Travelling waves for fragmentation processes.

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Motivation
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- **Super-diffusions**: Markov process $X = \{X_t : t \geq 0\}$ such that $X_t$ is a measure on $\mathbb{R}$, its probabilities denoted by $\mathbb{P}_\mu$ for measures $\mu$ on $\mathbb{R}$ where $X_0 = \mu$. 

Branching property: For two initial measures $\mu_1, \mu_2$, $\mathbb{P}_{\mu_1} + \mathbb{P}_{\mu_2} = \mathbb{P}_{\mu_1} \star \mathbb{P}_{\mu_2}$.

Non-linear semi-group: "Infinite divisibility" in the branching property suggests the natural object to describe the semi-group is the Laplace functional $\exp\{-u f(x,t)\} = E_\delta x (\exp\{-\langle f, X_t \rangle\})$ where $f : \mathbb{R} \rightarrow [0, \infty)$, $\langle f, X_t \rangle = \int f(y) X_t(dy)$ and one finds $\partial \partial_t u f(x,t) = Lu f(x,t) - \psi(u f(x,t))$ with $u f(x,0) = f(x)$, where $L$ is the infinitesimal generator of the "underlying motion" and $\psi$ necessarily respects the Lévy-Khintchine formula, $\psi(\lambda) = \alpha \lambda + \beta \lambda^2 + Z(0,\infty)(e^{-\lambda x} - 1 + \lambda x I\{x<1\}) \nu(dx)$ for $\lambda \geq 0$ where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and $\nu$ is a measure concentrated on $(0,\infty)$ which satisfies $\int (0,\infty)(1 \wedge x^2) \nu(dx) < \infty$. 


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\[
\exp\{-u_f(x,t)\} = \mathbb{E}_{\delta_x}(\exp\{-\langle f, X_t \rangle\})
\]

where $f : \mathbb{R} \to [0, \infty)$, $\langle f, X_t \rangle = \int_{\mathbb{R}} f(y)X_t(dy)$ and one finds

\[
\frac{\partial}{\partial t} u_f(x,t) = Lu_f(x,t) - \psi(u_f(x,t)) \quad \text{with} \quad u_f(x,0) = f(x),
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where $L$ is the infinitesimal generator of the “underlying motion" and $\psi$ necessarily respects the Lévy-Khintchine formula,

\[
\psi(\lambda) = \alpha \lambda + \beta \lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x 1_{\{x<1\}}) \nu(dx)
\]

for $\lambda \geq 0$ where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and $\nu$ is a measure concentrated on $(0, \infty)$ which satisfies $\int_{(0,\infty)} (1 \wedge x^2) \nu(dx) < \infty$. 
Linear semi-group:
\[ v_g(x,t) = E_{\delta x}(\langle g, X_t \rangle) \]
and it solves
\[ \frac{\partial}{\partial t} v_g(x,t) = L v_g(x,t) - \psi'(0) v_g(x,t) \]
with \( v_g(x,0) = g(x) \).

Multiplicative martingales:
Look for positive monotone "travelling" solutions with speed \( c \in \mathbb{R} \), i.e.
\[ u_f(x,t) = f(x - ct) \]
and consequently
\[ Lf + cf' - \psi(f) = 0 \].

Let \( X_c \) be the super-diffusion with added linear drift \( c \) to the support, then the associated motion operator is \( L + cd\frac{d}{dx} \) and
\[ e^{-f(x)} = E_{\delta x}(e^{-\langle f, X_c t \rangle}) \Rightarrow e^{-\langle f, X_c t \rangle} \text{is a martingale.} \]

Additive martingales:
Look for "travelling" solutions of the form \( v_g(x,t) = g(x - ct) \), i.e.
\[ Lg + cg' - \psi'(0) g = 0 \].
Then,
\[ g(x) = E_{\delta x}(\langle g, X_c t \rangle) \Rightarrow \langle g, X_c t \rangle \text{is a martingale.} \]

Martingale limits:
Positive martingales have limits so what does the relation between
\[ \lim_{t \to \infty} \langle f, X_c t \rangle, \lim_{t \to \infty} \langle g, X_c t \rangle \]
tell us (about \( f \) and \( g \))??
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- **Linear semi-group:** Set \( v_g(x, t) = \mathbb{E}_{\delta_x}(\langle g, X_t \rangle) \) and it solves

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BBM and BRW

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\[ \psi(\lambda) = -a\lambda + b\lambda^2, \]

in which case we see that for \( \lambda \in \mathbb{R} \), one may take

\[ g(x) = e^{-\lambda x} \]

and

\[ c_\lambda = \frac{c}{\lambda} = \frac{\lambda}{2} + \frac{a}{\lambda}. \]

Monotone travelling waves exist uniquely up to linear shift in the argument if and only if

\[ |c_\lambda| \geq \sqrt{2a}, \]

in which case, when

\[ |\lambda| < \sqrt{2a} \]

(\( \Rightarrow |c_\lambda| > \sqrt{2a} \)),

\[ \lim_{t \to \infty} \langle f, X_{c_\lambda t} \rangle = \lim_{t \to \infty} \langle e^{-\lambda \cdot}, X_{c_\lambda t} \rangle \preceq 0 \]

and

\[ f(x) \sim e^{-\lambda x}. \]

(Durrett/Liggett/Biggins/K./Liu) For BRW, if positions at generation \( n \) are given by

\[ \{ \zeta_n^i : i \geq 1 \} \]

then a "travelling wave" \( \phi : \mathbb{R} \to [0,1] \) is a solution to the functional equation

\[ \phi(x) = E Y_i \phi(x + \zeta_n^i + cn) \]

and can be similarly analysed by comparing against the behaviour of Biggins' martingale

\[ W_n(\lambda) := P_i e^{-\lambda \zeta_n^i / m(\lambda)} n. \]
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**BBM and BRW**

(McKean/Neveu/Chauvin/Lalley-Sellke/Harris/K./Murillo/Liu/Ren) All this works for branching Brownian motion/super-Brownian motion \((\psi(\lambda) = -a\lambda + b\lambda^2)\), in which case we see that for \(\lambda \in \mathbb{R}\), one may take \(g(x) = e^{-\lambda x}\) and \(c = c_\lambda = \lambda/2 + a/\lambda\). Monotone travelling waves exist uniquely up to linear shift in the argument if and only if \(|c_\lambda| \geq \sqrt{2a}\) in which case, when \(|\lambda| < \sqrt{2a}\) (\(\Rightarrow |c_\lambda| > \sqrt{2a}\)),

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\lim_{t \uparrow \infty} \langle f, X_t^{c_\lambda} \rangle = \lim_{t \uparrow \infty} \langle e^{-\lambda \cdot}, X_t^{c_\lambda} \rangle \geq 0 \text{ and } f(x) \sim e^{-\lambda x}.
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(Homogenous) Fragmentation Processes
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- **State space:** Let $\nabla = \{s = (s_1, s_2, \cdots) : s_1 \geq s_2 \geq \cdots \text{ and } \sum_i s_i = 1\}$. 

  - **Mass fragmentation:** $\mathcal{X} = \{\mathcal{X}(t) : t \geq 0\}$ is a $\nabla$-valued Markov process with $\mathcal{X}(0) = (1, 0, 0, \cdots)$ and otherwise we write $\mathcal{X}(t) = (\mathcal{X}_1(t), \mathcal{X}_2(t), \cdots)$.

  - Think of an object of unit mass falling apart into pieces such that the total mass is preserved.

  - **Notation:** Its probabilities are denoted by $\{P_s : s \in \nabla\}$ and, for $s \in (0, 1]$, we shall reserve the special notation $P_s$ as short hand for $P(s, 0, \cdots)$ and in particular write $P$ for $P_1$.

  - **Markov (fragmentation) property:** Given that $\mathcal{X}(t) = (s_1, s_2, \cdots)$, where $t \geq 0$, then for $u > 0$, $\mathcal{X}(t + u)$ has the same law as the variable obtained by ranking in decreasing order the sequences $\mathcal{X}_1(u)$, $\mathcal{X}_2(u)$, $\cdots$ where the latter are independent, random mass partitions with values in $\nabla$ having the same distribution as $\mathcal{X}(u)$ under $P_{s_1}, P_{s_2}, \cdots$ respectively.

  - **Rate of fragmentation:** Fragmentation is governed by a measure $\nu$ on $\nabla$ such that an individual block of mass $s \leq 1$ in the process $\mathcal{X}$ at time $t$ will dislocate into an array of fragments $s \times s$ with rate $\nu(ds) \times dt + o(dt)$. 

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Travelling waves for fragmentation processes.
Travelling wave equation for fragmentation

\[
\text{u}(x,t) := \mathbb{E} \left[ Y_i \text{u}(x - \log X_i(t), t) \right] = \mathbb{E} \left[ \text{u}(x, t) \right] - \mathbb{E} \left[ \nabla \left( Y_i \text{u}(x - \log s_i(t), t) - \text{u}(x, t) \right) \right] \nu(ds) \frac{1}{h} + o(h).
\]

As \( h \downarrow 0 \)
Travelling waves for fragmentation processes.

**Travelling wave equation for fragmentation**

- Natural analogue of \( \exp\{-u(x,t)\} = \mathbb{E}_{\delta_x} \left( \exp\{-\langle f, X_t \rangle\} \right) \):

\[
u(x, t) := \mathbb{E} \left( \prod_{i} g(x - \log X_i(t)) \right)
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with initial condition \( u(x, 0) = g(x) \).
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- Apply Markov (fragmentation) property:

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  u(x, t + h) = \mathbb{E} \left( \prod_i u(x - \log X_i(h), t) \right).
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\[
  u(x, t + h) - u(x, t) = \mathbb{E} \left( \prod_i u(x - \log X_i(h), t) \right) - u(x, t)
\]

\[
  = \int \nabla \left\{ \prod_i u(x - \log s_i, t) - u(x, t) \right\} \nu(ds) h + o(h).
\]
Travelling waves for fragmentation processes.

This suggestively leads us to the integro-differential equation, the KPP equation for fragmentation processes:

\[
\frac{\partial u}{\partial t}(x, t) = \int \left\{ \prod_i u(x - \log s_i, t) - u(x, t) \right\} \nu(ds)
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\[ \frac{\partial u}{\partial t}(x, t) = \int_\nabla \left\{ \prod_i u(x - \log s_i, t) - u(x, t) \right\} \nu(ds) \]

- Hence a travelling wave \( \psi : \mathbb{R} \to [0, 1] \) with wave speed \( c \in \mathbb{R} \) solves the equation

\[ -c\psi'(x) + \int_\nabla \left\{ \prod_i \psi(x - \log s_i) - \psi(x) \right\} \nu(ds) = 0 \]
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With some further restriction on the class in which \( \psi \) sits, one can show through stochastic calculus for semi-martingales (Poisson random fields) that \( \psi \) is a travelling wave with speed \( c \) iff

\[
M_t(c) := \prod_{i} \psi(x - \log X_i(t) - ct), \ t \geq 0
\]

is a martingale.
Spine
For each $t \geq 0$, $X(t)$ is a (random) probability distribution,

$$
\mathbb{E}\left( \sum_i X_i(t)g(- \log X_i(t)) \right) = E(g(\xi_t))
$$

where $\{\xi_t : t \geq 0\}$ under $P$ is a pure jump subordinator with Laplace exponent

$$
-\frac{1}{t} \log \mathbb{E}(e^{-q\xi_t}) = \Phi(q) = \int_\nabla_1 \left( 1 - \sum_{i=1}^{\infty} s_i^{q+1} \right) \nu(ds), \quad q > p,
$$

where

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p := \inf \left\{ p \in \mathbb{R} : \int_\nabla_1 \sum_{i=2}^{\infty} s_i^{p+1} \nu(ds) < \infty \right\} \leq 0.
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Without major restriction, we assume $p < 0$ and that $\Phi(p) = -\infty$. 
Permitted wave speeds
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- **Range of speeds**: Let $c_p = \Phi(p)/(p + 1)$. There exists a unique solution to the equation $(p + 1)\Phi'(p) = \Phi(p)$, denoted by $\bar{p}$. Then wave speeds exist for $c \in (c_p, c_\bar{p}]$.

Note

$$\lim_{t \uparrow \infty} \frac{-\log X_1(t)}{t} = c_\bar{p}, \text{ a.s.}$$
Travelling waves for fragmentation processes.

**Permitted wave speeds**

- **Range of speeds:** Let $c_p = \Phi(p)/(p + 1)$. There exists a unique solution to the equation $(p + 1)\Phi'(p) = \Phi(p)$, denoted by $\bar{p}$. Then wave speeds exist for $c \in (c_p, \bar{p}]$.

  Note

  $$\lim_{t \uparrow \infty} -\log X_1(t) \cdot t = c_{\bar{p}}, \ a.s.$$ 

- **Supercritical speeds:** Note that if $\psi$ for speeds $c > c_{\bar{p}}$,

  $$\prod_i \psi(x - \log X_i(t) - ct) \leq \psi(x - \log X_1(t) - ct) \xrightarrow{t \uparrow \infty} \psi(-\infty) = 0. (!)$$
Permitted wave speeds

- **Range of speeds**: Let \( c_p = \Phi(p)/(p + 1) \). There exists a unique solution to the equation \((p + 1)\Phi'(p) = \Phi(p)\), denoted by \( \overline{p} \). Then wave speeds exist for \( c \in (c_p, c_{\overline{p}}) \).

  Note

  \[
  \lim_{t \uparrow \infty} -\log \frac{X_1(t)}{t} = c_{\overline{p}}, \text{ a.s.}
  \]

- **Supercritical speeds**: Note that if \( \psi \) for speeds \( c > c_p \),

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- **Subcritical speeds**: Biggins’ martingale convergence theorem (Bertoin-Rouault) for additive martingales, \( p \in (p, \overline{p}) \),

  \[
  W(t, p) := \sum_i X_i(t)^{p+1} e^{\Phi(p)t} \xrightarrow{t \uparrow \infty} W(\infty, p), \text{ a.s., } L^1.
  \]

  \[
  \psi(x) = \mathbb{E}(\exp\{-e^{-(p+1)x}W(\infty, p)\}) \text{ is a travelling wave.}
  \]

- **Critical speeds**: Replace \( W(\infty, p) \) by \(-\partial W(\infty, p)/\partial p\).
Asymptotics and Uniqueness: basic ideas $p \in (p, \bar{p})$
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-\log M_t(c_p) \sim e^{- (p+1)x} \sum_i X_i(t)^{p+1} e^{\Phi(p)t} L_p(x - \log X_i(t) - ct)
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Asymptotics and Uniqueness: basic ideas $p \in (p, \bar{p})$

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Naively: Show that

$$\sum_i X_i(t)^{p+1} e^{\Phi(p)t} L_p(x - \log X_i(t) - ct) \sim L_p(\alpha t) \sum_i X_i(t)^{p+1} e^{\Phi(p)t}$$

for some $\alpha$, then $-\log M_t(c_p)/W(t, p) \sim L(\alpha t) \Rightarrow L_p \sim k_p \in (0, \infty)$ and uniqueness follows.
Asymptotics and Uniqueness: basic ideas $p \in (p, \bar{p})$

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for some $\alpha$, then $-\log M_t(c_p)/W(t, p) \sim L(\alpha t) \Rightarrow L_p \sim k_p \in (0, \infty)$ and uniqueness follows.

- Problem: 

"$-\log X_i(t) - ct$" behaves like a Lévy process with no positive jumps drifting to $+\infty$. Too difficult to control all of them uniformly.
Stopping lines

Figure: Freeze fragments as soon as $-\log X(t) - c_p t \geq z$ with $p \in (0, \bar{p})$. Collection of block sizes and their “freezing time” denoted $\{(B_i(z), \ell_i(z)) : i \geq 1\}$. 

[Diagram showing stopping lines and freezing times]
Stopping lines

Figure: Freeze fragments as soon as $-\log X(t) - c_p t \geq z$ with $p \in (0, p)$. Collection of block sizes and their “freezing time” denoted $\{ (B_i(z), \ell_i(z)) : i \geq 1 \}$. 
Working with stopping lines

\[ M_\ell(z)(c_p) := Y_i \psi(x - \log B_i(z) - c_p \ell_i(z)) \]

and

\[ W(\ell z, p) := X_i B_i(z)^{(p+1)} e^{\Phi(p) \ell_i(z)} \]

Now much easier to compare

\(-\log M_\ell z(c_p) \text{ against } W(\ell z, p)\)

\(-\log B_i(z) - c_p \ell_i(z) \geq x + z \text{ uniformly in } i\)

and deduce that, as \(z \uparrow \infty\),

\[-\log M_\ell z(c_p) W(\ell z, p) \sim e^{-(p+1)x} X_i B_i(z)^{(p+1)} e^{\Phi(p) \ell_i(z)} W(\ell z, p) \sim e^{-(p+1)x} L_p(x + z)\]

and our naive argument can be made rigorous.
Working with stopping lines

- All martingales concerned are uniformly integrable and their limits can be "projected back" on to the stopping lines to give "stopped" versions of martingales. For $z \geq 0$

$$M_{\ell_z}(c_p) := \prod_i \psi(x - \log B_i(z) - c_p \ell_i(z)) \quad \text{and} \quad W(\ell_z, p) := \sum_i B_i(z)^{(p+1)} e^{\Phi(p) \ell_i(z)}.$$
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- Now much easier to compare $-\log M_{\ell z}$ against $W(\ell_z, p)$

$(x - \log B_i(z) - c_p\ell_i(z) \geq x + z \text{ uniformly in } i)$ and deduce that, as $z \uparrow \infty$,

$$\frac{-\log M_{\ell z}(c_p)}{W(\ell_z, p)} \sim e^{-(p+1)x} \sum_i \frac{B_i(z)^{(p+1)} e^{\Phi(p)\ell_i(z)}}{W(\ell_z, p)} L_p(x - \log B_i(z) - c_p\ell_i(z))$$

$$\sim e^{-(p+1)x} L_p(x + z)$$

and our naive argument can be made rigorous.
Final technical note

To asymptotically replace

\[ P_i B_i(z) (p+1) e^{\Phi(p)} \ell_i(z) L_p(x - \log B_i(z) - c_p \ell_i(z)) \]

by

\[ L_p(x + z) W(\ell z, p) \]

we need the following technical lemma which echoes Nerman's classical strong law of large numbers for CMJ processes.

For all exponentially bounded positive functions \(f\) and \(p \in (p_L, p_U]\),

\[ X_i B_i(z) (p+1) e^{\Phi(p)} \ell_i(z) f(x - \log B_i(z) - c_p \ell_i(z)) \sim Q_p(f) W(\infty, p) \]

where \(Q_p(f)\) is the expectation of \(f\) with respect to the stationary overshoot distribution of a subordinator.

When \(p \in (0, p)\) this result can in fact be deduced from Nerman's classical strong law.
Final technical note

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