TWISTOR SPACES FOR RIEMANNIAN SYMMETRIC SPACES

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Abstract. We determine the structure of the zero-set of the Nijenhuis tensor of the natural almost complex structure J_1 on the total space of the bundle $J(G/K, g)$ of Hermitian structures on the tangent spaces of any even-dimensional Riemannian symmetric space G/K of compact or non-compact type.

1. Introduction

By a twistor space for a Riemannian manifold (M, g) we mean an (almost) complex manifold $\pi: Z \to M$, fibred over M with complex fibres, together with some additional properties; see section 2 for the details. A basic example is the space $J(M, g)$ consisting of all the complex structures on the tangent spaces of M which are compatible with the metric. $J(M, g)$ has fibre the Hermitian symmetric space $O(2n)/U(n)$ and the Riemannian connection allows this vertical complex structure on each fibre to be combined with the horizontal lift of the given complex structure on each tangent space to M to give $J(M, g)$ a natural almost complex structure. This almost complex structure is integrable only for M conformally flat [3], and for compact symmetric spaces this means only the spheres and real projective spaces. For more general twistor spaces Z we may have integrability under weaker assumptions, so it is desirable to find such spaces.

Any twistor space with an integrable complex structure will have an image in $J(M, g)$ which is a complex submanifold and so sits in the zero-set of the Nijenhuis tensor of the natural almost complex structure J_1 on $J(M, g)$. In [1], when $M = G/K$ is an inner Riemannian symmetric space and g the invariant metric, this zero-set was shown to consist of a finite number of connected components each of which was a flag space of G fibring over G/K in a 'minimal' way (thus, the components were generalized flag manifolds for G compact and flag domains for G non-compact). In particular, each of these flag spaces is a twistor space. We used the property that G/K was inner (or, equivalently, that $\text{rank}(G) = \text{rank}(K)$) in our analysis.

It is the purpose of this note to determine the zero-set of the Nijenhuis tensor for an arbitrary even-dimensional Riemannian symmetric space. Our analysis uses similar ideas to those of [1] but takes into account the more complicated relationship between the root structure of G with respect to a maximal torus maximally embedded in K (a so-called *fundamental* torus) and the symmetric space structure when the space is not inner.

The main difference from the inner case results from the fact that we cannot show that Z respects the de Rham decomposition of G/K into irreducible factors. Indeed, it does

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not, and the components of Z also turn out, in general, not to be homogeneous spaces of G. These twistor spaces appear to be new.

The components of the zero-set are expressible in terms of the τ-maximal parabolic subalgebras which were introduced in [1] where τ is the involution determining the symmetric space.

The paper is organized as follows. In section 2 we summarize the basic properties of twistor spaces for Riemannian manifolds. In section 3 we develop the properties of τ-maximal parabolic subalgebras needed in the sequel. In section 4 we show that each point in the zero-set corresponds with a τ-maximal parabolic together with a certain subspace and in section 5 we show that τ-maximal parabolics determine open subsets of the zero-set which are generalized twistor spaces in the sense of [5]. In section 6 we apply our analysis of the zero-set to some examples. Example 1 looks at the Calabi-Eckmann Hermitian structures [2] on the product of two odd-dimensional spheres and shows that the images of these structures exhaust the zero-set. In example 2 we show that the Hermitian structures found by Samelson [6] also exhaust the zero-set in the case of an even-dimensional Lie group. In example 3 we apply our theory to a less familiar example and describe the zero-set for the twistor space of the symmetric space $SU(2n)/Sp(n)$, n odd. By way of contrast with example 1, example 4 considers whether a product of odd-dimensional real Grassmannians might carry the analogue of a Calabi-Eckmann Hermitian structure. We show in theorem 6.1 that there can be no such Hermitian structures. Finally, in example 5 we apply our theory to obtain compact complex manifolds with the same fundamental group as certain compact locally symmetric spaces.

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2. Generalized twistor spaces

In this section we recall some of the basic facts about twistor spaces. See [5], for more details and some examples.

Let V denote a real vector space of even dimension $2n$ with an inner product $(\cdot,\cdot)$. A Hermitian structure on V is an endomorphism J of V with $J^2 = -1$ and which is compatible with the inner product in the sense that

$$(JX, JY) = (X, Y), \qquad \forall X,\, Y \in V.$$

We denote by $J(V)$ the set of all Hermitian structures on V.

The orthogonal group $O(V)$ acts transitively on $J(V)$ by conjugation:

$$g \cdot J = gJg^{-1}, \qquad g \in O(V).$$

The stabilizer at J of this action consists of elements of $O(V)$ which are complex linear with respect to J and so is a copy of the unitary group. We denote it by $U(V,J)$. Thus the set of all Hermitian structures on V coincides with the homogeneous space $O(V)/U(V,J)$. This is a Hermitian symmetric space, so has an invariant complex structure which we describe next.

Denote by $\mathfrak{o}(V)$, $\mathfrak{u}(V,J)$ the Lie algebras of $O(V)$ and $U(V,J)$, respectively. The tangent space at J is isomorphic to the quotient $\mathfrak{o}(V)/\mathfrak{u}(V,J)$ which in turn can be identified with the subspace of elements of $\mathfrak{o}(V)$ which anticommute with J. Multiplication of such elements on the left by J preserves this subspace and so induces an invariant almost complex structure on $O(V)/U(V,J)$ which is integrable by standard results.

Let (M,g) be any $2n$-dimensional Riemannian manifold. We denote by $J(M,g)$ the bundle of all Hermitian structures on the tangent spaces of M. This is a bundle associated to the orthonormal frame bundle $O(M,g)$ of the Riemannian metric g with fibre $J(\mathbb{R}^{2n})$. Since the fibre is homogeneous such an associated bundle can also be viewed as the quotient by the stabilizer: $O(M,g)/U(\mathbb{R}^{2n},J)$ where we pick some standard Hermitian structure J on $\mathbb{R}^{2n}$ as a base-point. The horizontal distribution on the frame bundle coming from the Levi-Civita connection will thus descend to $J(M,g)$ to give a horizontal distribution $\mathcal{H}$. We denote by $\mathcal{V}$ the vertical distribution. The latter has a Hermitian structure coming from the invariant Hermitian structure on each fibre. The horizontal distribution $\mathcal{H}$ also has a Hermitian structure since each horizontal space $\mathcal{H}_j$ is isomorphic to T_xM if j is a Hermitian structure on T_xM. Thus j can be lifted by this isomorphism to $\mathcal{H}_j$. We denote by J_1 the almost complex structure on $J(M,g)$ which we get by taking the direct sum of the natural horizontal and vertical Hermitian structures just defined.

By a twistor space for a Riemannian manifold (M,g) we mean an (almost) complex manifold $\pi\colon Z \to M$, fibred over M with complex fibres together with some additional properties which we shall come to in a moment. If Z is a twistor space then, for $x \in M$, each $z \in \pi^{-1}(x)$ defines a complex vector space structure $j(z)$ on T_xM by identifying the latter with $T_zZ/\mathcal{V}_z$ where $\mathcal{V}$ is the vertical tangent bundle. Thus we get a map $j\colon Z \to J(M,g)$ (in general the $j(z)$ are not automatically compatible with the metric g, but this is one of the extra assumptions we make).

Conversely, suppose we have a manifold Z which fibres over M with complex fibres and that we have a fibre-preserving map $j\colon Z \to J(M,g)$ which is holomorphic on each fibre. If we denote by $\mathcal{V}$ the vertical tangent bundle, as above, then the complex structure on each fibre transfers to $\mathcal{V}$. We suppose we have a complement $\mathcal{H}$ for $\mathcal{V}$, then, just as for $J(M,g)$, each point $z \in Z$ determines a complex structure on $\mathcal{H}_z$ as the horizontal lift of $j(z)$. The direct sum of these two gives Z an almost complex structure which we also call J_1. If j preserves the horizontal distributions on Z and $J(M,g)$ then it will be holomorphic with respect to J_1 on each of these spaces by construction.

In [5] we called a manifold Z with a horizontal distribution $\mathcal{H}$ and such a horizontal-preserving map $j\colon Z \to J(M,g)$ which is holomorphic on the fibres a *generalized twistor space*. Clearly $J(M,g)$ is itself a twistor space with j the identity map. As remarked in the introduction, J_1 on $J(M,g)$ is rarely integrable, so we look for generalized twistor spaces as possible candidates for developing Riemannian analogues of Penrose's Minkowskian twistor theory.

3. τ-maximal parabolic subalgebras

The results in this section extend those of the appendix to chapter 4 of [1]. We assume that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the symmetric decomposition of a compact Lie algebra with respect to an involution τ and denote by suffices the intersections of subspaces of $\mathfrak{g}$ with $\mathfrak{k}$ or $\mathfrak{p}$.

If $\mathfrak{q}$ is a τ-stable parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$ we denote its nil-radical by $\mathfrak{n}$ and set $\mathfrak{l} = \mathfrak{q} \cap \mathfrak{g}$ so that $\mathfrak{q} = \mathfrak{l}^{\mathbb{C}} + \mathfrak{n}$. $\mathfrak{n}$ and $\mathfrak{l}$ are also τ-stable so we have decompositions

$$\mathfrak{q} = \mathfrak{q}_k + \mathfrak{q}_p, \qquad \mathfrak{n} = \mathfrak{n}_k + \mathfrak{n}_p, \qquad \mathfrak{l} = \mathfrak{l}_k + \mathfrak{l}_p.$$

Denote the centre of $\mathfrak{l}$ by $\mathfrak{z}(\mathfrak{l})$ then we have the following definition taken from [1].

Definition 3.1. A parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}^{\mathbb{C}}$ is said to be τ-maximal if it is τ-stable and :

(i) $\mathfrak{l}_p \subset \mathfrak{z}(\mathfrak{l})$;
(ii) $\mathfrak{n} = \mathfrak{n}_p + [\mathfrak{n}_p, \mathfrak{q}_p]$.

In [1] we showed how to construct τ-maximal parabolics starting from a τ-stable Borel subalgebra. Indeed, in theorem 4.29 of [1], it was shown that if $\mathfrak{b}$ is such a Borel subalgebra and $\mathfrak{b}'$ is its nilradical then $\mathfrak{b}'_p + [\mathfrak{b}'_p, \mathfrak{b}_p]$ is the nilradical of a τ-maximal parabolic subalgebra $\mathfrak{q}$. Since $\mathfrak{b}'_p + [\mathfrak{b}'_p, \mathfrak{b}_p] \subset \mathfrak{b}'$, taking polars with respect to the Killing form gives $\mathfrak{b} \subset \mathfrak{q}$. In fact, we also have have the converse:

Lemma 3.2. *If $\mathfrak{q}$ is τ-maximal and $\mathfrak{b}$ is any τ-stable Borel subalgebra contained in $\mathfrak{q}$ then $\mathfrak{q}$ has nilradical $\mathfrak{b}'_p + [\mathfrak{b}'_p, \mathfrak{b}_p]$.*

Proof. $\mathfrak{t} = \mathfrak{b} \cap \mathfrak{g}$ is a τ-stable maximal toral subalgebra of $\mathfrak{g}$ which is contained in $\mathfrak{l} = \mathfrak{q} \cap \mathfrak{g}$. But $\mathfrak{q}$ is τ-maximal so that $\mathfrak{l}_p \subset \mathfrak{z}(\mathfrak{l})$ whence $\mathfrak{l}_p \subset \mathfrak{t}$ and thus $\mathfrak{l}_p = \mathfrak{t}_p$. Now $\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{b}' \oplus \overline{\mathfrak{b}'}$ so that $\mathfrak{p}^{\mathbb{C}} = \mathfrak{l}_p^{\mathbb{C}} \oplus \mathfrak{b}'_p \oplus \overline{\mathfrak{b}'}_p$. On the other hand, $\mathfrak{n} \subset \mathfrak{b}'$ so that $\mathfrak{n}_p \subset \mathfrak{b}'_p$ while $\mathfrak{p}^{\mathbb{C}} = \mathfrak{l}_p^{\mathbb{C}} \oplus \mathfrak{n}_p \oplus \overline{n}_p$. Thus $\mathfrak{n}_p = \mathfrak{b}'_p$. Moreover, $\mathfrak{q}_p = \mathfrak{l}_p^{\mathbb{C}} \oplus \mathfrak{n}_p$ so that we conclude that $\mathfrak{q}_p = \mathfrak{t}_p^{\mathbb{C}} \oplus \mathfrak{b}'_p = \mathfrak{b}_p$. Thus $[\mathfrak{n}_p, \mathfrak{q}_p] = [\mathfrak{b}'_p, \mathfrak{b}_p]$ and the result now follows immediately from the τ-maximality of $\mathfrak{q}$ $\square$

Remark 3.3. In the course of the proof of lemma 3.2 we have shown that for $\mathfrak{q}$ τ-maximal, $\mathfrak{q} \cap \mathfrak{p}$ is the $\mathfrak{p}$-part of a maximal toral subalgebra of $\mathfrak{g}$.

We now have a simple characterization of τ-maximal subalgebras given by the following theorem.

Theorem 3.4. *A τ-stable parabolic subalgebra $\mathfrak{q}$ is τ-maximal if and only if it contains a τ-stable Borel subalgebra $\mathfrak{b}$ with $\mathfrak{n} = \mathfrak{b}'_p + [\mathfrak{b}'_p, \mathfrak{b}_p]$.*

Proof. Let $\mathfrak{q}$ be a parabolic subalgebra with $\mathfrak{n} = \mathfrak{b}'_p + [\mathfrak{b}'_p, \mathfrak{b}_p]$ for some τ-stable Borel subalgebra $\mathfrak{b}$. Then $\mathfrak{b} \cap \mathfrak{g}$ is a τ-stable maximal toral subalgebra of $\mathfrak{g}$ which, by lemma 4.27 of [1], is fundamental. Thus theorem 4.29 of [1] says that $\mathfrak{q}$ is τ-maximal.

Conversely, if $\mathfrak{q}$ is a τ-maximal parabolic subalgebra, then it contains a τ-stable Borel subalgebra $\mathfrak{b}$ and Lemma 3.2 gives the required condition on its nilradical. □

4. Points in the zero-set

Let G/K be an even-dimensional Riemannian symmetric space of compact or non-compact type. The action of G as isometries on G/K lifts into $J(G/K, g)$ and preserves Z. Since G acts transitively on G/K then Z will be $G \cdot Z_k = G \times_K Z_k$ where Z_k denotes the intersection of Z with the fibre of $J(G/K, g)$ over the identity coset. If we identify the tangent space to G/K at the identity coset with $\mathfrak{p}$ where

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

is the usual symmetric space decomposition of the Lie algebra $\mathfrak{g}$ of G, then the fibre of $J(G/K, g)$ over the identity coset can be identified with $J(\mathfrak{p})$, the set of all skew-symmetric transformations j of $\mathfrak{p}$ with $j^2 = -I$. Such a transformation j has eigenvalues $\pm i$ and is determined by its $+i$-eigenspace which we denote by $\mathfrak{p}^+$. If $\mathfrak{g}$ is given an invariant bilinear form which induces the metric on G/K then $\mathfrak{p}^+$ is a maximal isotropic subspace of the complexification $\mathfrak{p}^{\mathbb{C}}$ of $\mathfrak{p}$. We shall use j and $\mathfrak{p}^+$ interchangeably without further comment. In [1] it is shown that the condition for j to be in the zero-set of the Nijenhuis tensor is

$$[[\mathfrak{p}^+, \mathfrak{p}^+], \mathfrak{p}^+] \subset \mathfrak{p}^+,$$

or equivalently that $[\mathfrak{p}^+, \mathfrak{p}^+]$ is an isotropic subspace of $\mathfrak{k}^{\mathbb{C}}$.

For connected G the components of Z will have the form $G \cdot Z_1$ where each Z_1 is a component of Z_k. Our goal is to describe the structure of the components of Z_k. Moreover, in view of the celebrated duality between symmetric spaces of compact and non-compact type, it suffices to take G compact. This is possible since, when G/K is of non-compact type, the space $\mathfrak{p}^{\mathbb{C}}$, the isotropic subspaces $\mathfrak{p}^+$ and their K-orbits coincide with those of the compact dual U/K and thus Z_k is the same for both spaces.

So let $\mathfrak{g}$ be compact and let $\mathfrak{p}^+$ be in Z_k. Set

$$\mathfrak{h} = \{\xi \in \mathfrak{g} : [\xi, \mathfrak{p}^+] \subset \mathfrak{p}^+ + [\mathfrak{p}^+, \mathfrak{p}^+]\}$$

then $\mathfrak{h}$ is τ-stable so $\mathfrak{h} = \mathfrak{h}_k + \mathfrak{h}_p$ where

$$\mathfrak{h}_k = \mathfrak{h} \cap \mathfrak{k} = \{\xi \in \mathfrak{k} : [\xi, \mathfrak{p}^+] \subset \mathfrak{p}^+\}$$

and

$$\mathfrak{h}_p = \mathfrak{h} \cap \mathfrak{p} = \{\xi \in \mathfrak{p} : [\xi, \mathfrak{p}^+] \subset [\mathfrak{p}^+, \mathfrak{p}^+]\}.$$

$\mathfrak{h}_k$ is then the Lie algebra of $H_k = \{k \in K : \mathrm{Ad}_G\, k\, \mathfrak{p}^+ \subset \mathfrak{p}^+\}$.

Lemma 4.1. *$\mathfrak{h}_p$ is an abelian subalgebra of $\mathfrak{g}$ and $[\mathfrak{h}_p, \mathfrak{h}_k] = 0$.*

Proof. Let $\xi \in \mathfrak{h}_k$, $\eta \in \mathfrak{h}_p$ and $\zeta \in \mathfrak{p}^+$ then $[\eta, \zeta] = \sum_i [\lambda_i, \mu_i]$ for some λ_i and μ_i in $\mathfrak{p}^+$. Thus

$$([\xi, \eta], \zeta) = (\xi, [\eta, \zeta]) = \sum_i (\xi, [\lambda_i, \mu_i]) = \sum_i ([\xi, \lambda_i], \mu_i) = 0$$

so $[\mathfrak{h}_p, \mathfrak{h}_k] = 0$. Obviously $[\mathfrak{h}_p, \mathfrak{h}_p] \subset \mathfrak{h}_k$ and if $\xi, \eta \in \mathfrak{h}_p$, $\zeta \in \mathfrak{h}_k$ then

$$([\xi, \eta], \zeta) = (\xi, [\eta, \zeta]) = 0$$

and hence $[\mathfrak{h}_p, \mathfrak{h}_p] = 0$. □

Let $\mathfrak{m}$ denote the orthogonal complement of $\mathfrak{h}_k$ in $\mathfrak{k}$ and $\mathfrak{m}^{\mathbb{C}}$ its complexification.

Lemma 4.2. *We have $\mathfrak{m}^{\mathbb{C}} = [\mathfrak{p}^+, \mathfrak{p}^+] + \overline{[\mathfrak{p}^+, \mathfrak{p}^+]}$ and $\mathfrak{h}_k$ is the centralizer of a torus in $\mathfrak{k}$.*

Proof. Lemma 5.1 and Proposition 5.2 of [1] still apply since these are proven without the assumption that G/K is inner. □

Take a maximal toral subalgebra $\mathfrak{t}_k$ of $\mathfrak{k}$ in $\mathfrak{h}_k$. Such a toral subalgebra exists by Lemma 4.2. Then $\mathfrak{t} = \mathfrak{t}_k + \mathfrak{t}_p$ is a fundamental toral subalgebra of $\mathfrak{g}$ where $\mathfrak{t}_p$ is the centralizer of $\mathfrak{t}_k$ in $\mathfrak{p}$. It is clear that $\mathfrak{t}_p^{\mathbb{C}}$ is the zero weight space (relative to $\mathfrak{t}_k$) for $\mathfrak{p}^{\mathbb{C}}$ as a representation of $\mathfrak{k}^{\mathbb{C}}$ and so as of $\mathfrak{h}_k^{\mathbb{C}}$. $\mathfrak{p}^{\mathbb{C}}$ splits into $\mathfrak{p}^+ + \overline{\mathfrak{p}^+}$ as a representation of $\mathfrak{h}_k^{\mathbb{C}}$ and so $\mathfrak{t}_p^{\mathbb{C}}$ is the sum of the zero weight space $\mathfrak{t}^+$ on $\mathfrak{p}^+$ and its complex conjugate. Hence $\mathfrak{t}^+$ is a maximal isotropic subspace of $\mathfrak{t}_p^{\mathbb{C}}$.

Let Δ be the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ and let I denote the set of roots which vanish on $\mathfrak{t}_p$, II those which do not. If $\alpha \in I$ then the root space $\mathfrak{g}_\alpha$ lies in $\mathfrak{k}^{\mathbb{C}}$ or $\mathfrak{p}^{\mathbb{C}}$. Let I_k and I_p denote the corresponding sets of roots, so $\Delta = I_k \cup I_p \cup II$ is a disjoint union.

Lemma 4.3. *Each root of type II is non-zero on $\mathfrak{t}^+$.*

Proof. The roots of a compact torus take imaginary values, so a root α of type II will be imaginary on $\mathfrak{t}_p$ and hence if it vanishes on $\mathfrak{t}^+$ it will vanish on the complex conjugate and so on $\mathfrak{t}_p$. This is impossible. □

For each root α choose a non-zero vector e_α in $\mathfrak{g}_\alpha$. If a root α is in II then $\mathfrak{g}_\alpha$ cannot lie entirely in $\mathfrak{k}^{\mathbb{C}}$ nor in $\mathfrak{p}^{\mathbb{C}}$. Thus there are non-zero elements $x_\alpha \in \mathfrak{k}^{\mathbb{C}}$ and $y_\alpha \in \mathfrak{p}^{\mathbb{C}}$ with $e_\alpha = x_\alpha + y_\alpha$.

Lemma 4.4. *Let $\mathfrak{p}^+$ be in Z_k, choose $\mathfrak{t}_k$, $\mathfrak{t}_p$ as above and let Δ be the roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$. Set*

$$\Phi = \{\alpha \in \Delta : \mathfrak{g}_\alpha \subset \mathfrak{p}^+ + [\mathfrak{p}^+, \mathfrak{p}^+]\}$$

then Φ is closed under root addition and there exists a subset $\mathfrak{t}^+$ of $\mathfrak{t}_p^{\mathbb{C}}$ such that

$$\mathfrak{p}^+ = \mathfrak{t}^+ + \sum_{\alpha \in \Phi \cap II} \mathbb{C} y_\alpha + \sum_{\alpha \in \Phi \cap I_p} \mathfrak{g}_\alpha.$$

Proof. We examine $\mathfrak{p}^{\mathbb{C}}$ in terms of its weight spaces as a representation of $\mathfrak{t}_k$. The zero weight space is $\mathfrak{t}_p^{\mathbb{C}}$ by definition, so the zero weight space on $\mathfrak{p}^+$ will be a maximal isotropic subspace $\mathfrak{t}^+$ of $\mathfrak{t}_p^{\mathbb{C}}$. To finish the proof we need to show that the $\mathfrak{t}_k$-invariant complement of $\mathfrak{t}^+$ in $\mathfrak{p}^+$ consists of 1-dimensional weight spaces. This depends on knowing how the roots of $\mathfrak{g}^{\mathbb{C}}$ may coincide when they are restricted to $\mathfrak{t}_k$. Obviously the restrictions of no two type I roots can coincide. Equally obviously if α is of type II then α and $\tau\alpha$ coincide if τ is the involution, but this is the only way two type II roots can coincide when they are restricted. This follows since x_α will be a root vector of $\mathfrak{k}^{\mathbb{C}}$ for the restriction of α of type II. If two type II roots α, β have coincident restrictions, then both x_α and x_β would be in the same $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}_k^{\mathbb{C}})$-root space. Thus x_α and x_β are proportional, and by rescaling we can assume they are equal. Then for any ξ in $\mathfrak{t}_p$ we have

$$\alpha(\xi)^2 x_\alpha = [\xi, [\xi, x_\alpha]] = [\xi, [\xi, x_\beta]] = \beta(\xi)^2 x_\beta$$

so $\alpha = \pm\beta$ on $\mathfrak{t}_p$. Hence $\alpha = \beta$ or $\alpha = \tau\beta$. Thus the only remaining coincidence that can happen is that the restriction of a type I root β coincides with the restrictions of a pair of type II roots α and $\tau\alpha$.

The weight spaces for non-zero weights will be one-dimensional unless there are coincidences when roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ are restricted to $\mathfrak{t}_k$. By the above, weight vectors will be either type I_p root vectors, or the y_α of type II roots or a combination of these when there happens to be a coincidence for restricted roots. So suppose that $\beta \in I_p$ with $e_\beta \in \mathfrak{g}_\beta$ and $\alpha \in II$ and also that $\alpha = \beta$ on $\mathfrak{t}_k$ with $e_\beta + y_\alpha \in \mathfrak{p}^+$. By Lemma 4.3 we can pick an element ξ of $\mathfrak{t}^+$ with $\alpha(\xi) \neq 0$ then $[\xi, [\xi, e_\beta + y_\alpha]] \in \mathfrak{p}^+$. But this is $\alpha(\xi)^2 y_\alpha$ and so $y_\alpha \in \mathfrak{p}^+$. Thus $e_\beta \in \mathfrak{p}^+$ also.

Thus we have shown that $\mathfrak{p}^+$ is composed of a maximal isotropic subspace $\mathfrak{t}^+$ of $\mathfrak{t}_p^{\mathbb{C}}$ together with a sum of type I_p root spaces and a sum of spaces of the form $\mathbb{C}y_\alpha$ for type II roots α. We may now define

$$\Phi = \{\alpha \in \Delta : \mathfrak{g}_\alpha \subset \mathfrak{p}^+ + [\mathfrak{p}^+, \mathfrak{p}^+]\}$$

and we have

$$\mathfrak{p}^+ = \mathfrak{t}^+ + \sum_{\alpha \in \Phi \cap II} \mathbb{C}y_\alpha + \sum_{\alpha \in \Phi \cap I_p} \mathfrak{g}_\alpha$$

as required. We note that since $\mathfrak{p}^+ + [\mathfrak{p}^+, \mathfrak{p}^+]$ is an algebra, Φ will be closed under root addition. □

With this we have the main result relating points in the zero-set of the Nijenhuis tensor to parabolic subalgebras:

Theorem 4.5. *If $\mathfrak{p}^+ \in Z_k$ then there exists a τ-maximal parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}^{\mathbb{C}}$ with $\mathfrak{q} \cap \mathfrak{g} = \mathfrak{h}$ and such that $\mathfrak{p}^+ = \mathfrak{n} \cap \mathfrak{p}^{\mathbb{C}} + \mathfrak{h}^+$ with $\mathfrak{h}^+$ a maximal isotropic subspace of $\mathfrak{h}_p^{\mathbb{C}}$ where $\mathfrak{h}_p = \mathfrak{p} \cap \mathfrak{q}$.*

Proof. We use the subset Φ of the roots defined in lemma 4.4. Since $\mathfrak{p}^+ \cap \overline{\mathfrak{p}^+} = 0$ we have $\Phi \cap -\Phi = \emptyset$, whilst $II \subset \Phi \cup -\Phi$. Since Φ is closed under root addition it follows that

$$\mathfrak{n} = \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

is the nilradical of a τ-stable parabolic $\mathfrak{q}$ with Levi factor $\mathfrak{h}^{\mathbb{C}}$ and that $\mathfrak{p}^+ = \mathfrak{t}^+ + \mathfrak{n} \cap \mathfrak{p}^{\mathbb{C}}$. Finally note that $[\mathfrak{t}_p, \mathfrak{p}^+] = \sum_{\alpha \in \Phi \cap II} \mathbb{C}x_\alpha \subset [\mathfrak{p}^+, \mathfrak{p}^+]$ and so $\mathfrak{t}_p \subset \mathfrak{h}$. Hence we must have $\mathfrak{t}_p = \mathfrak{h}_p$. In particular $\mathfrak{t}^+$ is a maximal isotropic subspace of $\mathfrak{h}_p^{\mathbb{C}}$.

In order to see that the parabolic subalgebra $\mathfrak{q}$ constructed above is τ-maximal we observe that condition (i) of definition 3.1 is a consequence of lemma 4.1. Condition (ii) follows since $\mathfrak{n}_k = [\mathfrak{p}^+, \mathfrak{p}^+]$ by lemma 4.2. But $\mathfrak{n}$ is an ideal in $\mathfrak{q}$ so $[\mathfrak{n}_p, \mathfrak{t}_p^{\mathbb{C}} + \mathfrak{n}_p] \subset \mathfrak{n} \cap \mathfrak{k}^{\mathbb{C}} = \mathfrak{n}_k$. $\square$

We also have a converse to this result. Suppose we have a τ-maximal parabolic $\mathfrak{q}$ with Levi factor $\mathfrak{l} = \mathfrak{q} \cap \mathfrak{g}$ then we know that $\mathfrak{l}_p = \mathfrak{q} \cap \mathfrak{p}$ is even dimensional and if $\mathfrak{n}$ is the nilradical then $\mathfrak{p}^{\mathbb{C}} = \mathfrak{n}_p + \overline{\mathfrak{n}_p} + \mathfrak{l}_p^{\mathbb{C}}$. If we take a maximal isotropic subspace $\mathfrak{l}^+$ of $\mathfrak{l}_p^{\mathbb{C}}$ then $\mathfrak{p}^+ = \mathfrak{n}_p + \mathfrak{l}^+$ is maximal isotropic. In fact:

Theorem 4.6. *If $\mathfrak{q}$ is a τ-maximal parabolic of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{l}^+$ is a maximal isotropic subspace of $\mathfrak{l}_p^{\mathbb{C}}$ (defined as above) then $\mathfrak{p}^+ = \mathfrak{n}_p + \mathfrak{l}^+$ is in Z_k.*

Proof. We have seen that $\mathfrak{p}^+$ is maximal isotropic. Since $\mathfrak{n}$ is an ideal in $\mathfrak{q}$ and $\mathfrak{l}_p$ is abelian then $[\mathfrak{p}^+, \mathfrak{p}^+] \subset [\mathfrak{n}_p, \mathfrak{n}_p + \mathfrak{l}^+] \subset \mathfrak{n}_k$. Further $[\mathfrak{n}_k, \mathfrak{p}^+] \subset \mathfrak{n}_p \subset \mathfrak{p}^+$, so $[[\mathfrak{p}^+, \mathfrak{p}^+], \mathfrak{p}^+] \subset \mathfrak{p}^+$. $\square$

Consider the set $\widetilde{Z_k}$ consisting of pairs $(\mathfrak{q}, \mathfrak{l}^+)$ where $\mathfrak{q}$ is a τ-maximal parabolic and $\mathfrak{l}^+$ is a maximal isotropic subspace of $(\mathfrak{q} \cap \mathfrak{p})^{\mathbb{C}}$. Theorem 4.5 gives us a map $a\colon Z_k \to \widetilde{Z_k}$ $a(\mathfrak{p}^+) = (\mathfrak{h}^{\mathbb{C}} + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha, \mathfrak{h}^+)$ and theorem 4.6 gives us a map $b\colon \widetilde{Z_k} \to Z_k$ defined by $b(\mathfrak{q}, \mathfrak{l}^+) = \mathfrak{n} \cap \mathfrak{p}^{\mathbb{C}} + \mathfrak{l}^+$ where $\mathfrak{n}$ is the nilradical of $\mathfrak{q}$.

Theorem 4.7. *The maps a, b, defined above, are inverses of each other.*

Proof. $b \circ a$ is clearly the identity. To see the converse, suppose we have a τ-maximal parabolic $\mathfrak{q}$ and a maximal isotropic subspace $\mathfrak{l}^+$ of $\mathfrak{l}_p^{\mathbb{C}}$ (notation as in section 3) and we set $\mathfrak{p}^+ = \mathfrak{n}_p + \mathfrak{l}^+$. Take a maximal toral subalgebra $\mathfrak{t}_k$ of $\mathfrak{l}_k$ then $\mathfrak{t} = \mathfrak{t}_k + \mathfrak{l}_p$ is maximal toral in $\mathfrak{g}$ (see remark 3.3). Take the roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$ and divide them into types I and II as usual. Type I roots vanish on $\mathfrak{l}_p$, so $[\mathfrak{g}_\alpha, \mathfrak{l}_p^{\mathbb{C}}] = 0 = [\mathfrak{g}_\alpha, \mathfrak{l}^+]$ for α of type I. As in lemma 4.3, a root α of type II does not vanish on $\mathfrak{l}^+$, and so $[\mathfrak{g}_\alpha, \mathfrak{l}^+] = \mathfrak{g}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{l}_p^{\mathbb{C}}]$. Thus for all roots α we have $[\mathfrak{g}_\alpha, \mathfrak{l}^+] = [\mathfrak{g}_\alpha, \mathfrak{l}_p^{\mathbb{C}}]$ and so summing over root spaces in $\mathfrak{n}$ we have $[\mathfrak{n}, \mathfrak{l}^+] = [\mathfrak{n}, \mathfrak{l}_p^{\mathbb{C}}]$. Intersecting with $\mathfrak{k}^{\mathbb{C}}$ we conclude that $[\mathfrak{n}_p, \mathfrak{l}^+] = [\mathfrak{n}_p, \mathfrak{l}_p^{\mathbb{C}}]$ and so $[\mathfrak{p}^+, \mathfrak{p}^+] = [\mathfrak{n}_p, \mathfrak{n}_p + \mathfrak{l}^+] = [\mathfrak{n}_p, \mathfrak{n}_p + \mathfrak{l}_p^{\mathbb{C}}] = [\mathfrak{n}_p, \mathfrak{q}_p] = \mathfrak{n}_k$ since $\mathfrak{q}$ is τ-maximal.

This means that the $\mathfrak{h}_k$ determined by $\mathfrak{p}^+$ will be equal to the $\mathfrak{l}_k$ of $\mathfrak{q}$, and so $\mathfrak{h}_p = \mathfrak{l}_p$ and then $\mathfrak{h}^+ = \mathfrak{p}^+ \cap \mathfrak{h}_p^{\mathbb{C}} = \mathfrak{l}^+$. Then the $\mathfrak{p}$-part of the nilradical of the parabolic determined by $\mathfrak{p}^+$ will be $\mathfrak{n}_p$ and so we recover both $\mathfrak{q}$ and $\mathfrak{l}^+$ from $\mathfrak{p}^+$ showing that $a \circ b = \mathrm{id}$. $\square$

5. The structure of the zero-set

We now associate a subset $Z_{\mathfrak{q}}$ of the zero-set of the Nijenhuis tensor of J_1 on $J(G/K, g)$ to the K-conjugacy class of a τ-maximal parabolic $\mathfrak{q}$; we continue with the notation above.

Let $J(\mathfrak{l}_p)$ denote the almost complex structures on the vector space $\mathfrak{l}_p$ compatible with the Killing form and give $J(\mathfrak{l}_p)$ its natural structure of a complex manifold as in section 2. Let L_k be the stabilizer in K of $\mathfrak{q}$ in the adjoint representation of G on $\mathfrak{g}^{\mathbb{C}}$. Then L_k has Lie algebra the normalizer of $\mathfrak{q}$ in $\mathfrak{k}$. Since a parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$ is its own normalizer, it follows that L_k has Lie algebra $\mathfrak{q} \cap \mathfrak{k} = \mathfrak{l}_k$. Obviously, L_k also preserves $\mathfrak{q}_k$ and so the latter defines an invariant complex structure on K/L_k. Give $K/L_k \times J(\mathfrak{l}_p)$ the product complex structure. Define a map $\phi\colon K/L_k \times J(\mathfrak{l}_p) \to J(\mathfrak{p})$ by $\phi(kL_k, \mathfrak{l}^+) = \mathrm{Ad}\, k(b(\mathfrak{q}, \mathfrak{l}^+))$ where b is the map defined in section 4. Then we have the following proposition:

Proposition 5.1. *The map ϕ defined above is holomorphic.*

Proof. That the map is holomorphic follows by considering the two variables separately. The inclusion of $J(\mathfrak{l}_p)$ into $J(\mathfrak{p})$ given by adding on a fixed isotropic subspace $\mathfrak{n}_p$ is clearly holomorphic. Keeping the point in $J(\mathfrak{l}_p)$ fixed we need to see finally that the map from K/L_k to $J(\mathfrak{p})$ given by conjugating a fixed element j_0 of $J(\mathfrak{p})$ is holomorphic. This follows from the following more general lemma.

Lemma 5.2. *Let the reductive homogeneous space K/H have a complex structure given by the subspace $\mathfrak{m}^+$ of $\mathfrak{m}^{\mathbb{C}}$ where $\mathfrak{m}$ is the reductive summand. Let j_0 be an H-invariant element of $J(\mathfrak{p})$ where $\mathfrak{p}$ is an even-dimensional representation of K. Then the map $kH \mapsto k\, j_0\, k^{-1}$ is holomorphic if and only if $\mathfrak{m}^+ \cdot \mathfrak{p}^+ \subset \mathfrak{p}^+$ where $\cdot$ denotes the infinitesimal action and $\mathfrak{p}^+$ is the $+i$ eigenspace of j_0.*

Proof. Denote the map by ϕ and let $\widetilde{\xi}$ denote the vector-field on K/H generated by an element ξ of the Lie algebra $\mathfrak{k}$ of K. Then

$$d\phi(\widetilde{\xi}_{eH}) = [\xi\cdot, j_0].$$

Since ϕ is equivariant it will be holomorphic if its differential at the identity coset preserves the spaces of (1,0) vectors. Thus, for $\xi \in \mathfrak{m}^+$, we need to have $[\xi\cdot, j_0]$ in the (1,0) space at j_0. The latter consists of endomorphisms A of $\mathfrak{p}^{\mathbb{C}}$ which anticommute with j_0 and satisfy $(j_0 - i)A = 0$. Thus take ξ in $\mathfrak{m}^+$ and consider

$$\begin{aligned}(j_0 - i)\circ[\xi\cdot, j_0] &= (j_0 - i)\circ(\xi\cdot)\circ j_0 - (j_0 - i)\circ j_0\circ(\xi\cdot) \\ &= (j_0 - i)\circ(\xi\cdot)\circ(j_0 + i)\end{aligned}$$

This will vanish if and only if $\mathfrak{m}^+ \cdot \mathfrak{p}^+ \subset \mathfrak{p}^+$. □

To complete the proof of proposition 5.1, we observe that in our case $\mathfrak{m}^+ = \mathfrak{n}_k$ and $\mathfrak{p}^+ = \mathfrak{n}_p + \mathfrak{l}^+$ so $\mathfrak{m}^+ \cdot \mathfrak{p}^+ = [\mathfrak{n}_k, \mathfrak{n}_p + \mathfrak{l}^+] \subset \mathfrak{n}_p \subset \mathfrak{p}^+$. □

Theorem 4.7 shows that ϕ is injective so we may use it to view $K/L_k \times J(\mathfrak{l}_p)$ as a subset of the fibre of $J(G/K, g)$ over the identity coset and take its orbit $Z_{\mathfrak{q}}$ under G. Clearly this orbit will be in the zero-set of the Nijenhuis tensor and depends only on the K-conjugacy class of $\mathfrak{q}$. As a manifold it is just the homogeneous fibre bundle associated to the principal K-bundle $G \to G/K$ with fibre $K/L_k \times J(\mathfrak{l}_p)$ and as such it is an example of a generalized twistor space as considered in section 2. There it is shown that such spaces have a natural almost complex structure J_1 with respect to which the natural map to $J(G/K, g)$ is holomorphic. In our case this map is just the inclusion map, so that we have immediately that $Z_{\mathfrak{q}}$ is an almost-complex submanifold of $J(G/K, g)$. Since Nijenhuis tensors are natural with respect to almost-complex maps, and $Z_{\mathfrak{q}}$ is in the zero-set of the Nijenhuis tensor of $J(G/K, g)$ it follows that the Nijenhuis tensor of J_1 on $Z_{\mathfrak{q}}$ also vanishes and hence that J_1 is integrable on $Z_{\mathfrak{q}}$. We have thus shown:

Proposition 5.3. *To each τ-maximal parabolic $\mathfrak{q}$ is associated a subset $Z_{\mathfrak{q}}$ of $J(G/K, g)$ which lies in the zero-set of the Nijenhuis tensor of J_1. The latter induces an integrable complex structure on $Z_{\mathfrak{q}}$. $Z_{\mathfrak{q}}$ depends only on the K-conjugacy class of $\mathfrak{q}$.*

Proposition 5.4. *There are only a finite number of K-conjugacy classes of τ-maximal parabolics.*

Proof. Each τ-maximal parabolic $\mathfrak{q}$ determines a parabolic $\mathfrak{q}_k$ of $\mathfrak{k}^{\mathbb{C}}$ and there are only a finite number of K-conjugacy classes of these. It suffices to show, therefore, that the $\mathfrak{k}^{\mathbb{C}}$ parabolic $\mathfrak{q}_k$ can be contained in only a finite number of τ-maximal parabolics. But this is the case since we can choose a maximal toral subalgebra $\mathfrak{t}_k$ of $\mathfrak{l}_k$ which is also maximal in $\mathfrak{k}$. We take $\mathfrak{l}_p$ to be the centralizer of $\mathfrak{l}_k$ in $\mathfrak{p}$ (so dependent only on $\mathfrak{q}_k$ and not $\mathfrak{q}$). Then we know $\mathfrak{t}_k + \mathfrak{l}_p$ is a maximal toral subalgebra of $\mathfrak{g}$. Since a maximal toral subalgebra may only be contained in a finite number of parabolics of any kind this means that the extensions $\mathfrak{q}$ of $\mathfrak{q}_k$ are finite in number. □

We summarize these results as

Theorem 5.5. *The zero-set of the Nijenhuis tensor of J_1 on $J(G/K, g)$ is a finite union of complex manifolds of the form $Z_{\mathfrak{q}}$ where $\mathfrak{q}$ is a τ-maximal parabolic of $\mathfrak{g}^{\mathbb{C}}$.*

6. Examples and applications

Let us see what our analysis tells us about the geometry of Z and examine some examples. First we note that the situation is rather more complicated for non-inner Riemannian symmetric spaces than for the inner spaces treated in [1]: for instance, G does not act transitively on the components of Z except when $J(\mathfrak{l}_p)$ is zero-dimensional, which is the case precisely when $\dim \mathfrak{l}_p = 2$. We remark that $\dim \mathfrak{l}_p = \operatorname{rank} G - \operatorname{rank} K$ and so only depends on τ rather than the particular τ-maximal parabolic $\mathfrak{q}$. Thus G will be transitive on all components of Z if it is transitive on one.

Moreover, if $G/K = G_1/K_1 \times G_2/K_2$ is an isometric splitting of G/K into a pair of even-dimensional non-inner Riemannian symmetric spaces, then $\mathfrak{l}_p = \mathfrak{l}_{p_1} + \mathfrak{l}_{p_2}$ but $J(\mathfrak{l}_p) \neq J(\mathfrak{l}_{p_1}) \times J(\mathfrak{l}_{p_2})$ so that, in general, $j \in Z_{\mathfrak{q}}$ will not split as $j = j_1 + j_2$ with $j_i \in J(G_i/K_i)$. Thus Z does not respect the de Rham decomposition of G/K in contrast to the case of inner symmmetric spaces (compare theorem 5.3 of [1]). Despite this, the $Z_{\mathfrak{q}}$ do not behave too badly with respect to the de Rham decomposition: if $\mathfrak{b} \subset \mathfrak{g}^{\mathbb{C}}$ is a τ-stable Borel subalgebra and

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

is the decomposition of $\mathfrak{g}$ into irreducible orthogonal symmetric Lie algebras, then it is straightforward to show that $\mathfrak{b} = \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_k$ with each $\mathfrak{b}_i$ a τ-stable Borel subalgebra of $\mathfrak{g}_i^{\mathbb{C}}$. Thus τ-stable parabolic subalgebras also commute with this decomposition.

Let us now consider some examples:

Example 1. Let us take our symmetric space to be a product of odd-dimensional spheres $S^{2n-1} \times S^{2m-1} = SO(2n) \times SO(2m)/SO(2n-1) \times SO(2m-1)$. In this case, $\operatorname{rank} G - \operatorname{rank} K = 2$ so that the connected components of Z are G-orbits. To find the $Z_{\mathfrak{q}}$, we note from the above discussion that a τ-maximal parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$ is a sum of τ-maximal parabolic subalgebras for the factors $\mathfrak{so}(2n)$ and $\mathfrak{so}(2m)$ and so it suffices to find these. For this, fix $x \in S^{2n-1}$ and set $V = \{x\}^{\perp}$. Let τ be the involution at x and then, under the usual identification of $\mathfrak{so}(2n)$ with $\Lambda^2\mathbb{R}^{2n}$, we have as symmetric decomposition:

$$\Lambda^2\mathbb{R}^{2n} = \Lambda^2 V \oplus V \otimes \mathbb{R}x.$$

A τ-maximal parabolic is equivalent to a maximal isotropic subspace V^+ of $V^{\mathbb{C}}$: we have an orthogonal direct sum

$$V^{\mathbb{C}} = V^+ \oplus V^0 \oplus V^-$$

with $V^{\pm}$ mutually conjugate and V^0 real and 1-dimensional and then the corresponding parabolic subalgebra $\mathfrak{q}$ has nilradical

$$(\Lambda^2 V^+ \oplus V^+ \otimes V^0) \oplus V^+ \otimes \mathbb{C}x.$$

We note that such a parabolic determines (and is determined by) a choice of complex structure on $\mathbb{R}^{2n}$ (equivalently, a choice of isomorphism $\mathbb{R}^{2n} \cong \mathbb{C}^n$). Indeed, if $j : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a complex structure, take $V^0 = \mathbb{C}jx$ and V^+ to be the $\sqrt{-1}$-eigenspace of j on $\{x, jx\}^{\perp}$. We denote the corresponding parabolic by $\mathfrak{q}_{x,j}$.

Thus, if $(x, y) \in S^{2n-1} \times S^{2m-1}$ and τ is the involution at (x, y) then any τ-maximal parabolic is of the form $\mathfrak{q}_{x,j} \oplus \mathfrak{q}_{y,k}$ with j, k complex structures on $\mathbb{R}^{2n}$ and $\mathbb{R}^{2m}$ respectively. The $\mathfrak{p}$-part of the Levi-factor is then given by

$$\mathfrak{l}_p = \mathbb{R}jx \otimes x + \mathbb{R}ky \otimes y$$

giving just two choices for $\mathfrak{l}^+$: $\mathbb{C}(jx \otimes x \pm \sqrt{-1}ky \otimes y)$.

Observe that fixing j, k and a choice of $\mathfrak{l}^+$ while letting x and y vary gives rise to a globally defined section of Z—this section is easily checked to be an integrable Hermitian structure on $S^{2n-1} \times S^{2m-1}$ and is that discovered by Calabi-Eckmann [2].

From lemma 5.4 of [1], it is known that any integrable Hermitian structure on an even dimensional manifold M, when viewed as a section of $J(M)$, has image in Z. In the case at hand then, we conclude from the above development that for $S^{2n-1} \times S^{2m-1}$, Z is exhausted by the images of globally defined Hermitian structures.

Example 2. Let G be an even-dimensional compact semisimple Lie group. We view G as a symmetric $G \times G$-space $G \cong (G \times G)/\Delta G$. The involution at the identity coset is then $\tau : (x, y) \mapsto (y, x)$ so the the symmetric decomposition has

$$\mathfrak{k} = \Delta\mathfrak{g}, \qquad \mathfrak{p} = \{(\xi, -\xi) : \xi \in \mathfrak{g}\}.$$

Now a τ-stable Borel subalgebra of $\mathfrak{g}^{\mathbb{C}} \oplus \mathfrak{g}^{\mathbb{C}}$ is of the form $\mathfrak{b} \oplus \mathfrak{b}$ with Levi-factor $\mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{t}^{\mathbb{C}}$ for a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{t} = \mathfrak{b} \cap \mathfrak{g}$. We now apply theorem 3.4 to find the τ-maximal parabolic subalgebras: let $\mathfrak{b}'$ be the nilradical of $\mathfrak{b}$ and $\mathfrak{n}$ that of $\mathfrak{b} \oplus \mathfrak{b}$. We have

$$\mathfrak{n}_p = \{(\xi, -\xi) : \xi \in \mathfrak{b}'\} \qquad (\mathfrak{b} \oplus \mathfrak{b})_p = \{(\eta, -\eta) : \eta \in \mathfrak{b}\},$$

so that

$$[\mathfrak{n}_p, (\mathfrak{b} \oplus \mathfrak{b})_p] = \Delta[\mathfrak{b}', \mathfrak{b}] = \Delta\mathfrak{b}' = \mathfrak{n}_k.$$

From this we see that the τ-maximal parabolic containing $\mathfrak{b} \oplus \mathfrak{b}$ is $\mathfrak{b} \oplus \mathfrak{b}$ itself, that is, the τ-maximal parabolics are precisely the τ-stable Borels.

We now use projection onto the first factor to identify $\mathfrak{p}$ with $\mathfrak{g}$ and conclude that $\mathfrak{p}^+ \in Z_k$ if and only if it is of the form

$$\mathfrak{p}^+ = \mathfrak{b}' \oplus \mathfrak{t}^+$$

with $\mathfrak{t}^+$ maximal isotropic in $\mathfrak{t}^{\mathbb{C}}$. Observe that such a $\mathfrak{p}^+$is in fact a subalgebra of $\mathfrak{g}^{\mathbb{C}}$ and so gives rise, by left (or right) translation, to a globally defined Hermitian structure on G. These Hermitian structures were first discovered by Samelson [6] (see also Wang [7]). Once again, we conclude that Z is exhausted by the images of the globally defined Hermitian structures.

One may observe that in both the previous examples there is but a single K-conjugacy class of τ-maximal parabolic subalgebras and thus at most two components of the zero-set of the Nijenhuis tensor. This is a consequence of the fact that both products of spheres and semisimple Lie groups are *split-rank* symmetric spaces as we now explain.

A symmetric space G/K is said to be split-rank if its rank is the difference between the ranks of G and K. In this case there cannot be any type I_p roots since $\mathfrak{t}_p$ is now maximal abelian in $\mathfrak{p}$. It follows that any τ-stable Borel subalgebra $\mathfrak{b}$ is determined by its $\mathfrak{k}$-part $\mathfrak{b}_k$ by $\mathfrak{b} = \mathfrak{b}_k + [\mathfrak{b}_k, \mathfrak{t}_p]$ and hence that there is a single K-conjugacy class of τ-stable Borel subalgebras. The resulting τ-maximal parabolics built from these Borels by theorem 4.29 of [1] will thus also form a single K-conjugacy class, strengthening proposition 5.4. Since $J(\mathfrak{t}_p)$ has two components, it follows that in the split-rank case there are just one or two components to the zero-set.

Example 3.

Another example of this situation is the symmetric space $SU(2n)/Sp(n)$ which has dimension $(2n+1)(n-1)$. This is even for n odd. Let us illustrate the previous sections by determining explicitly the components of its zero-set.

So fix n odd and let $V = \mathbb{C}^{2n}$ with its usual Hermitian metric $\langle\ ,\ \rangle$ and fix a normalized complex volume form $\varepsilon \in \Lambda^{2n}V^*$. A *quaternionic structure* on V is an antilinear map $j\colon V \to V$ with $j^2 = -1$ which is compatible with the metric in the sense that

$$\langle ju, jv\rangle = \langle v, u\rangle$$

for all u, v in V. Such a j gives rise to a non-degenerate 2-form $\omega_j \in \Lambda^2 V^*$ by

$$\omega_j(u, v) = \langle u, jv\rangle$$

and we further demand that j be compatible with ε in the sense that

$$\omega_j^n = \varepsilon.$$

Let N be the collection of all such quaternionic structures. Then $SU(2n)$ acts transitively on N by conjugation. A choice of base point $j \in N$ allows us to identify $\mathbb{C}^{2n}$ with $\mathbb{H}^n$ and hence its stabilizer with $Sp(n)$. The involution τ corresponding with j is then given by

$$\tau(g) = -jgj$$

for $g \in SU(2n)$. Thus N is a model for $SU(2n)/Sp(n)$.

Fix j and denote ω_j by ω. For $A \in \mathrm{End}(V)$ define the *quaternion transpose* A^T by

$$\omega(Au, v) = \omega(u, A^T v),$$

then the symmetric decomposition of the Lie algebra $su(2n)$ is given by

$$su(2n) = \mathfrak{k} \oplus \mathfrak{p}$$

with

$$sp(n) = \mathfrak{k} = \{A \in su(2n) | A + A^T = 0\}, \qquad \mathfrak{p} = \{A \in su(2n) | A = A^T\}.$$

It is useful to have another model for $su(2n)$ and its complexification. For this we use ω to identify V with V^* by

$$u(v) = \omega(u, v)$$

so that $\mathrm{End}(V) \cong V \otimes V$. Under this identification, it is easy to check that

$$\mathfrak{k}^{\mathbb{C}} = S^2 V, \qquad \mathfrak{p}^{\mathbb{C}} = \Lambda_0^2 V$$

where $\Lambda_0^2 V$ is the orthogonal complement of $\omega \in \Lambda^2 V^* \cong \Lambda^2 V$. Moreover, conjugation with respect to the real form $su(2n)$ becomes

$$u \otimes v \mapsto jv \otimes ju$$

while the involution is given by

$$u \otimes v \mapsto -(u \otimes v)^T = v \otimes u.$$

With these preliminaries, let us fix a maximal torus $\mathfrak{t}_k$ of $\mathfrak{k}$. This amounts to fixing a j-stable orthogonal decomposition of V into one-dimensional subspaces

$$V = j\mathcal{L}_n \oplus \cdots \oplus j\mathcal{L}_1 \oplus \mathcal{L}_1 \cdots \oplus \mathcal{L}_n.$$

The fundamental toral subalgebra $\mathfrak{t}$ of $su(2n)$ containing $\mathfrak{t}_k$ is then the stabilizer in $su(2n)$ of this decomposition.

Any Borel subalgebra $\mathfrak{b}$ of $sl(2n, \mathbb{C})$ containing $\mathfrak{t}$ is the stabilizer of a full flag of subspaces $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{2n} = V$ with $\dim V_i = i$ and each V_i a direct sum of some of the $\mathcal{L}_i$ and $j\mathcal{L}_i$. The condition that $\mathfrak{b}$ be τ-stable amounts to the demand that

$$V_i^0 = V_{2n-i}$$

where V_i^0 denotes the polar of V_i with respect to ω. From this we conclude that, after relabelling the $\mathcal{L}_i$ if necessary, a τ-stable Borel subalgebra is the stabilizer of a flag given by

$$V_i = \bigoplus_{k=1}^{i} j\mathcal{L}_{n+1-k}, \qquad i \leq n,$$

$$V_{n+i} = V_{n-i}^0 = V_n \oplus \bigoplus_{k=1}^{i} \mathcal{L}_k, \qquad i \leq n.$$

Denoting $j\mathcal{L}_k$ by $\mathcal{L}_{-k}$, it is now straightforward to check that

$$\mathfrak{b} = \bigoplus_{i+j \leq 0} \mathcal{L}_i \otimes \mathcal{L}_j$$

with nilradical

$$\mathfrak{b}' = \bigoplus_{i+j<0} \mathcal{L}_i \otimes \mathcal{L}_j.$$

From 3.5, the τ-maximal parabolic containing $\mathfrak{b}$ has nilradical given by

$$\mathfrak{n} = \mathfrak{b}'_p + [\mathfrak{b}'_p, \mathfrak{b}_p].$$

The following bracketing relations are easy to verify

$$\begin{aligned}
[\mathcal{L}_i \wedge \mathcal{L}_{-i}, \mathcal{L}_i \wedge \mathcal{L}_j] &= \mathcal{L}_i \vee \mathcal{L}_j, && \text{for } i+j<0, i \neq j;\\
[\mathcal{L}_i \wedge \mathcal{L}_{-k}, \mathcal{L}_{-i} \wedge \mathcal{L}_{-j}] &= \mathcal{L}_{-k} \vee \mathcal{L}_{-j}, && \text{for } i \neq j, k \geq 0;\\
[\mathcal{L}_i \wedge \mathcal{L}_{-j}, \mathcal{L}_j \wedge \mathcal{L}_{-k}] &= \mathcal{L}_i \vee \mathcal{L}_{-k}, && \text{for } 0<i<j<k;
\end{aligned}$$

while all other brackets between summands of $\mathfrak{b}'_p$ vanish. In particular, $S^2\mathcal{L}_{-k} = [\mathcal{L}_i \wedge \mathcal{L}_{-k}, \mathcal{L}_{-i} \wedge \mathcal{L}_{-k}]$ only lies in $[\mathfrak{b}'_p, \mathfrak{b}_p]$ for $1 \leq i < k$. We therefore conclude that

$$[\mathfrak{b}'_p, \mathfrak{b}_p] = \bigoplus_{\substack{i+j<0\\ i\neq j}} \mathcal{L}_i \vee \mathcal{L}_j \oplus \bigoplus_{i \geq 2} S^2\mathcal{L}_i.$$

Thus the $\mathfrak{k}$-part of the Levi factor $\mathfrak{l}_k$ is given by

$$\mathfrak{l}_k^{\mathbb{C}} = \mathfrak{t}_k^{\mathbb{C}} + S^2\mathcal{L}_1 \oplus S^2\mathcal{L}_{-1}$$

so $\mathcal{L}_k = \overbrace{U(1) \times \cdots \times U(1)}^{n-1 \text{ times}} \times Sp(1)$.

Note that in this case, all τ-stable Borels and hence τ-maximal parabolics are K-conjugate. Thus there are at most two components of the zeroset of the Nijenhuis tensor of $J(N)$, each a copy of the same $Z_{\mathfrak{q}}$.

In summary, our analysis shows that any $\mathfrak{p}^+$ in the zero set arises from an orthogonal decomposition

$$V = \bigoplus_{1-n \leq i \leq n-1} E_i$$

with $\dim E_0 = 2$, $\dim E_i = 1$, $|i| \geq 1$ with $jE_i = E_{-i}$ (in our previous notation, $E_0 = \mathcal{L}_1 \oplus \mathcal{L}_{-1}$, $E_1 = \mathcal{L}_2, \ldots$) and then

$$\mathfrak{p}^+ = \bigoplus_{i+j<0} E_i \wedge E_j \oplus \mathfrak{t}^+$$

where $\mathfrak{t}^+$ is a maximal isotropic subspace of

$$\sum_i F_i \wedge E_{-i} \cap \{\omega\}^\perp$$

which is $(n-1)$-dimensional.

Example 4. Consideration of example 1 might lead one to enquire as to whether there were Calabi-Eckmann type complex structures on products of odd-dimensional oriented Grassmannians. In fact, this is far from being the case: in this setting, there are, in general, not even any *continuous* sections of Z as the following theorem shows.

Theorem 6.1. *Let $M = M_1 \times \cdots \times M_r$ be an even-dimensional product of connected Riemannian symmetric spaces of semisimple type with $M_1 = G_k(\mathbb{R}^{k+n})$ a Grassmannian of oriented k-planes in $\mathbb{R}^{k+n}$ with n, k odd and $n \geq k > 1$. Then Z has no globally defined continuous sections.*

In particular, M admits no Hermitian complex structures.

Proof. A continuous section of Z must lie in some $Z_{\mathfrak{q}}$ and so gives a reduction of the K-bundle $G \to G/K = M$ to some H_k. However, τ-maximal parabolic subalgebras commute with the de Rham decomposition of M so that restricting attention to a slice $M_1 \subset M$, we get a reduction of the $SO(k) \times SO(n)$-bundle $SO(n+k) \to G_k(\mathbb{R}^{k+n})$ to the centralizer of a maximal torus in $SO(k) \times SO(n)$. However, such a reduction would induce a splitting of the tautological k-plane bundle $W \to G_k(\mathbb{R}^{k+n})$ into a line sub-bundle and its complement:

$$W = L \oplus L^{\perp}$$

and such splittings do not exist for topological reasons. Indeed, such a splitting would give a factorization of Stiefel-Whitney classes

$$w_k(W) = w_1(L) w_{k-1}(L^{\perp}),$$

but $w_1(L) = 0$ since $H^1(G_k(\mathbb{R}^{n+k}), \mathbb{Z}_2)$ vanishes while $w_k(W)$ is known to be non-zero, see [4] for example. □

Example 5. Finally, we prove a result of a different nature, relating the topology of G/K to that of the components of Z under the simplifying assumption that K is connected (this involves no loss of generality when G/K is of non-compact type).

We prove

Theorem 6.2. *Let G/K be an even-dimensional Riemannian symmetric space of compact or non-compact type with G, K connected and let X be a connected component of $Z \subset J(G/K)$. Then*

$$\pi_1(G/K) = \pi_1(X).$$

Proof. From theorem 5.5, we know that any component of Z arises in the following manner: fix a τ-maximal parabolic subalgebra $\mathfrak{q}$ and let L_k be the normalizer of $\mathfrak{q}$ in K. Let $\mathfrak{l}_p$ be the centralizer of $\mathfrak{l}_k$ in $\mathfrak{p}$ and take a connected component $J_0(\mathfrak{l}_p) \subset J(\mathfrak{l}_p)$. Then $X = G \times_K (K/L_k \times J_0(\mathfrak{l}_p))$ is a connected component of Z and all components arise this way.

Now L_k coincides with the normalizer in K of the parabolic subalgebra $\mathfrak{q}_k$ of $\mathfrak{k}^{\mathbb{C}}$ and so is the centralizer of a torus in K whence K/L_k is simply connected, as is $J_0(\mathfrak{l}_p)$. The homotopy long exact sequence of

$$K/L_k \times J_0(\mathfrak{l}_p) \to X \to G/K$$

now gives

$$0 \to \pi_1(X) \to \pi_1(G/K) \to \pi_0(K/L_k \times J_0(\mathfrak{l}_p)) = 0$$

whence $\pi_1(X) \cong \pi_1(G/K)$. □

As a corollary, we see that certain compact quotients of Riemannian symmetric spaces of non-compact type have the same fundamental group as a compact complex manifold. This partially answers a question posed to us by D. Toledo.

Theorem 6.3. *Let D be an even-dimensional compact Riemannian locally symmetric space with universal cover M a Riemannian symmetric space of non-compact type. Suppose that, viewed as deck translations, $\Gamma = \pi_1(D) \subset I_0(M)$. Then there is a compact complex manifold with fundamental group Γ.*

Proof. Let $G = I_0(M)$. Then $M = G/K$ with K connected and M simply connected. Let X be a component of $Z \subset J(M)$. From theorem 6.2 we know that X is simply connected and, moreover, X is a complex manifold on which G acts holomorphically. Thus $\Gamma \backslash X$ is the required complex manifold. □

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