Capped American Lookback

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Consider a financial market consisting of a bank account and a risky asset.

- Bank account $R = (R_t)_{t \geq 0}$ satisfies
  \[
  dR_t = rR_t \, dt, \quad R_0 = 1, r \geq 0,
  \]
  that is, $R_t = e^{rt}$, $t \geq 0$.

- Risky asset under $\mathbb{P}$ is modeled as exponential Lévy process
  \[
  S_t = S_0 e^{X_t}, \quad S_0 > 0, t \geq 0.
  \]
Motivation

■ A (perpetual) American lookback option gives the holder the right to exercise at any finite stopping time $\tau$ yielding payout

$$e^{-\alpha \tau} \left( M_0 \vee \sup_{0 \leq u \leq \tau} S_u - K \right)^+, \quad M_0 \geq S_0, \alpha > 0.$$  

■ Which translates to the optimal stopping problem

$$V^{AL}(x, s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x, s} \left[ e^{-q \tau} (e^{\overline{X}_\tau} - K)^+ \right], \quad q > 0, K > 0,$$

where $\overline{X}_\tau = \sup_{s \leq \tau} X_s, x \leq s$

$$\mathbb{P}_{x, s}(\cdot) = \mathbb{P}(\cdot | X_0 = x, \overline{X}_0 = s)$$

and $\mathcal{M}$ is the set of all stopping times (not necessarily finite).

■ This problem has been earlier considered in a diffusive setting by Conze and Viswanathan (1991), Pedersen (2000), Guo and Shepp (2001) and Gapeev (2007).
A (perpetual) American lookback option with cap gives the holder the right to exercise at any finite stopping time $\tau$ yielding payouts

$$e^{-\alpha \tau} \left( M_0 \lor \sup_{0 \leq u \leq \tau} S_u \wedge C - K \right)^+, \quad C \geq M_0 \geq S_0, \alpha > 0.$$ 

Which translates to the optimal stopping problem

$$V^{AL}_\epsilon(x, s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x, s} \left[ e^{-q \tau} (e^{X_{\tau} \wedge \epsilon} - K)^+ \right], \quad q > 0, K > 0,$$

where $x \leq s$ and $\epsilon \in (\log(K), \infty]$. 

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Notation

- $X$ is a **spectrally negative** Lévy process.
- The Laplace exponent $\psi$ of $X$ is defined by

$$\psi(\lambda) := \frac{1}{t} \log \mathbb{E}[e^{\lambda X_t}], \quad \lambda \geq 0$$

- For $q \geq 0$, its right-inverse $\Phi$ is given by

$$\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}.$$  

- For $q \geq 0$, the $q$-scale function $W^{(q)} : \mathbb{R} \rightarrow [0, \infty)$ is the unique function whose restriction to $(0, \infty)$ is continuous and has Laplace transform

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x) \, dx = \frac{1}{\psi(\lambda) - q},$$

for $\lambda$ suff. large, and is defined to be zero for $x \leq 0$. 
For $q \geq 0$, we define $Z^{(q)} : \mathbb{R} \longrightarrow [1, \infty)$ by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(z) \, dz.$$
How do Russian-type stopping problems work?

- Capped American Lookback:
  \[ V_{\epsilon}^{AL}(x, s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}[e^{-q\tau}\left(e^{X_\tau \wedge \epsilon} - K\right)^+] , \]

- Russian:
  \[ V_{\epsilon}^{R}(x, s) := \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}[e^{-q\tau+X_\tau}] . \]

Recall the Russian option was introduced and studied by Shepp and Shiryaev (1993, 1994) in the Black-Scholes setting and was studied in the current spectrally negative setting by Avram, K. and Pistorius (2004).

- As \( \epsilon \uparrow \infty \) we expect to see \( V_{\epsilon}^{AL}(x, s) \) look more and more like the value function of \( V^{AL} \). Moreover as \( s \uparrow \infty \) we expect to see \( V^{AL} \) look more and more like \( V^{R} \).

- Roughly speaking all of these optimal stopping problems appear to fit the following setting:
  \[ V^f(x, s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}[e^{-q\tau f(X_\tau)}] , \]

where \( f \) is an increasing function.
How do Russian-type stopping problems work?

$$V^R(x, s) := \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s} \left[ e^{-q\tau + X_\tau} \right].$$

**Theorem** [Shepp, Shiryaev, Avram, K., Pistorius]: Suppose that $q > \psi(1)$. Then

$$V^R(x, s) = e^s Z^{(q)}(x - s + k^*)$$

with optimal strategy

$$\tau^R = \inf\{ t \geq 0 : \overline{X}_t - X_t \geq k^* \}$$

for some constant $k^* \in (0, \infty)$, where $k^*$ is the unique solution to $Z^{(q)}(z) - qW^{(q)}(z) = 0.$
Figure: Stopping region $D^*$ and continuution region $C^*$ for the Russian optimal stopping problem.
How do Russian-type stopping problems work?

\[ V^f(x, s) = \sup_{\tau \in \mathcal{M}} E_{x,s} \left[ e^{-q\tau} f(\overline{X}_\tau) \right], \quad x \leq s. \]

Assuming the optimal strategy is of the form

\[ \tau^f = \inf \{ t > 0 : \overline{X}_t - X_t > g(\overline{X}_t) \} : \]

- Let \( \tau^+_s = \inf \{ t > 0 : X_t > s \} \) and \( \tau^-_z = \inf \{ t > 0 : X_t < z \} \),

\[ V^f(x, s) = f(s) E_{x,s} \left( e^{-q\tau^-_s g(s)} \mathbf{1}_{(\tau^-_s - g(s) < \tau^+_s)} \right) + E_{x,s} \left( e^{-q\tau^+_s} \mathbf{1}_{(\tau^-_s - g(s) > \tau^+_s)} \right) V^f(s, s) \]
How do Russian-type stopping problems work?

- Hence

\[ V^f(x, s) = f(s) \left( Z^{(q)}(x - s + g(s)) - W^{(q)}(x - s + g(s)) \frac{Z^{(q)}(g(s))}{W^{(q)}(g(s))} \right) \]

\[ + \frac{W^{(q)}(x - s + g(s))}{W^{(q)}(g(s))} V^f(s, s) \]

- Smooth fit:

\[ 0 = \lim_{x \downarrow s - g(s)} \frac{\partial V^f}{\partial x}(x, s) \]

\[ = \lim_{x \downarrow s - g(s)} \frac{W^{(q)'}(x - s + g(s))}{W^{(q)}(g(s))} \left[ V^f(s, s) - f(s)Z^{(q)}(g(s)) \right] . \]

\[ \implies V^f(x, s) = f(s)Z^{(q)}(x - s + g(s)). \]

(Russian): \[ V^R(x, s) = e^sZ^{(q)}(x - s + k^*) \]
How do Russian-type stopping problems work?

Once we know $V^f(x, s) = f(s)Z^{(q)}(x - s + g(s))$, normal reflection at $(s, s)$ tells us

$$\frac{\partial V^f}{\partial s}(s-, s) = 0 \implies g'(s) = 1 - \frac{f'(s)Z^{(q)}(g(s))}{f(s)qW^{(q)}(g(s))}$$

(Russian) : $(k^*)' = 0 = 1 - \frac{e^sZ^{(q)}(k^*)}{e^sqW^{(q)}(k^*)} \implies Z^{(q)}(k^*) - qW^{(q)}(k^*) = 0$
Figure: Expected shape of optimal boundary for the Capped American Lookback when $\epsilon = (\log(K), \infty)$ and $\epsilon = \infty$ respectively.
Lemma (Solution of ODE)

There exists a unique solution $g$ of the ODE

$$g'(s) = 1 - \frac{e^s Z(q)(g(s))}{(e^s - K)qW(q)(g(s))} \quad \text{on } (\log(K), \epsilon) \quad (1)$$

satisfying the boundary conditions $g(\log(K) +) = \infty$ and

$$\lim_{s \uparrow \epsilon} g(s) = \begin{cases} 0, & \epsilon \in (\log(K), \infty), \\ k^*, & \epsilon = \infty, \end{cases}$$

where $k^* \in (0, \infty)$ is the unique root of $Z(q)(s) - qW(q)(s) = 0$.

See below for sketch of proof.
Theorem

Suppose that $q > \psi(1)$. The solution of the American Lookback OSP is given by

$$V^*(x, s) = \begin{cases} (e^{s\wedge\epsilon} - K)Z(q)(x - s + g(s)), & (x, s) \in C^*_I \cup D^*, \\ e^{-\Phi(q)(\log(K) - x)}A, & (x, s) \in C^*_II, \end{cases}$$

where $A = \lim_{s \downarrow \log(K)}(e^s - K)Z(q)(g(s)) > 0$, with optimal strategy

$$\tau^* = \inf\{t \geq 0 : \overline{X}_t - X_t \geq g(\overline{X}_t) \text{ and } \overline{X}_t > \log(K)\},$$

where $g$ is given in the Lemma above.
Consider the ODE

\[ g'(s) = 1 - \frac{e^s Z^{(q)}(g(s))}{(e^s - K)qW^{(q)}(g(s))} \quad \text{on} \quad (\log(K), \infty). \]

The 0-isocline is given by the graph of

\[ f(H) = \log \left( K \left( 1 - \frac{Z^{(q)}(H)}{qW^{(q)}(H)} \right)^{-1} \right), \]

where \( H \in (k^*, \infty) \). It can be shown that \( f \) is strictly decreasing, \( \eta := f(\infty) = \log(K(1 - \Phi(q)^{-1})^{-1}) \) and \( f(k^*+) = \infty \).
Figure: A qualitative picture of the direction field.
Figure: The solutions to the ODE and the corresponding possible stopping boundaries.
The solutions exhibit a behavior parallel to Peskir’s maximality principle in both cases $\epsilon = \infty$ and $\epsilon \in (\log(K), \infty)$.

If $\epsilon = \infty$, the “red” curves correspond to the so-called “bad-good” solutions in Peskir’s maximality principle (see Peskir (1998)); “bad” because they do not give the optimal boundary, “good” as they can be used to approximate the optimal boundary.

The same can be observed in the capped Russian stopping problem.
Current/future work (Curdin!!!):

- American lookback with floating strike:

\[ V^*(x, s) = \sup_{\tau} \mathbb{E}_{x,s} \left[ e^{-q\tau} (e^{\overline{X}_\tau} - Ke^{X_\tau})^+ \right] \]

Cap \( \overline{X} \) or \( X \), both?

- \( \pi \)-option:

\[ V^*(x, s) = \sup_{\tau} \mathbb{E}_{x,s} \left[ e^{-q\tau} (e^{aX_\tau + b\overline{X}_\tau} - K)^+ \right] , \]

where \( a, b > 0 \).


