ISOTHERMIC SURFACES IN ARBITRARY CO-DIMENSION

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Introduction

Since its inception, the theory of integrable systems has been intimately related to that of differential geometry: both the integrable PDE that arise and their infinite-dimensional symmetry groups often have a rich geometrical interpretation. Perhaps the most famous example of this is the relation between constant negative Gauss surfaces in $\mathbb{R}^3$, their Bäcklund transformations and the integrable sin-Gordon equation.

In this paper, I describe another system of this kind also known to the great differential geometers of the 19th century: isothermic surfaces. That isothermic surfaces in $\mathbb{R}^3$ constitute an integrable system was first observed in modern times by Cieśliński–Goldstein–Sym [CGS] whose work was taken up by Burstall, Hertrich-Jeromin, Pedit and Pinkall [BHPP], [HP] who emphasized the conformal invariance of the theory and the relation with the general theory of curved flats [FP_1].

Here, I will indicate how the entire theory of isothermic surfaces goes through in arbitrary codimension with no loss of integrable structure. Along the way, we shall find an extraordinarily beautiful and efficient method, originally due to Vahlen [V], of doing conformal geometry via 2×2 matrices with values in a Clifford algebra.

1. Isothermic surfaces

We begin in the setting of classical local differential geometry: let $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^n$ be an immersion of a simply connected subset of the plane.

Say that f is *isothermic* if there is a second non-constant map $f^c : \Omega \to \mathbb{R}^n$, the *dual surface* or *Christoffel transform*, such that

$$\mathrm{d}f \wedge \mathrm{d}f^c = 0. \tag{1.1}$$

Here we view $\mathrm{d}f$ and $\mathrm{d}f^c$ as $\mathbb{R}^n$-valued 1-forms and $\mathrm{d}f \wedge \mathrm{d}f^c$ as a 2-form with values in the Clifford algebra Cl_n of $\mathbb{R}^n$.

Recall that Cl_n is the unital associative algebra generated by $\mathbb{R}^n$ subject only to the relations

$$x \cdot y + y \cdot x = -2(x, y)1,$$

for $x, y \in \mathbb{R}^n$. (Here $(\ ,\ )$ is the Euclidean inner product of $\mathbb{R}^n$.) Thus (1.1) reads

$$\mathrm{d}f_X \cdot \mathrm{d}f^c_Y = \mathrm{d}f_Y \cdot \mathrm{d}f^c_X,$$

for $X, Y \in T_p\Omega$ and $p \in \Omega$.

To clarify the meaning of (1.1), equip Ω with the conformal structure induced by f. Then

1. For z a conformal co-ordinate, (1.1) holds if and only if $\partial f^c/\partial z$ is proportional to $\partial f/\partial \bar{z}$.

   In particular, f^c is weakly conformal.
2. (1.1) implies that $(\mathrm{d}f, \mathrm{d}f^c)^{2,0}$ is a non-zero holomorphic quadratic differential whence f^c is an immersion off a discrete set.

To summarise: f^c is a branched conformal immersion with tangent planes parallel to those of f but with the opposite orientation. Moreover f^c is isothermic also with

$$(f^c)^c = f.$$

The structure equations for f now give:

3. f (and hence f^c) has flat normal bundle so that all shape operators commute.
4. Choose $z = x+iy$ so that $(\mathrm{d}f, \mathrm{d}f^c)^{2,0} = \mathrm{d}z^2$ (such co-ordinates clearly exist away from the discrete zero-set of this differential). Then $\partial/\partial x$ and $\partial/\partial y$ simultaneously diagonalises all the shape operators of f (and f^c).

   Otherwise said, (x, y) are *conformal curvature line* (CCL) co-ordinates for f and f^c.

In the classical literature, a surface was said to be isothermic if it admitted CCL co-ordinates about each point. The following result, due to Christoffel [C] when $n = 3$, shows that this formulation coincides, at least locally, with ours:

Theorem 1.1. *Let $f : \Omega \to \mathbb{R}^n$ be an immersion with CCL co-ordinate $z = x + iy$ and corresponding conformal factor u: $(\mathrm{d}f, \mathrm{d}f) = e^{2u}\mathrm{d}z\mathrm{d}\bar{z}$. Then the $\mathbb{R}^n$-valued 1-form given by*

$$e^{-2u}\left(\frac{\partial f}{\partial z}\mathrm{d}\bar{z} + \frac{\partial f}{\partial \bar{z}}\mathrm{d}z\right) \tag{1.2}$$

is closed so that there is a dual surface f^c with $\partial f^c/\partial z = e^{-2u}\partial f/\partial \bar{z}$.

When $n = 3$, there are many examples of isothermic surfaces available in the classical literature:

- surfaces of revolution: here (1.2) amounts to an ODE for the profile curve of the dual.
- quadrics: in this case there may not be a globally defined dual surface f^c since $(\mathrm{d}f, \mathrm{d}f^c)^{2,0}$ is a non-zero holomorphic differential.
- surfaces of constant mean curvature $H \neq 0$: let n be a field of unit normals to such an f. Then the parallel surface given by

  $$f^c = f + \frac{2}{H}n$$

  is a dual surface of constant mean curvature $1/H$.

- (non-umbilic) minimal surfaces: here the dual surface is just the Gauss map $n : \Omega \to S^2$.

  In this case, the converse provided by Theorem 1.1 has independent interest: let $g : \Omega \subset \mathbb{C} \to \mathbb{C} \cup \{\infty\} = S^2 \subset \mathbb{R}^3$ be a meromorphic function viewed, via stereographic projection, as a map into S^2. Since S^2 is totally umbilic, the standard co-ordinate z on Ω is CCL so that Theorem 1.1 provides a dual *minimal* surface $g^c = f$ via (1.2) which reads:

  $$\frac{\partial f}{\partial z} = \frac{1}{g'}\big(\tfrac{1}{2}(1 - g^2), \tfrac{i}{2}(1 + g^2), g\big)$$

  which we recognize as the Enneper–Weierstrass formula in co-ordinates for which the Hopf differential is $-\mathrm{d}z^2$.

So far, we have been doing Riemannian geometry but it turns out that isothermic surfaces belong in the wider arena of conformal geometry:

Theorem 1.2. *Let f be isothermic and $T : S^n = \mathbb{R}^n \cup \{\infty\} \to S^n$ be a conformal diffeomorphism. Then $T \circ f$ is isothermic also.*

2. Conformal geometry and Clifford algebras

A standard model for the conformal geometry of the n-sphere $S^n = \mathbb{R}^n \cup \{\infty\}$ is to view S^n as the projective light cone in a Minkowski space (for a modern account, see [Bry]).

Thus, let $\mathbb{R}^{n+1,1}$ be the standard $(n+2)$-dimensional vector space equipped with the inner product $(\ ,\ ) = x_1^2 + \cdots + x_{n+1}^2 - x_{n+2}^2$ and set

$$\mathcal{L} = \{v \in \mathbb{R}^{n+1,1} : (v, v) = 0\}.$$

Then the projective light cone $\mathbb{P}\mathcal{L}$ is diffeomorphic to S^n and the linear action of the orthogonal group $\mathrm{O}(n+1,1)$ descends to an action on $\mathbb{P}\mathcal{L}$ by conformal diffeomorphisms giving a finite cover

$$\mathrm{O}(n+1,1) \to \mathrm{Mob}(n) \to 1$$

of the Möbius group $\mathrm{Mob}(n)$ of conformal diffeomorphisms of S^n.

In this context, the inverse of stereographic projection requires a choice of "zero" and "infinity": $v_0, v_\infty \in \mathcal{L}$ with $(v_0, v_\infty) = -\frac{1}{2}$, and then reads

$$x \mapsto [x + v_0 + (x, x)v_\infty] \in \mathbb{P}\mathcal{L} \setminus \langle v_\infty\rangle, \tag{2.1}$$

for $x \in \langle v_0, v_\infty\rangle^\perp \cong \mathbb{R}^n$.

It will be convenient for us to work with Vahlen's reformulation of conformal geometry [V] (see also [A] and the detailed account in [P]) which amounts to replacing $\mathrm{O}(n+1,1)$ with $\mathrm{Pin}_{n+1,1}$.

For this, let $Cl_{n+1,1}$ be the Clifford algebra of $\mathbb{R}^{n+1,1}$ with group of units $Cl^{\times}_{n+1,1}$. We distinguish two involutions of $Cl_{n+1,1}$:

(a) The *grading involution*: $a \mapsto \tilde{a}$, the unique automorphism with $\tilde{v} = -v$ for $v \in \mathbb{R}^{n+1,1}$.

(b) The *transpose*: $a \mapsto a^t$, the unique anti-automorphism with $v^t = v$ for $v \in \mathbb{R}^{n+1,1}$.

The *Clifford group* $\Gamma_{n+1,1}$ is defined by

$$\Gamma_{n+1,1} = \{a \in Cl^{\times}_{n+1,1} : av\tilde{a}^{-1} \in \mathbb{R}^{n+1,1} \text{ for all } v \in \mathbb{R}^{n+1,1}\}.$$

Clearly, $\Gamma_{n+1,1}$ has a representation on $\mathbb{R}^{n+1,1}$ which turns out to be orthogonal and we have an exact sequence

$$1 \to \mathbb{R}^{\times} \to \Gamma_{n+1,1} \to \mathrm{O}(n+1,1) \to 1.$$

Vahlen's theory rests on the following basic observation: $Cl_{n+1,1}$ is isomorphic to the algebra $M_2(Cl_n)$ of 2×2 matrices with entries in Cl_n. Indeed, define $\mathbb{R}^{n+1,1} \subset M_2(Cl_n)$ by

$$\mathbb{R}^{n+1,1} = \{\begin{pmatrix} v & \lambda \\ \mu & -v \end{pmatrix} : v \in \mathbb{R}^{n+1,1},\ \lambda, \mu \in \mathbb{R}\}$$

and note that

$$\begin{pmatrix} v & \lambda \\ \mu & -v \end{pmatrix}^2 = \begin{pmatrix} v^2 + \lambda\mu & 0 \\ 0 & v^2 + \lambda\mu \end{pmatrix} \in \mathbb{R}$$

so that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are light-like with inner product $-\frac{1}{2}$.

We now have

Theorem 2.1 [V]. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{n+1,1}$ *if and only if* $a, b, c, d \in \Gamma_n \cup \{0\}$ *and*

(i) $ad^t - bc^t \in \mathbb{R}^{\times}$*;*

(ii) $ac^t, bd^t, a^t b, c^t d \in \mathbb{R}^n$

In this setting, the map (2.1) $\mathbb{R}^n \to \mathbb{P}\mathcal{L}$ reads

$$x \mapsto \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix}$$

and one finds that the action of $\Gamma_{n+1,1}$ by conformal diffeomorphisms of $\mathbb{R}^n \cup \{\infty\}$ is given by the following attractive formula:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = (ax+b)(cx+d)^{-1} \in \mathbb{R}^n \cup \{\infty\}.$$

To make contact with our theory of isothermic surfaces, we first realise the Lie algebra $\mathfrak{o}_{n+1,1}$ of $\mathrm{O}(n+1,1)$ as

$$\mathfrak{o}_{n+1,1} = \{\begin{pmatrix} \xi & \eta \\ \zeta & -\xi^t \end{pmatrix} : \eta, \zeta \in \mathbb{R}^n,\ \xi \in [\mathbb{R}^n, \mathbb{R}^n] \oplus \mathbb{R}\}. \tag{2.2}$$

We now see that our main equation (1.1) is a Maurer–Cartan (zero-curvature) equation: indeed, let $\alpha = \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix}$ be an $\mathfrak{o}_{n+1,1}$-valued 1-form on Ω (thus the α_i are $\mathbb{R}^n$-valued). The Maurer–Cartan equation $\mathrm{d}\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$ reads

$$\mathrm{d}\alpha_i = 0, \qquad [\alpha_1 \wedge \alpha_2] = 0$$

so that $\alpha_1 = \mathrm{d}f$ and $\alpha_2 = \mathrm{d}f^c$ for f, f^c a dual pair of isothermic surfaces!

These Maurer–Cartan equations are a special case of a general class of integrable systems to which we now turn.

3. Curved flats: an integrable system

According to Ferus–Pedit [FP$_1$], a map $\phi : M \to N = G/K$ of a manifold into a pseudo-Riemannian symmetric space is a *curved flat* if the curvature tensor R^N of N vanishes on TM:

$$R^N|_{\bigwedge^2 \mathrm{d}\phi(TM)} = 0.$$

Definition. A *framing* of ϕ is a map $F : M \to G$ such that $\pi \circ F = \phi$ where $\pi : G \to G/K$ is the coset projection.

The curved flat condition amounts to a condition on the Maurer–Cartan form $\alpha = F^{-1}\mathrm{d}F$ of a framing F. Indeed, use the symmetric decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ to write

$$\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{p}}$$

and then ϕ is a curved flat if and only if $[\alpha_{\mathfrak{p}} \wedge \alpha_{\mathfrak{p}}] = 0$ in which case the Maurer–Cartan equations decouple to give:

$$\begin{aligned} \mathrm{d}\alpha_{\mathfrak{k}} + \tfrac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] &= 0 \\ \mathrm{d}\alpha_{\mathfrak{p}} + [\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{p}}] &= 0 \\ [\alpha_{\mathfrak{p}} \wedge \alpha_{\mathfrak{p}}] &= 0. \end{aligned} \tag{3.1}$$

The main observation is that (3.1) holds if and only if each connection in the pencil $\mathrm{d} + \alpha_{\mathfrak{k}} + \lambda\alpha_{\mathfrak{p}}$, $\lambda \in \mathbb{R}$ is *flat*. Thus the curved flat system amounts to a zero-curvature equation with spectral parameter and the whole machinery of integrable systems theory (loop groups, dressing actions, Hamiltonian formulation and algebro-geometric solutions) is available for its study.

In particular, when M is simply connected, we can integrate to get $F_\lambda : M \to G$ with

$$F_\lambda^{-1}\mathrm{d}F_\lambda = \alpha_{\mathfrak{k}} + \lambda\alpha_{\mathfrak{p}}$$

and thus a curved flat $\phi_\lambda = \pi \circ F_\lambda : M \to N$. So we see that curved flats come in 1-parameter families.

With a suitable choice of gauge transformation, the system (3.1) can be put into several canonical forms. For example, under the assumption that all images of $\alpha_{\mathfrak{p}}$

are K-conjugate maximal abelian subspaces of $\mathfrak{p}$, the system amounts to the *n-dimensional system associated to the rank n symmetric space G/K* introduced by Terng to study isometric immersions between space forms [T] (see also [FP_2]) and flat Egoroff metrics [TU].

Again, with no extra assumptions, $F_0 : M \to K$ can be used to gauge $\alpha_{\mathfrak{k}}$ away giving a new frame with $\tilde{\alpha} = \tilde{a}_{\mathfrak{p}}$ and structure equations

$$\mathrm{d}\tilde{\alpha}_{\mathfrak{p}} = 0 = [\tilde{\alpha}_{\mathfrak{p}} \wedge \tilde{\alpha}_{\mathfrak{p}}].$$

We call this the *flat frame* of the curved flat ϕ.

In the case at hand, we take $N = \tilde{G}_{1,1}(\mathbb{R}^{n+1,1})$, the Grassmannian of oriented 2-planes in $\mathbb{R}^{n+1,1}$ on which the metric has signature $(1,1)$—this is an $\mathrm{O}(n+1,1)$-symmetric space with invariant metric of signature (n,n). An element of N is uniquely determined by the ordered pair of light-lines it contains so that we have a diffeomorphism

$$N \cong S^n \times S^n \setminus \Delta.$$

Under the identification (2.2) the corresponding symmetric decomposition of $\mathfrak{o}_{n+1,1}$ is the usual decomposition into diagonal and off-diagonal 2×2 matrices.

Pulling all this together, we see that a flat frame F of a curved flat $\phi : \Omega \to N$ amounts to a pair of dual isothermic surfaces f, f^c via

$$F^{-1}\mathrm{d}F = \begin{pmatrix} 0 & \mathrm{d}f \\ \mathrm{d}f^c & 0 \end{pmatrix}.$$

Moreover, such a pair gives rise to a 1-parameter family of maps $\phi_\lambda = (f_\lambda, \hat{f}_\lambda) : \Omega \to S^n \times S^n \setminus \Delta$ framed by F_λ satisfying

$$F_\lambda^{-1}\mathrm{d}F_\lambda = \begin{pmatrix} 0 & \lambda\mathrm{d}f \\ \lambda\mathrm{d}f^c & 0 \end{pmatrix}.$$

Finally, we can recover our original dual pair from the family $(f_\lambda, \hat{f}_\lambda)$ by observing that $f_0 \equiv 0$, $\hat{f}_0 \equiv \infty$ and

$$\begin{aligned} f &= \left.\frac{\partial f_\lambda}{\partial \lambda}\right|_{\lambda=0} : \Omega \to T_0 S^n \cong \mathbb{R}^n \\ f^c &= \left.\frac{\partial \hat{f}_\lambda}{\partial \lambda}\right|_{\lambda=0} : \Omega \to T_\infty S^n \cong \mathbb{R}^n \end{aligned}$$

Let us now turn to the geometry of the surfaces $f_\lambda, \hat{f}_\lambda : \Omega \to S^n$.

4. The Bianchi–Darboux transform

Consider a curved flat $\phi = (f, \hat{f}) : \Omega \to S^n \times S^n \setminus \Delta$. Use stereographic projection to view $f, \hat{f} : \Omega \to \mathbb{R}^n$ and set $g = \hat{f} - f$. One shows that ϕ has a framing given by

$$F = \begin{pmatrix} \hat{f}g^{-1} & f \\ g^{-1} & 1 \end{pmatrix}$$

and then the structure equations (3.1) give

$$\mathrm{d}f \wedge g^{-1}\mathrm{d}\hat{f}g^{-1} = 0 = \mathrm{d}(g^{-1}\mathrm{d}\hat{f}g^{-1})$$

so that f is isothermic with dual surface f^c satisfying

$$\mathrm{d}f^c = g^{-1}\mathrm{d}\hat{f}g^{-1}. \tag{4.1}$$

Similarly $\hat{f}$ is isothermic and

$$\mathrm{d}(\hat{f})^c = g^{-1}\mathrm{d}fg^{-1}$$

from which one deduces that

$$\hat{f}^c = f^c + g^{-1} \tag{4.2}$$

Moreover, one can easily check that

$$(\mathrm{d}f, \mathrm{d}f^c) = (\mathrm{d}\hat{f}, \mathrm{d}\hat{f}^c)$$

from which it follows that f and $\hat{f}$ have the same CCL co-ordinates and we have proved:

Theorem 4.1. *f and $\hat{f}$ are both isothermic surfaces with the same CCL co-ordinates.*

Remark. For $n = 3$ this result is due to Burstall–Hertrich-Jeromin–Pedit–Pinkall [BHPP] and for $n = 4$ it is due to Hertrich-Jeromin–Pedit [HP].

There is a beautiful geometric relationship between f and $\hat{f}$: we know that

$$\operatorname{Im}\mathrm{d}f = \operatorname{Im}\mathrm{d}f^c = \operatorname{Im} g^{-1}\mathrm{d}\hat{f}g^{-1}$$

from which it follows that the tangent spaces of f are obtained by reflecting those of $\hat{f}$ in the affine hyperplane orthogonal to g through $f + g/2$. Thus, both f and $\hat{f}$ are tangent to a 2-sphere centered in that hyperplane and we have:

Theorem 4.2. *f and $\hat{f}$ are the enveloping surfaces of a 2-parameter family of 2-spheres and have the same conformal structure and curvature lines.*

2-parameter families of spheres (*sphere congruences*) with the property that the enveloping surfaces have the same conformal structure and curvature lines are known in the classical literature as *conformal Ribeaucour sphere congruences.* It turns out that, modulo some degenerate situations, all such sphere congruences arise this way and we have the following result proved in [BHPP] when $n = 3$:

Theorem 4.3. *$\phi = (f, \hat{f}) : \Omega \to N$ is a curved flat if and only if f, $\hat{f}$ are the enveloping surfaces of a non-degenerate conformal Ribeaucour 2-sphere congruence.*

When $n = 3$, it is a theorem of Darboux [D] that any isothermic surface f arises as an enveloping surface in this way. Bianchi [B_1][B_2] took up these ideas, christened the second enveloping surface $\hat{f}$ the *Darboux transform* of f and showed how Darboux transforms are obtained by solving a linear differential system indexed by an auxiliary parameter $r \in \mathbb{R}^\times$.

In arbitrary co-dimension, these results still hold as can be seen by using ideas of Hertrich-Jeromin–Pedit for the $n = 4$ case: given isothermic f with dual f^c and $r \in \mathbb{R}^\times$, we note that rf^c is also a dual surface of f and we can find a Darboux transform by rearranging (4.1) into a Riccati equation for g:

$$\mathrm{d}g = gr\mathrm{d}f^c g - \mathrm{d}f. \tag{4.3}$$

This equation is easily shown to be integrable and, for any solution, we have that $f + g$ is a Darboux transform of f. Moreover, if g solves (4.3) then

$$\mathrm{d}(rg)^{-1} = (rg)^{-1}\mathrm{d}f(rg)^{-1} - \mathrm{d}f^c$$

so that $f^c + (rg)^{-1}$ is a Darboux transform of f^c and, comparing with (4.2), a Christoffel transform of $f + g$. Thus we have proved a result of Bianchi [B_2] ($n = 3$) and Hertrich-Jeromin–Pedit [HP] ($n = 4$):

Theorem 4.4. *Darboux and Christoffel transforms commute.*

There is a second formulation of the equation (4.3) which coincides with Bianchi's: our Riccati equation is the projectivisation of the integrable linear system

$$\mathrm{d}\omega + \alpha\omega = 0$$

where $\omega : \Omega \to \mathbb{R}^{n+1,1}$ and, as usual, α is the flat one form $\begin{pmatrix} 0 & \mathrm{d}f \\ r\mathrm{d}f^c & 0 \end{pmatrix}$. Solving for ω with initial condition in $\mathcal{L}$ gives $\omega : \Omega \to \mathcal{L}$ and $g = [\omega]$. This linear system is gauge equivalent to that of Bianchi.

At this point, everything becomes very reminiscent of the general theory of Bäcklund transforms developed by Terng–Uhlenbeck [TU]: in fact, the framing F_λ may be viewed as a map into a certain loop group and then the Darboux transform can be viewed as dressing this map by a "simple factor" in the sense of [TU] with poles at $\pm\sqrt{r}$. Once this is established, results like the Bianchi permutability theorem [B_1] for iterated Darboux transforms follow simply from entirely general principles. The details of this story will appear elsewhere.

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