r,\theta))\,\mathrm{d}v-\int_0^s\Psi_\infty(u(v+r,s+r,\theta))\,\mathrm{d}v\Big|\\ \le&\ \epsilon\, s.\end{aligned} \tag{3.3.17}$$

Now assume for contradiction that $\limsup_{r\to\infty}u(r,s+r,\theta)=+\infty$. Since Ψ_∞ is convex and $\Psi'_\infty(0+)\ge0$ (with $\Psi'_\infty(0+)=0$ in the supercritical case), the integrand in the first line of (3.3.17) is bounded from below by $\Psi_\infty(0)$. Therefore, the expression in the first line of (3.3.17) tends to ∞ along a subsequence of r which is an obvious contradiction.
Hence, $u(r,s+r,\theta)$ is bounded as a sequence in r. It therefore contains a con-

vergent subsequence, say $u(r_n, s + r_n, \theta)$ where $(r_n, n \geq 1)$ is a strictly monotone sequence which tends to ∞.
Let us show that every subsequence converges to the same limit. Let $(r'_n, n \geq 1)$ be another strictly monotone sequence which tends to ∞. To begin with we note that $\sup_{n\in\mathbb{N}}\{u(v + r_n, v + r_n, \theta)\} < \infty$ by boundedness. Then we set

$$\bar{u} = \sup_{v\in(0,s)} \sup_{n\in\mathbb{N}}\{u(v + r_n, v + r_n, \theta)\} < \infty$$

and define $\bar{u}'$ accordingly using the sequence $(r'_n, n \geq 1)$ in place of $(r_n, n \geq 1)$. By (3.3.17), for any $\epsilon > 0$, we can find an $N \in \mathbb{N}$ large enough such that for all $n \geq N$

$$\begin{aligned} &|u(r_n, s + r_n, \theta) - u(r'_n, s + r'_n, \theta)| && (3.3.18)\\ &\quad\leq 2\epsilon s + \int_0^s \Big|\Psi_\infty(u(v + r_n, s + r_n, \theta)) - \Psi_\infty(u(v + r'_n, s + r'_n, \theta))\Big|\, \mathrm{d}v \\ &\quad\leq 2\epsilon s + \int_0^s M|u(v + r_n, s + r_n, \theta) - u(v + r'_n, s + r'_n, \theta)|\, \mathrm{d}v. && (3.3.19)\end{aligned}$$

where $M := \sup\{\Psi'_\infty(w) : w \in (0, \max\{\bar{u}, \bar{u}'\}\} < \infty$. Set

$$F_n(s') = M \int_0^{s'} |u(v + r_n, s + r_n, \theta) - u(v + r'_n, s + r'_n, \theta)|\, \mathrm{d}v, \quad \text{for } 0 \leq s' \leq s,$$

and note that $\partial F_n(s')/\partial s' = M|u(s' + r_n, s + r_n, \theta) - u(s' + r'_n, s + r'_n, \theta)|$. By (3.3.18),

$$\frac{\partial}{\partial s'}F_n(s') - 2\epsilon M(s - s') - M(F_n(s) - F_n(s')) \leq 0.$$

Multiplying by $e^{Ms'}$, we derive $\partial[\big(F_n(s) - F_n(s') + 2\epsilon(s - s') + \frac{2\epsilon}{M}\big)e^{Ms'}]/\partial s' \geq 0$. Therefore,

$$\big(F_n(s) - F_n(s') + 2\epsilon(s - s') + \frac{2\epsilon}{M}\big)e^{Ms'} \leq \frac{2\epsilon}{M}e^{Ms}, \quad \text{for any } 0 \leq s' \leq s.$$

Hence, $F_n(s) - F_n(s') \leq 2\epsilon\big(\frac{1}{M}(e^{M(s-s')} - 1) - (s - s')\big)$, for $0 \leq s' \leq s$. Since $\epsilon > 0$ can be chosen arbitrarily small, we conclude from the definition of $F_n(s')$ that $u(r'_n, s' + r'_n, \theta)$ converges to the same limit as $u(r_n, s' + r_n, \theta)$ as $n \to \infty$. We have thus shown that, considered as a sequence in r, all subsequences of $u(r, s + r, \theta)$ converge to the same limit. Therefore $u^\infty(s, \theta) = \lim_{r\to\infty} u(r, s + r, \theta)$ exists and, with (3.3.17), it satisfies (3.3.15). By uniqueness of solutions to (3.3.15), $u^\infty(s, \theta)$ agrees with the Laplace functional associated with Z^∞ which in turn implies the desired convergence. □

3.4 Derivation of the differential equation (3.2.5) corresponding to the integral equation (3.2.4)

The reader familiar with the superprocess literature will readily believe that any solution to the differential equation (3.2.5) also solves the integral equation (3.2.4) and conversely that solutions to (3.2.4) also solve (3.2.5). Results of this fashion can be found for instance in the work of Dynkin, see [17], Section 3 in [18] or Section 5.2 in [21]. However, in these references only (sub)critical branching mechanism are allowed and we are unaware of a rigorous proof in the literature for the case of a supercritical branching mechanism. Although it seems possible to adapt Dynkin's arguments to the supercritical case, we will offer a self-contained proof here instead.

Recall from (3.2.4) that the Laplace functional u of Z, defined in (3.1.5), is the unique non-negative solution to the equation

$$u(r,s,\theta) = \theta - \mathbb{E}_r^{\mathrm{R}} \int_0^{\tau_s} \psi(u(R_l,s,\theta))\,\mathrm{d}l,\ 0 < r \le s,\ \theta \ge 0, \tag{3.4.1}$$

where $(R, \mathbb{P}^{\mathrm{R}})$ is a d-dimensional Bessel process and $\tau_s := \inf\{l > 0 : R_l > s\}$ its first passage time above level s.
Fix $0 < r \le s$ and $\theta \ge 0$ from now on. Let us apply a Lamperti transform to the d-Bessel process R in the integral on the right-hand side of (3.4.1). Define $\varphi(s) = \int_0^{r^2 s} R_l^{-2}\mathrm{d}l$, $s \ge 0$, then

$$B_s = \log(r^{-1}R_{r^2\varphi^{-1}(s)}),\ s \ge 0,$$

is a one-dimensional Brownian motion with drift $\frac{d}{2} - 1$ starting from 0. Let us denote the law of $B = (B_s, s \ge 0)$ by $\bar{\mathbb{P}}_0^d$. Thus we get

$$\begin{aligned}
\mathbb{E}_r^{\mathrm{R}} \int_0^{\tau_s} \psi(u(R_l,s,\theta))\,\mathrm{d}l &= \mathbb{E}_r^{\mathrm{R}} \int_0^{\varphi(r^{-2}\tau_s)} \psi(u(R_{r^2\varphi^{-1}(l)},s,\theta))R^2_{r^2\varphi^{-1}(l)}\,\mathrm{d}l \\
&= \bar{\mathbb{E}}_0^d \int_0^{T_{\log(s/r)}} \psi(u(e^{B_l+\log r},s,\theta))e^{2(B_l+\log r)}\,\mathrm{d}l \\
&= \bar{\mathbb{E}}_{\log r}^d \int_0^{T_{\log s}} \psi(u(e^{B_l},s,\theta))e^{2B_l}\,\mathrm{d}l,
\end{aligned}$$

where $T_{\log s}$ is the first time B crosses level $\log s$. Equation (3.4.1) becomes

$$u(r,s,\theta) = \theta - \bar{\mathbb{E}}_{\log r}^d \int_0^{T_{\log s}} \psi(u(e^{B_l},s,\theta))e^{2B_l}\,\mathrm{d}l. \tag{3.4.2}$$

We split the integral on the right hand side into its excursions away from the maximum. This gives

$$\bar{\mathbb{E}}^d_{\log r} \int_0^{T_{\log s}} \psi(u(e^{B_l}, s, \theta))e^{2B_l}\ \mathrm{d}l$$
$$= \bar{\mathbb{E}}^d_{\log r} \sum_{\log r \le u \le \log s} \int_0^{\zeta^{(u)}} \psi(u(e^{u-e_u(l)}, s, \theta))e^{2(u-e_u(l))}\ \mathrm{d}l,$$

where e_u is an excursion away from the maximum with lifetime $\zeta^{(u)}$ and the sum is taken over all left end-points u of the excursion intervals in $(T_{\log r}, T_{\log s})$. It follows from the Compensation formula for excursions (Bertoin [6], Cor. 11, p.110) that

$$\bar{\mathbb{E}}^d_{\log r} \sum_{\log r \le u \le \log s} \int_0^{\zeta^{(u)}} \psi(u(e^{u-e_u(l)}, s, \theta))e^{2(u-e_u(l))}\ \mathrm{d}l$$
$$= \int_{\log r}^{\log s} \eta\left(\int_0^{\zeta} \psi(u(e^{u-e(l)}, s, \theta))e^{2(u-e(l))}\ \mathrm{d}l\right)\ \mathrm{d}u,$$

where η denotes the excursion measure and e is a generic excursion with length ζ. Then we apply Exercise 5, chapter VI, [6], to get

$$\int_{\log r}^{\log s} \eta\left(\int_0^{\zeta} \psi(u(e^{u-e(s)}, s, \theta))e^{2(u-e(l))}\ \mathrm{d}l\right)\ \mathrm{d}u$$
$$= \int_{\log r}^{\log s} \int_0^{\infty} \psi(u(e^{u-y}, s, \theta))e^{2(u-y)}\ \hat{V}(\mathrm{d}y)\ \mathrm{d}u,$$

where $\hat{V}$ is the renewal function of the dual ladder height process (the dual process is here simply Brownian motion with drift $-(\frac{d}{2}-1)$). We see from equation (4),

p. 196 in [6] that $\hat{V}(\mathrm{d}y) = 2e^{-2(\frac{d}{2}-1)y}\mathrm{d}y$ and obtain

$$\begin{aligned}
&\int_{\log r}^{\log s} \int_0^\infty \psi(u(e^{u-y}, s, \theta))e^{2(u-y)}\ \hat{V}(\mathrm{d}y)\ \mathrm{d}u \\
&= \quad 2 \int_{\log r}^{\log s} e^{2u} \int_0^\infty \psi(u(e^{u-y}, s, \theta))\ e^{-dy}\ \mathrm{d}y\ \mathrm{d}u \\
&\overset{z=e^{u-y}}{=} \quad -2 \int_{\log r}^{\log s} e^{2u} \int_{e^u}^0 \psi(u(z, s, \theta))z^d e^{-du}\ z^{-1}\ \mathrm{d}z\ \mathrm{d}u \\
&\overset{v=e^u}{=} \quad -2 \int_r^s v^2 \int_v^0 \psi(u(z, s, \theta))\ z^{d-1}v^{-d}\ \mathrm{d}z\ v^{-1}\ \mathrm{d}v \\
&= \quad 2 \int_r^s v^{1-d} \int_0^v \psi(u(z, s, \theta))\ z^{d-1}\ \mathrm{d}z\ \mathrm{d}v.
\end{aligned}$$

Thus the characterising integral equations (3.4.1) and (3.4.2) become

$$u(r, s, \theta) = \theta - 2 \int_r^s v^{1-d} \int_0^v \psi(u(z, s, \theta))\ z^{d-1}\ \mathrm{d}z\ \mathrm{d}v. \tag{3.4.3}$$

Differentiation in r gives

$$\begin{aligned}
\frac{\partial}{\partial r}u(r, s, \theta) &= 2r^{1-d} \int_0^r \psi(u(z, s, \theta))z^{d-1}\ \mathrm{d}z, \\
\frac{\partial^2}{\partial r^2}u(r, s, \theta) &= 2(1-d)r^{-d} \int_0^r \psi(u(z, s, \theta))z^{d-1}\ \mathrm{d}z + 2\psi(u(r, s, \theta)).
\end{aligned}$$

Hence, we obtain the differential equation in (3.2.5), i.e. for $\theta \geq 0$,

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2}{\partial r^2}u(r, s, \theta) + \frac{d-1}{2r}\frac{\partial}{\partial r}u(r, s, \theta) &= \psi(u(r, s, \theta)) \quad 0 < r \leq s, \\
u(r, r, \theta) &= \theta.
\end{aligned}$$

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