

Capped American Lookback

Andreas E. Kyprianou and Curdin Ott

University of Bath

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Consider a financial market consisting of a bank account and a risky asset.

- Bank account $R = (R_t)_{t \geq 0}$ satisfies

$$dR_t = rR_t dt, \quad R_0 = 1, r \geq 0,$$

that is, $R_t = e^{rt}, t \geq 0$.

- Risky asset under \mathbb{P} is modeled as exponential Lévy process

$$S_t = S_0 e^{X_t}, \quad S_0 > 0, t \geq 0.$$

- A (perpetual) American lookback option gives the holder the right to exercise at any finite stopping time τ yielding payout

$$e^{-\alpha\tau} \left(M_0 \vee \sup_{0 \leq u \leq \tau} S_u - K \right)^+, \quad M_0 \geq S_0, \alpha > 0.$$

- Which translates to the optimal stopping problem

$$V^{AL}(x, s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s} [e^{-q\tau} (e^{\bar{X}_\tau} - K)^+], \quad q > 0, K > 0,$$

where $\bar{X}_\tau = \sup_{s \leq \tau} X_s$, $x \leq s$

$$\mathbb{P}_{x,s}(\cdot) = \mathbb{P}(\cdot | X_0 = x, \bar{X}_0 = s)$$

and \mathcal{M} is the set of all stopping times (not necessarily finite).

- This problem has been earlier considered in a diffusive setting by Conze and Viswanathan (1991), Pedersen (2000), Guo and Shepp (2001) and Gapeev (2007).

- A (perpetual) American lookback option **with cap** gives the holder the right to exercise at any finite stopping time τ yielding payouts

$$e^{-\alpha\tau} \left(M_0 \vee \sup_{0 \leq u \leq \tau} S_u \wedge C - K \right)^+, \quad C \geq M_0 \geq S_0, \alpha > 0.$$

- Which translates to the optimal stopping problem

$$V_{\epsilon}^{AL}(x, s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s} [e^{-q\tau} (e^{\bar{X}_{\tau} \wedge \epsilon} - K)^+], \quad q > 0, K > 0,$$

where $x \leq s$ and $\epsilon \in (\log(K), \infty]$.

- X is a **spectrally negative** Lévy process.
- The Laplace exponent ψ of X is defined by

$$\psi(\lambda) := \frac{1}{t} \log \mathbb{E}[e^{\lambda X_t}], \quad \lambda \geq 0$$

- For $q \geq 0$, its right-inverse Φ is given by

$$\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}.$$

- For $q \geq 0$, the q -scale function $W^{(q)} : \mathbb{R} \rightarrow [0, \infty)$ is the unique function whose restriction to $(0, \infty)$ is continuous and has Laplace transform

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x) dx = \frac{1}{\psi(\lambda) - q},$$

for λ suff. large, and is defined to be zero for $x \leq 0$.

- For $q \geq 0$, we define $Z^{(q)} : \mathbb{R} \rightarrow [1, \infty)$ by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(z) dz.$$

How do Russian-type stopping problems work?

- Capped American Lookback:

$$V_{\epsilon}^{AL}(x, s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x, s} [e^{-q\tau} (e^{\overline{X}_{\tau} \wedge \epsilon} - K)^+],$$

- Russian:

$$V_{\epsilon}^R(x, s) := \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x, s} [e^{-q\tau + \overline{X}_{\tau}}].$$

Recall the Russian option was introduced and studied by Shepp and Shiryaev (1993,1994) in the Black-Scholes setting and was studied in the current spectrally negative setting by Avram, K. and Pistorius (2004).

- As $\epsilon \uparrow \infty$ we expect to see $V_{\epsilon}^{AL}(x, s)$ look more and more like the value function of V^{AL} . Moreover as $s \uparrow \infty$ we expect to see V^{AL} look more and more like V^R .
- Roughly speaking all of these optimal stopping problems appear to fit the following setting:

$$V^f(x, s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x, s} [e^{-q\tau} f(\overline{X}_{\tau})],$$

where f is an increasing function.

How do Russian-type stopping problems work?

$$V^R(x, s) := \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x, s} [e^{-q\tau + \bar{X}_\tau}].$$

Theorem [Shepp, Shiryaev, Avram, K., Pistorius]: Suppose that $q > \psi(1)$. Then

$$V^R(x, s) = e^s Z^{(q)}(x - s + k^*)$$

with optimal strategy

$$\tau^R = \inf\{t \geq 0 : \bar{X}_t - X_t \geq k^*\}$$

for some constant $k^* \in (0, \infty)$, where k^* is the unique solution to $Z^{(q)}(z) - qW^{(q)}(z) = 0$.

Russian stopping problem

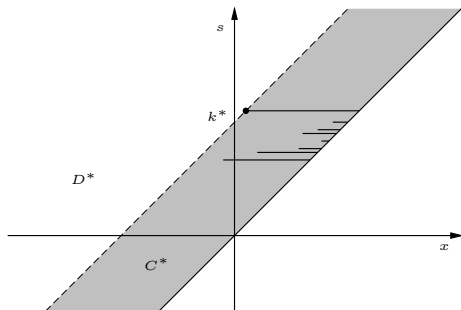


Figure: Stopping region D^* and continuation region C^* for the Russian optimal stopping problem.

How do Russian-type stopping problems work?

$$V^f(x, s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s} [e^{-q\tau} f(\bar{X}_\tau)], \quad x \leq s.$$

Assuming the optimal strategy is of the form

$$\tau^f = \inf\{t > 0 : \bar{X}_t - X_t > g(\bar{X}_t)\} :$$

■ Let $\tau_s^+ = \inf\{t > 0 : X_t > s\}$ and $\tau_s^- = \inf\{t > 0 : X_t < z\}$,

$$V^f(x, s) = f(s) \mathbb{E}_{x,s} (e^{-q\tau_s^-} \mathbf{1}_{(\tau_{s-g(s)}^- < \tau_s^+)}) + \mathbb{E}_{x,s} (e^{-q\tau_s^+} \mathbf{1}_{(\tau_{s-g(s)}^- > \tau_s^+)}) V^f(s, s)$$

How do Russian-type stopping problems work?

■ Hence

$$\begin{aligned} V^f(x, s) &= f(s) \left(Z^{(q)}(x - s + g(s)) - W^{(q)}(x - s + g(s)) \frac{Z^{(q)}(g(s))}{W^{(q)}(g(s))} \right) \\ &\quad + \frac{W^{(q)}(x - s + g(s))}{W^{(q)}(g(s))} V^f(s, s) \end{aligned}$$

■ Smooth fit:

$$\begin{aligned} 0 &= \lim_{x \downarrow s - g(s)} \frac{\partial V^f}{\partial x}(x, s) \\ &= \lim_{x \downarrow s - g(s)} \frac{W^{(q)'}(x - s + g(s))}{W^{(q)}(g(s))} [V^f(s, s) - f(s)Z^{(q)}(g(s))]. \end{aligned}$$

$$\implies V^f(x, s) = f(s)Z^{(q)}(x - s + g(s)).$$

$$\text{(Russian) : } V^R(x, s) = e^s Z^{(q)}(x - s + k^*)$$

How do Russian-type stopping problems work?

- Once we know $V^f(x, s) = f(s)Z^{(q)}(x - s + g(s))$, normal reflection at (s, s) tells us

$$\frac{\partial V^f}{\partial s}(s-, s) = 0 \implies g'(s) = 1 - \frac{f'(s)Z^{(q)}(g(s))}{f(s)qW^{(q)}(g(s))}$$

$$\text{(Russian)} : (k^*)' = 0 = 1 - \frac{e^s Z^{(q)}(k^*)}{e^s q W^{(q)}(k^*)} \implies Z^{(q)}(k^*) - q W^{(q)}(k^*) = 0$$

Guess solution

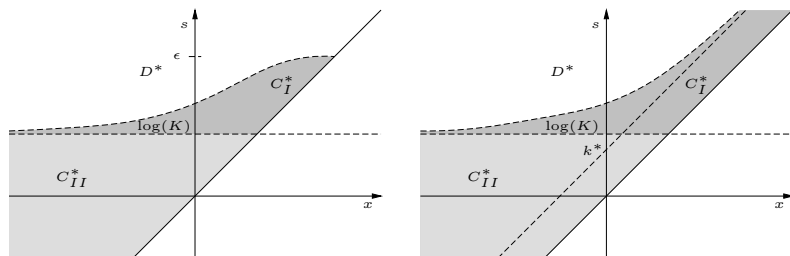


Figure: Expected shape of optimal boundary for the Capped American Lookback when $\epsilon = (\log(K), \infty)$ and $\epsilon = \infty$ respectively.

Lemma (Solution of ODE)

There exists a unique solution g of the ODE

$$g'(s) = 1 - \frac{e^s Z^{(q)}(g(s))}{(e^s - K)qW^{(q)}(g(s))} \quad \text{on } (\log(K), \epsilon) \quad (1)$$

satisfying the boundary conditions $g(\log(K)+) = \infty$ and

$$\lim_{s \uparrow \epsilon} g(s) = \begin{cases} 0, & \epsilon \in (\log(K), \infty), \\ k^*, & \epsilon = \infty, \end{cases}$$

where $k^ \in (0, \infty)$ is the unique root of $Z^{(q)}(s) - qW^{(q)}(s) = 0$.*

See below for sketch of proof.

Theorem

Suppose that $q > \psi(1)$. The solution of the American Lookback OSP is given by

$$V^*(x, s) = \begin{cases} (e^{s \wedge \epsilon} - K)Z^{(q)}(x - s + g(s)), & (x, s) \in C_I^* \cup D^*, \\ e^{-\Phi(q)(\log(K) - x)}A, & (x, s) \in C_{II}^*, \end{cases}$$

where $A = \lim_{s \downarrow \log(K)} (e^s - K)Z^{(q)}(g(s)) > 0$, with optimal strategy

$$\tau^* = \inf\{t \geq 0 : \bar{X}_t - X_t \geq g(\bar{X}_t) \text{ and } \bar{X}_t > \log(K)\},$$

where g is given in the Lemma above.

Sketch of proof of ODE lemma

Consider the ODE

$$g'(s) = 1 - \frac{e^s Z^{(q)}(g(s))}{(e^s - K)qW^{(q)}(g(s))} \quad \text{on } (\log(K), \infty).$$

The 0-isocline is given by the graph of

$$f(H) = \log \left(K \left(1 - \frac{Z^{(q)}(H)}{qW^{(q)}(H)} \right)^{-1} \right),$$

where $H \in (k^*, \infty)$. It can be shown that f is strictly decreasing, $\eta := f(\infty) = \log(K(1 - \Phi(q)^{-1})^{-1})$ and $f(k^*+) = \infty$.

Direction field

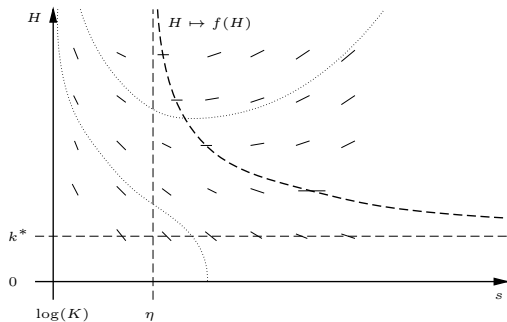


Figure: A qualitative picture of the direction field.

Sketch of proof of ODE lemma / maximality principle

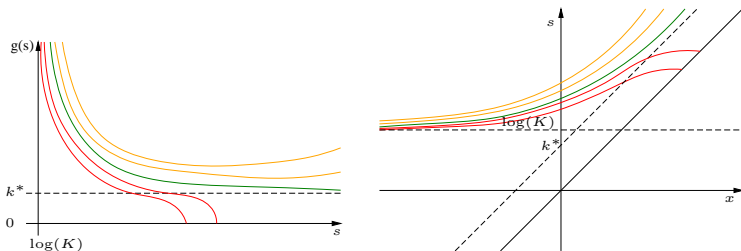


Figure: The solutions to the ODE and the corresponding possible stopping boundaries.

Link to maximality principle

- The solutions exhibit a behavior parallel to Peskir's maximality principle in both cases $\epsilon = \infty$ and $\epsilon \in (\log(K), \infty)$.
- If $\epsilon = \infty$, the “red” curves correspond to the so-called “bad-good” solutions in Peskir's maximality principle (see Peskir (1998)); “bad” because they do not give the optimal boundary, “good” as they can be used to approximate the optimal boundary.
- The same can be observed in the capped Russian stopping problem.

Current/future work (Curdin!!!):

- American lookback with floating strike:







$$V^*(x, s) = \sup_{\tau} \mathbb{E}_{x,s} [e^{-q\tau} (e^{\bar{X}_{\tau}} - Ke^{X_{\tau}})^+]$$







Cap \bar{X} or X , both?

- π -option:

$$V^*(x, s) = \sup_{\tau} \mathbb{E}_{x,s} [e^{-q\tau} (e^{aX_{\tau} + b\bar{X}_{\tau}} - K)^+],$$

where $a, b > 0$.

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