Capped American Lookback

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Motivation

Consider a financial market consisting of a bank account and a risky asset.

- Bank account $R = (R_t)_{t \geq 0}$ satisfies

$$dR_t = rR_t dt, \quad R_0 = 1, r \geq 0,$$

  that is, $R_t = e^{rt}, t \geq 0$.

- Risky asset under $\mathbb{P}$ is modeled as exponential Lévy process

$$S_t = S_0 e^{X_t}, \quad S_0 > 0, t \geq 0.$$

Motivation

- A (perpetual) American lookback option gives the holder the right to exercise at any finite stopping time τ yielding payout

$$e^{-\alpha\tau}\left(M_0 \vee \sup_{0\le u\le\tau} S_u - K\right)^+, \quad M_0 \ge S_0, \alpha > 0.$$

- Which translates to the optimal stopping problem

$$V^{AL}(x,s) = \sup_{\tau\in\mathcal{M}} \mathbb{E}_{x,s}\left[e^{-q\tau}\left(e^{\overline{X}_\tau} - K\right)^+\right], \quad q > 0, K > 0,$$

where $\overline{X}_\tau = \sup_{s\le\tau} X_s$, $x \le s$

$$\mathbb{P}_{x,s}(\cdot) = \mathbb{P}(\cdot|X_0 = x, \overline{X}_0 = s)$$

and $\mathcal{M}$ is the set of all stopping times (not necessarily finite).

- This problem has been earlier considered in a diffusive setting by Conze and Viswanathan (1991), Pedersen (2000), Guo and Shepp (2001) and Gapeev (2007).

Motivation

- A (perpetual) American lookback option **with cap** gives the holder the right to exercise at any finite stopping time τ yielding payouts

$$e^{-\alpha\tau}\left(M_0 \vee \sup_{0\le u\le\tau} S_u \wedge C - K\right)^+, \quad C \ge M_0 \ge S_0, \alpha > 0.$$

- Which translates to the optimal stopping problem

$$V_\epsilon^{AL}(x,s) = \sup_{\tau\in\mathcal{M}} \mathbb{E}_{x,s}\left[e^{-q\tau}\left(e^{\overline{X}_\tau\wedge\epsilon} - K\right)^+\right], \quad q>0, K>0,$$

where $x \le s$ and $\epsilon \in (\log(K), \infty]$.

Notation

- X is a **spectrally negative** Lévy process.
- The Laplace exponent ψ of X is defined by

$$\psi(\lambda) := \frac{1}{t} \log \mathbb{E}\big[e^{\lambda X_t}\big], \qquad \lambda \geq 0$$

- For $q \geq 0$, its right-inverse Φ is given by

$$\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}.$$

- For $q \geq 0$, the q-scale function $W^{(q)} : \mathbb{R} \longrightarrow [0, \infty)$ is the unique function whose restriction to $(0, \infty)$ is continuous and has Laplcae transform

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)\, dx = \frac{1}{\psi(\lambda) - q},$$

for λ suff. large, and is defined to be zero for $x \leq 0$.

Notation

- For $q \geq 0$, we define $Z^{(q)} : \mathbb{R} \longrightarrow [1, \infty)$ by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(z)\, dz.$$

How do Russian-type stopping problems work?

- Capped American Lookback:

$$V_\epsilon^{AL}(x,s) = \sup_{\tau\in\mathcal{M}} \mathbb{E}_{x,s}\left[e^{-q\tau}\left(e^{\overline{X}_\tau\wedge\epsilon} - K\right)^+\right],$$

- Russian:

$$V_\epsilon^{R}(x,s) := \sup_{\tau\in\mathcal{M}} \mathbb{E}_{x,s}\left[e^{-q\tau+\overline{X}_\tau}\right].$$

  Recall the Russian option was introduced and studied by Shepp and Shiryaev (1993,1994) in the Black-Scholes setting and was studied in the current spectrally negative setting by Avram, K. and Pistorius (2004).
- As $\epsilon\uparrow\infty$ we expect to see $V_\epsilon^{AL}(x,s)$ look more and more like the value function of V^{AL}. Moreover as $s\uparrow\infty$ we expect to see V^{AL} look more and more like V^R.
- Roughly speaking all of these optimal stopping problems appear to fit the following setting:

$$V^f(x,s) = \sup_{\tau\in\mathcal{M}} \mathbb{E}_{x,s}\left[e^{-q\tau}f(\overline{X}_\tau)\right],$$

  where f is an increasing function.

How do Russian-type stopping problems work?

$$V^R(x,s) := \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}\left[e^{-q\tau + \overline{X}_\tau}\right].$$

Theorem [Shepp, Shiryaev, Avram, K., Pistorius]: Suppose that $q > \psi(1)$. Then

$$V^R(x,s) = e^s Z^{(q)}(x - s + k^*)$$

with optimal strategy

$$\tau^R = \inf\{t \geq 0 \,:\, \overline{X}_t - X_t \geq k^*\}$$

for some constant $k^* \in (0,\infty)$, where k^* is the unique solution to $Z^{(q)}(z) - qW^{(q)}(z) = 0$.

Russian stopping problem

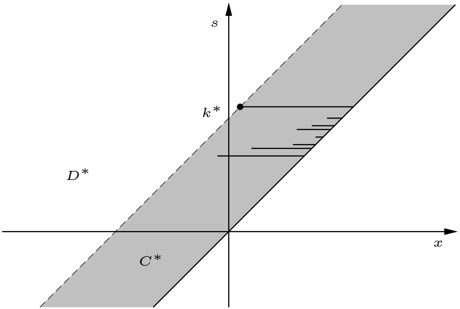

Figure: Stopping region D^* and continuatuion region C^* for the Russian optimal stopping problem.

How do Russian-type stopping problems work?

$$V^f(x,s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}\big[e^{-q\tau} f(\overline{X}_\tau)\big], \qquad x \leq s.$$

Assuming the optimal strategy is of the form

$$\tau^f = \inf\{t > 0 : \overline{X}_t - X_t > g(\overline{X}_t)\}:$$

- Let $\tau_s^+ = \inf\{t > 0 : X_t > s\}$ and $\tau_z^- = \inf\{t > 0 : X_t < z\}$,

$$V^f(x,s) = f(s)\mathbb{E}_{x,s}(e^{-q\tau^-_{s-g(s)}}\mathbf{1}_{(\tau^-_{s-g(s)} < \tau^+_s)}) + \mathbb{E}_{x,s}(e^{-q\tau^+_s}\mathbf{1}_{(\tau^-_{s-g(s)} > \tau^+_s)})V^f(s,s)$$

How do Russian-type stopping problems work?

- Hence

$$\begin{aligned} V^f(x,s) &= f(s)\left(Z^{(q)}(x-s+g(s)) - W^{(q)}(x-s+g(s))\frac{Z^{(q)}(g(s))}{W^{(q)}(g(s))}\right) \\ &\quad + \frac{W^{(q)}(x-s+g(s))}{W^{(q)}(g(s))}V^f(s,s) \end{aligned}$$

- Smooth fit:

$$\begin{aligned} 0 &= \lim_{x\downarrow s-g(s)} \frac{\partial V^f}{\partial x}(x,s) \\ &= \lim_{x\downarrow s-g(s)} \frac{W^{(q)'}(x-s+g(s))}{W^{(q)}(g(s))}\left[V^f(s,s) - f(s)Z^{(q)}(g(s))\right]. \end{aligned}$$

$$\Longrightarrow V^f(x,s) = f(s)Z^{(q)}(x-s+g(s)).$$

$$(\text{Russian}) : V^R(x,s) = e^s Z^{(q)}(x-s+k^*)$$

How do Russian-type stopping problems work?

- Once we know $V^f(x,s) = f(s)Z^{(q)}(x - s + g(s))$, normal reflection at (s,s) tells us

$$\frac{\partial V^f}{\partial s}(s-,s) = 0 \Longrightarrow g'(s) = 1 - \frac{f'(s)Z^{(q)}(g(s))}{f(s)qW^{(q)}(g(s))}$$

$$(\text{Russian}) : (k^*)' = 0 = 1 - \frac{e^s Z^{(q)}(k^*)}{e^s qW^{(q)}(k^*)} \Rightarrow Z^{(q)}(k^*) - qW^{(q)}(k^*) = 0$$

Guess solution

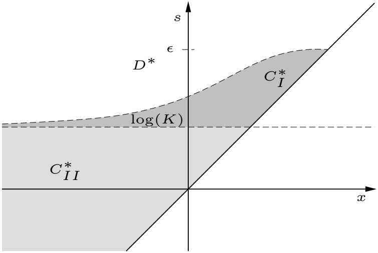

Figure: Expected shape of optimal boundary for the Capped American Lookback when $\epsilon = (\log(K), \infty)$ and $\epsilon = \infty$ respectively.

Result

Lemma (Solution of ODE)

There exists a unique solution g of the ODE

$$g'(s) = 1 - \frac{e^s Z^{(q)}(g(s))}{(e^s - K)qW^{(q)}(g(s))} \quad \text{on } (\log(K), \epsilon) \tag{1}$$

satisfying the boundary conditions $g(\log(K)+) = \infty$ *and*

$$\lim_{s \uparrow \epsilon} g(s) = \begin{cases} 0, & \epsilon \in (\log(K), \infty), \\ k^*, & \epsilon = \infty, \end{cases}$$

where $k^* \in (0, \infty)$ *is the unique root of* $Z^{(q)}(s) - qW^{(q)}(s) = 0$.

See below for sketch of proof.

Result

Theorem

Suppose that $q > \psi(1)$. *The solution of the American Lookback OSP is given by*

$$V^*(x,s) = \begin{cases} (e^{s\wedge\epsilon} - K)Z^{(q)}(x - s + g(s)), & (x,s) \in C_I^* \cup D^*, \\ e^{-\Phi(q)(\log(K)-x)}A, & (x,s) \in C_{II}^*, \end{cases}$$

where $A = \lim_{s\downarrow\log(K)}(e^s - K)Z^{(q)}(g(s)) > 0$*, with optimal strategy*

$$\tau^* = \inf\{t \geq 0 : \overline{X}_t - X_t \geq g(\overline{X}_t) \text{ and } \overline{X}_t > \log(K)\},$$

where g *is given in the Lemma above.*

Sketch of proof of ODE lemma

Consider the ODE

$$g'(s) = 1 - \frac{e^s Z^{(q)}(g(s))}{(e^s - K)qW^{(q)}(g(s))} \quad \text{on } (\log(K), \infty).$$

The 0-isocline is given by the graph of

$$f(H) = \log\left(K\left(1 - \frac{Z^{(q)}(H)}{qW^{(q)}(H)}\right)^{-1}\right),$$

where $H \in (k^*, \infty)$. It can be shown that f is strictly decreasing, $\eta := f(\infty) = \log(K(1 - \Phi(q)^{-1})^{-1})$ and $f(k^*+) = \infty$.

Direction field

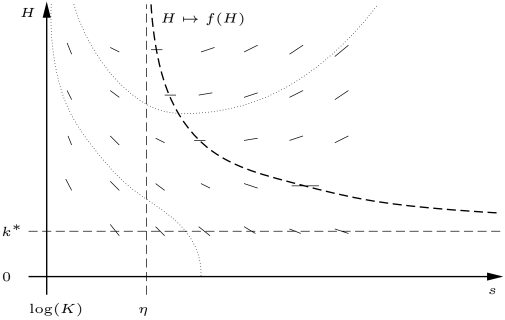

Figure: A qualitative picture of the direction field.

Sketch of proof of ODE lemma / maximality principle

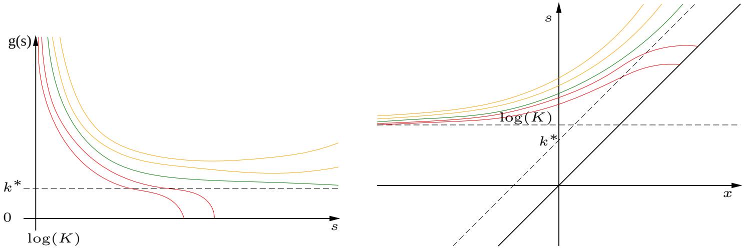

Figure: The solutions to the ODE and the corresponding possible stopping boundaries.

Link to maximality principle

- The solutions exhibit a behavior parallel to Peskir's maximality principle in both cases $\epsilon = \infty$ and $\epsilon \in (\log(K), \infty)$.
- If $\epsilon = \infty$, the "red" curves correspond to the so-called "bad-good" solutions in Peskir's maximality principle (see Peskir (1998)); "bad" because they do not give the optimal boundary, "good" as they can be used to approximate the optimal boundary.
- The same can be observed in the capped Russian stopping problem.

Current/future work (Curdin!!!):

- American lookback with floating strike:

$$V^*(x,s) = \sup_\tau \mathbb{E}_{x,s}\left[e^{-q\tau}\left(e^{\overline{X}_\tau} - Ke^{X_\tau}\right)^+\right]$$

  Cap $\overline{X}$ or X, both?

- π-option:

$$V^*(x,s) = \sup_\tau \mathbb{E}_{x,s}\left[e^{-q\tau}\left(e^{aX_\tau + b\overline{X}_\tau} - K\right)^+\right],$$

  where $a, b > 0$.

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