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A remark on parabolic subalgebras

$\mathfrak{g}$ a cmplx semisimple Lie algebra & $\mathfrak{p} \leq \mathfrak{g}$

$\mathfrak{p}$ is **parabolic** if it contains a Borel (thus max. solvable) subalgebra.

Let $\mathfrak{p}^\perp$ denote the Killing polar of $\mathfrak{p}$ in $\mathfrak{g}$. Then it is well-known that $\mathfrak{p}^\perp \trianglelefteq \mathfrak{p}$ is the nilradical of $\mathfrak{p}$.

We prove a converse:

Prop$^{\underline{n}}$ $\mathfrak{p} \leq \mathfrak{g}$ is parabolic iff $\mathfrak{p}^\perp$ is a nilpotent subalgebra of $\mathfrak{g}$.

Remark The same result holds for real semisimple $\mathfrak{g}$ since $\mathfrak{p} \leq \mathfrak{g}$ is parabolic by def$^{\underline{n}}$ iff $\mathfrak{p}^{\mathbb{C}}$ is parabolic in $\mathfrak{g}^{\mathbb{C}}$, and ~~our~~ result is preserved by complexification.

We first prove:

Special case: assume $\mathfrak{p}^\perp \subseteq \mathfrak{p}$

Define a filtration on $\mathfrak{g}$ by setting

$$\mathfrak{g}^{(-1)} = \mathfrak{p}^\perp \subseteq \mathfrak{g}^{(0)} = \mathfrak{p}$$

$$\mathfrak{g}^{(i)} = [\mathfrak{g}^{(-1)}, \mathfrak{g}^{(i+1)}] \qquad i < -1$$

$$\mathfrak{g}^{(i)} = \mathfrak{g}^{(-1-i)\perp} \qquad i > 0$$

(thus $\dots \leq \mathfrak{g}^{(-2)} \leq \mathfrak{g}^{(-1)}$ is the central descending series of $\mathfrak{p}^\perp$)

Lemma A $[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \leq \mathfrak{g}^{(i+j)} \quad \forall i,j \qquad (*)_{i,j}$

Pf The Jacobi identity + def^{n} of $\mathfrak{g}^{(j)}$, $j<0$, gives $(*)_{i,j}$ for $i,j<0$

Then, for $j \geq 0$

$$([\mathfrak{g}^{(-1)}, \mathfrak{g}^{(j)}], \mathfrak{g}^{(-j)}) = (\mathfrak{g}^{(j)}, [\mathfrak{g}^{(-1)}, \mathfrak{g}^{(-j)}])$$
$$\subseteq (\mathfrak{g}^{(j)}, \mathfrak{g}^{(-1-j)}) = 0$$

so $[\mathfrak{g}^{(-1)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(j-1)} \quad \forall j.$

$\therefore$ By Jacobi $[\mathfrak{g}^{(-i)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(j-i)} \quad \forall i \geq 0.$

Finally, for $i,j \geq 0$

$$([\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}], \mathfrak{g}^{(-1-i-j)}) = (\mathfrak{g}^{(j)}, [\mathfrak{g}^{(-1-i-j)}, \mathfrak{g}^{(i)}])$$
$$\subseteq (\mathfrak{g}^{(j)}, \mathfrak{g}^{(-1-j)}) = 0$$

so $[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(i+j)} \quad \forall i,j.$ ∎

In particular, each $\mathfrak{z} \in \mathfrak{p}^{\perp}$ is ad-nilpotent on $\mathfrak{g}$.

We now have

Lemma B ([Grot, Lemme 4.2]) If $\mathfrak{p}^{\perp}$ is ad-nilpt on $\mathfrak{g}$ then there is a Borel subalg. $\mathfrak{b}$ with $\mathfrak{p}^{\perp} \leq \mathfrak{b} \subseteq \mathfrak{p}$

[we do not even need $\mathfrak{p}$ a subalg for this!]

Pf [Grot] $\mathfrak{p}^{\perp}$ is nilpt $\therefore$ solvable $\therefore$ $\exists$ max. solv. $\mathfrak{b}$ with $\mathfrak{p}^{\perp} \leq \mathfrak{b}$.

However $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ with $\mathfrak{h}$ CSA & $\mathfrak{n} = \mathfrak{b}^{\perp}$ nilrad

& any $X \in \mathfrak{b}$ with $\mathrm{ad}_{\mathfrak{g}} X$ nilpt lies in $\mathfrak{n}$

$\therefore$ $\mathfrak{p}^{\perp} \leq \mathfrak{b}^{\perp}$ whence $\mathfrak{b} \leq \mathfrak{p}$. ∎

This completes the proof of the special case $\mathfrak{p}^\perp \leq \mathfrak{p}$ and, in particular, we deduce that in this case, $\mathfrak{p}^\perp$ is the nilradical of $\mathfrak{p}$.

Now we treat the general case: so let $\mathfrak{p}, \mathfrak{p}^\perp \leq \mathfrak{g}$ with $\mathfrak{p}^\perp$ nilpotent. We have

$$([\mathfrak{p}^\perp, \mathfrak{p}], \mathfrak{p}) = (\mathfrak{p}^\perp, [\mathfrak{p}, \mathfrak{p}]) \subset (\mathfrak{p}^\perp, \mathfrak{p}) = 0$$

so that $[\mathfrak{p}, \mathfrak{p}^\perp] \subset \mathfrak{p}^\perp$.

Thus (1) $\mathfrak{p} + \mathfrak{p}^\perp \leq \mathfrak{g}$

(2) $\mathfrak{p}^\perp \trianglelefteq \mathfrak{p} + \mathfrak{p}^\perp$

Now $(\mathfrak{p}+\mathfrak{p}^\perp)^\perp = \mathfrak{p}^\perp \cap \mathfrak{p} \leq \mathfrak{p} + \mathfrak{p}^\perp$ & $\mathfrak{p}^\perp \cap \mathfrak{p} \leq \mathfrak{p}^\perp$ is nilpotent. The special case tells us then that $\mathfrak{p}+\mathfrak{p}^\perp$ is parabolic with nilradical $\mathfrak{p} \cap \mathfrak{p}^\perp$.

However $\mathfrak{p}^\perp$ is a nilpotent ideal of $\mathfrak{p}+\mathfrak{p}^\perp$ whence

$\mathfrak{p}^\perp \cap \mathfrak{p} = \mathfrak{p}^\perp$ & $\mathfrak{p} + \mathfrak{p}^\perp = \mathfrak{p}$. Thus $\mathfrak{p}$ is parabolic & we are done.

∎