On the Hirsch-Plotkin radical of stability groups

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Abstract

We study the stability group of subspace series of infinite dimensional vector spaces. In [1], the authors proved that when the vector space has countable dimension then the Hirsch-Plotkin radical of the stability group coincides with the set of all space automorphisms that fix a finite subseries and they conjectured that this would hold in all dimensions. We give a counter example of dimension $2^\aleph_0$. We however show that in general the result remains true if the Hirsch-Plotkin radical is replaced by the Fitting group, the product of all the normal nilpotent subgroups of the stability group. We also show that the Hirsch-Plotkin radical has a certain strong local nilpotence property.

0 Introduction

We start with a precise definition, taken from [1], of what we will mean generally by series of subspaces in a given vector space $V$.

Definition. Let $V$ be a vector space over a field $\mathbb{F}$. A set $\mathcal{L}$, consisting of subspaces of $V$, is said to be a series in $V$ if

(1) Both 0 and $V$ belong to $\mathcal{L}$. 

1
(2) The set $L$ is linearly ordered with respect to inclusion.
(3) For every $F \subseteq L$ both $\bigcap \{W : W \in F\}$ and $\bigcup \{W : W \in F\}$ are in $L$.

The definition thus follows the Kurosh notion of a series rather than the version of P. Hall (see for example pages 9-10 in [6] for a discussion of these). For our purposes it doesn’t really matter which definition one uses and the main results of the paper are true for both versions.

A subseries of $L$ is any subset of $L$ which is a series. In particular every finite subset of $L$ containing 0 and $V$ is a subseries.

**Definition.** Let $L$ be a series in a vector space $V$. A jump of $L$ is an ordered pair $(B, T)$ of elements $B, T \in L$ such that $B < T$ and there is no $U \in L$ where $B < U < T$.

Now let $0 \neq v \in V$ and let

$$T = \bigcap \{U \in L : v \in U\}, \quad B = \bigcup \{U \in L : v \notin U\}.$$ 

Notice that $(B, T)$ is a jump and $v \in T \setminus B$. Now let $J$ be the collection of all the jumps of $L$. It follows from the discussion above that

$$V \setminus 0 = \bigcup_{(B,T) \in J} T \setminus B$$

where the union is a disjoint union.

**Definition.** Let $L$ be a series in a vector space $V$. We say that an element $g \in GL(V)$ stabilises $L$ if $T(g - 1) \leq B$ for all jumps $(B, T)$ of $L$.

To a given series $L$, there are certain associated subgroups of $GL(V)$ that will play a role later on. We let $S(L)$ be the subgroup consisting of all the elements in $GL(V)$ that stabilise $L$. We let $HP(L)$ be the Hirsch-Plotkin radical of $S(L)$ and $Fit(L)$ be the Fitting subgroup of $S(L)$, that is the product of all the normal nilpotent subgroups of $S(L)$. Finally we let $F(L)$ be the normal subgroup of $S(L)$ consisting of the elements that stabilise some finite subseries of $L$.

It is well known, and not difficult to see, that any subgroup $G$ of the group of
linear automorphisms $GL(V)$ that stabilises a finite series is nilpotent. Here
the elements of $G$ act unipotently on $V$, that is $(g-1)$ is a nilpotent linear op-
erator for all $g \in G$. For finite dimensional vector spaces the converse is true
(see for example 8.1.10 in [5]) and thus any subgroup of $GL(V)$, consisting
of elements that act unipotently on $V$, stabilises a finite series of subspaces.
The situation is very different for infinite dimensional vector spaces and is
studied in [1]. To start with it is easy to come up with examples of nilpotent
subgroups of $GL(V)$, where $V$ has countably infinite dimension, that stabilise
some series of $V$ but stabilise no finite series of subspaces.

**Example.** Let $V$ be a vector space with basis $v_i$, $i \in \mathbb{Z}$ and let $V_i = \langle v_j : j \leq i \rangle$. Let

$$L = \{V_i : i \in \mathbb{Z}\} \cup \{0, V\}.$$  

The linear automorphism $g$ that maps $v_i$ to $v_i + v_{i-1}$ for $i \in \mathbb{Z}$ is in $S(L)$
but as it is not unipotent, it can’t stabilise any finite series in $V$. Thus we
have a cyclic subgroup, $\langle g \rangle$, of a series stabiliser that stabilises no finite series.

**Remark.** In [1] the authors however show that all the elements of $HP(L)$ are
unipotent and thus stabilise a finite subseries. The proof of this is quite tricky
and delicate. Using this result and a classic result of Gruenberg [3], that tells
us that any finitely generated Engel group is nilpotent, it is not difficult to
obtain the following generalisation: if $H$ is a finitely generated subgroup of
$HP(L)$, then there exists a positive integer $m$ such that $[V_m H] = 0$. In
particular $H$ stabilises the finite series $S = \{[V_k H] : 0 \leq k \leq m\}$. We will
do this in Section 2.

In [1] there is also an example given of a group acting unipotently on
a vector space that doesn’t stabilise any series of subspaces. Let $G$ be a Tarski
$p$-group and let $\mathbb{F}$ be any field of characteristic $p$. Then $G$ acts unipotently
on $\mathbb{F}G$. It cannot however stabilise a series in $V$. To see this one argues by
contradiction and assumes that $G$ stabilises some series in $V$. According to
a result of [4], $G$ itself then has a series with abelian factors. This is however
clearly not the case.

In fact the structure of the group $S(L)$ is in general not much restricted
and the authors of [1] demonstrate that every countable free group faithfully
stabilises an ascending series in a suitable vector space.
In view of the remark above however, HP(üzü) has a restricted structure as we have mentioned that its elements act unipotently on $V$. As a result of their study, the authors of [1] came up with the conjecture that $F(üzü) = HP(üzü)$ in general. They proved this conjecture for any vector space of countable dimension as well as for few other special situations. Quite surprisingly, it turns out that the conjecture is however not true in general and in Section 2 we show that there exists a vector space $V$ of dimension $2^\aleph_0$ with a series $ üzü$ such that $F(üzü) < HP( üzü)$. We will however show in Section 1 that in general the weaker equality $F( üzü) = \text{Fit}( üzü)$ holds.

1 A characterisation of elements that stabilise a finite subseries

We start this section by introducing an associated ‘quotient series’ to a given pair $(üzü, U)$, where $ üzü$ is a series of a vector space $V$ and $ U \in üzü$. There is a natural associated series of subspaces in $V/U$, namely

$$üzü/U = \{ W + U/U : W \in üzü \}.$$ 

There is another associated series that can be defined with respect to any element $U$ of $ üzü$. This is the series

$$üzü \cap U : \{ W \in üzü : W \leq U \}$$

that is a series in $U$. We will need both these terms later on.

It is clear that if $g$ stabilises a finite subseries of $ üzü$ then $\langle g \rangle^{ üzü}$ is nilpotent. We will now see that the converse holds. In other words we will show that $F( üzü) = \text{Fit}( üzü)$. Let $E = S( üzü)$ and let $g$ be an element in $E$ where $\langle g \rangle^E$ is nilpotent. We will argue by contradiction and and assume that $g$ stabilises no finite subseries and show that $\langle g \rangle^E$ is not nilpotent. Notice that if $ U \in üzü$ then $S(üzü/U) = E/N$ where $N$ is the normal subgroup consisting of all the elements $h$ in $E$ where $V(h-1) \leq U$. If $\langle gN \rangle^{E/N}$ is not nilpotent then clearly the same is true for $\langle g \rangle^E$. Also if $ üzü$ is some subseries of $ üzü$ that is stabilised by $g$, then $S( üzü) \leq E$ and again we have that non-nilpotence of $\langle g \rangle^{ üzü}$ implies non-nilpotence of $\langle g \rangle^E$. In view of this we will first reduce our problem to a specific series where we can more easily obtain a contradiction. We need the
following lemma whose proof is derived from arguments uses in the proofs of Theorem 3.9 and Theorem 3.10 from [1].

**Lemma 1.1** Let $\mathcal{L}$ be a series in a vector space $V \neq 0$ and let $g \in HP(\mathcal{L})$. Then there exists $V \neq W \in \mathcal{L}$ such that $[V, g] \leq W$.

**Proof** We argue by contradiction and suppose that there is no such $W \in \mathcal{L}$. This is saying that there is no jump of the form $(W, V)$. As $g \in S(\mathcal{L})$ this implies that $V / U = \bigcup_{W \leq U < V} W / U$ for all $U \in \mathcal{L}$ such that $U < V$. In particular, taking $U = W_0 = 0$, we see that there is $W_0 < W_1 < W$ such that $[W_1, g] \neq 0$. Repeating this argument we get $W_1 < W_2 < V$ such that $[W_2, g] \leq W_1$ and continuing in this manner we get a strictly ascending sequence of subspaces $0 = W_0 < W_1 < W_2 < \ldots$, in $\mathcal{L}$ such that $[W_{i+1}, g] \leq W_i$ for $i \geq 0$. Notice that this implies that $g$ does not stabilise any subseries of $\mathcal{L} \cap W_n$ of length less than or equal to $n$. The same argument works for the series $\mathcal{L} / U$ for any $U \in \mathcal{L}$ such that $U < V$. We can thus come of with a strictly ascending chain of subspaces $0 = U_0 < U_1 < \ldots$, in $\mathcal{L}$ such that $g$ stabilises no subseries of $\mathcal{L} \cap U_n / U_{n-1}$ that is of length less than or equal to $n$. Let $M$ be a complement to $\bigcup U_i$. Let $A_i$ be a complement to $U_{i-1}$ in $U_i$ for $i \geq 1$ and for each $i$ fix an isomorphism $\sigma_i : U_i / U_{i-1} \rightarrow A_i$. These isomorphisms can be used to define an action of $g$ on each $A_i$. Moreover we can use the isomorphism $\sigma_i$ to transfer the series $\mathcal{L} \cap U_i / U_{i-1}$ into an equivalent series $S_i$ in $A_i$. As $g \in HP(\mathcal{L})$ we know from the introduction that $g$ acts unipotently on $V$. Suppose $(g - 1)^k = 0$. By Lemma 3.8 from [1], there exists for each $i > k + 2$, a $h_i \in S(S_i)$ such that $(gg^{h_i} - 1)^{(i-2)/k} - 1 \neq 0$ on $A_i$. We define an automorphism $h$ on $V = M \oplus \bigoplus_{i \leq k+2} A_i$ by setting $h = 1$ on $M \oplus \bigoplus_{i \leq k+2} A_i$ and $h = h_i$ on $A_i$ when $i > k + 2$. Now for each positive integer $m$ there is an $i$ such that $(gg^h - 1)^m$ acts non-trivially on $A_i$. Hence it follows that $gg^h$ does not act unipotently on $V$. But this contradicts the fact that all elements in $HP(\mathcal{L})$ act unipotently on $V$. □
For an element \( g \in \text{Fit}(\mathcal{L}) \) we say that \( g \) has unipotence degree \( n \) if \([v, n g] = 0\) for all \( v \in V \). Suppose that we have an element \( g \in \text{Fit}(\mathcal{L}) \) that does not stabilise any finite subseries of \( \mathcal{L} \) and pick \( g, V \) and \( \mathcal{L} \) such that the unipotence degree \( n \) of \( g \) is the smallest possible. Notice that \( n \geq 2 \).

We let \( V_0 = V \) and let

\[ V_1 = \bigcap\{U \in \mathcal{L} : [V_0, g] \leq U\}. \]

By Lemma 1.1 we know that \( V_1 < V_0 \). As \( g \) stabilises no finite subseries, we also have that \( V_1 > 0 \). Let \( g_1 \) be the restriction of \( g \) on \( V_1 \). Then it is not difficult to see that \( g_1 \) is in \( \text{HP}(\mathcal{L} \cap V_1) \) and clearly \( g_1 \) cannot stabilise any finite subseries of \( \mathcal{L} \cap V_1 \). We can thus apply Lemma 1.1 again. In this manner we obtain recursively a strictly descending sequence of subspaces such that

\[ V_{i+1} = \bigcap\{U \in \mathcal{L} : [V_i, g] \leq U\}. \]

Let \( U = \bigcap_{n \in \mathbb{N}} V_n \) and consider the subseries

\[ \{V_i/U : i \in \mathbb{N}\} \cup \{0\} \]

of \( \mathcal{L}/U \). If \( S(\mathcal{L}/U) = E/N \), then \( gN \) stabilises the new series.

**Theorem 1.2** Let \( \mathcal{L} \) be any series. We then have that \( F(\mathcal{L}) = \text{Fit}(\mathcal{L}) \).

**Proof** In view of what we said at the beginning of this section, we can assume without loss of generality that

\[ \mathcal{L} = \{V_i : i \in \mathbb{N}\} \cup \{0\} \]

where \( V_0 > V_1 > \ldots \) and \( \bigcap_{i \in \mathbb{N}} V_i = 0 \). Notice that we can’t have \([V_i, n g] = 0\) for any \( i \in \mathbb{N} \) since otherwise, by our choice of \( n \), there would be a finite subseries in \( V_i \) of \( \{V_j : j \geq i\} \cup \{0\} \) that is stabilised by \( g \) but this is not the case. We now construct a sequence of elements \( v_1, v_2, \ldots \) and a descending chain \( V = A_0 > A_1 > A_2 > \ldots \), with \( A_1, A_2, \ldots \in \mathcal{L} \), recursively such that

\[ [v_{i+n-2} g] \in A_{2i-1} \setminus A_{2i}, \quad [v_{i+n-1} g] \in A_{2i} \setminus A_{2i+1} \]

for \( i = 1, 2, \ldots \). To start with we let \( v_1 \in V \) such that \([v_{1, n-1} g] \neq 0\). Let

\[ A_1 = \bigcap\{V_i \in \mathcal{L} : [v_{1, n-2} g] \in V_i\} \]
and then
\[ A_2 = \bigcap \{ V_i \in \mathcal{L} : [v_{1,n-1} g] \in V_i \} \].

Next pick \( v_2 \in A_2 \) where \([v_{2,n-1} g] \neq 0\) and repeat the process by letting
\[ A_3 = \bigcap \{ V_i \in \mathcal{L} : [v_{2,n-2} g] \in V_i \} \]
and then
\[ A_4 = \bigcap \{ V_i \in \mathcal{L} : [v_{2,n-1} g] \in V_i \} \].

Continuing like this gives us the sequences of elements and subspaces we wanted.

Pick now an integer \( k = 1 + 2^m \) where \( m \) is arbitrarily large. Let
\[ W = W_0 \oplus W_1 \oplus \cdots \oplus W_k \]
be a complement to \( A_{k+1} \) such that \( A_i = W_i \oplus \cdots \oplus W_k \oplus A_{k+1} \). We pick a basis \( \beta_i \) for \( W_i \) such that \([v_{i,n-2} g] \in \beta_{i-1}\) and \([v_{i,n-1} g] \in \beta_i\). We let \( z : V \to V \) be the endomorphism that maps \( A_{k+1} \) to 0 and all the basis elements in \( \beta_0 \cup \cdots \cup \beta_k \) also to 0 apart from
\[ [v_{i,n-1} g]z = [v_{i+1,n-2} g] \]
for \( i = 1, \ldots, k \). Notice that \( V_j z \leq V_{j+1} \) for all \( j \in \mathbb{N} \) and thus \( \phi = \phi(m) = 1 + z \) stabilises \( \mathcal{L} \). Let \( x = g - 1 \) and let
\[ W = \langle [v_{i,n-2} g], [v_{i,n-1} g] : i = 1, 2, \ldots \rangle \]
Notice that \( W \) is \( z \) and \( x \) invariant and that the restrictions of \( z \) and \( x \) to \( W \) satisfy \( x^2 = z^2 = 0 \) (we will be using the same notation for these as we will be from now on working in \( W \) only). Let \( \phi_r = [\phi_r g] \). To finish the proof of the theorem we first prove a lemma.

**Lemma 1.3** We have
\[ x\phi_r^{-1}x = -x\phi_rx \]
and
\[ x\phi_{r+1}x = x\phi_rx\phi_rx \]
for all \( r \).
**Proof** Notice first that
\[ x\phi^{-1}_0 x = x(1 - z)x = -xzx = -x(1 + z)x = -x\phi_0 x \]

For \( r \geq 0 \) we then have
\[
x\phi_{r+1} x = x[\phi_r, g] x \\
= x\phi_r^{-1}(1 - x)\phi_r(1 + x) x \\
= -x\phi_r^{-1} x\phi_r x \\
= -x(1 - x)\phi_r^{-1}(1 + x)\phi_r x \\
= -x[g, \phi_r] x \\
= -x\phi_{r+1}^{-1} x.
\]

This proves the first claim. We use this to prove the second claim. We saw from the calculations above that
\[ x\phi_{r+1} x = -x\phi_r^{-1} x\phi_r x \]

and by the first claim this is equal to \( x\phi_r x\phi_r x \). \( \square \)

We can now finish the proof of Theorem 1.2. It follows from the previous lemma by induction that
\[ x\phi_r x = (x\phi_0)^{2^r} x = (x(1 + z))^{2^r} x = (xz)^{2^r} x \]

for all \( r \). Now
\[ [v_{1, n-2} g](xz)^{2^m} x = [v_{1+2^m, n-1} g] \neq 0. \]

This shows that
\[ 0 \neq x\phi_m x = x[\phi_m g] x \]

and hence \([\phi_m g] \neq 1\). For a given \( k = 1 + 2^m \) we can thus find \( \phi(m) \in E \) such that \( \langle g, g^{\phi(m)} \rangle \) is nilpotent of class at least \( m - 1 \). Thus
\[ \langle g, g^{\phi(1)}, g^{\phi(2)}, \ldots \rangle \]

is not nilpotent that gives the contradiction that \( \langle g \rangle^E \) is not nilpotent. This finishes the proof of Theorem 1.1. \( \square \)
2 Counterexamples for spaces of uncountable dimension

When $V$ is a vector space of countable dimension the authors in [1] proved the stronger result that $F(L) = HP(L)$. They then made the conjecture that this result holds for all vector spaces. In this section we give a general example that shows that the conjecture does not hold in any vector space of dimension greater than or equal to $2^{\aleph_0}$. Let $F$ be any field. Let $U = \bigoplus_{i=0}^{\infty} F u_i$ be a direct sum of countably many 1-dimensional vector spaces over $F$ and let $W = \prod_{i=0}^{\infty} F w_j$ be a cartesian sum of countably many 1-dimensional vector spaces over $F$. Then let

$$V = U \oplus W.$$ 

Remark. The dimension of $V$ is the cardinality of $W$ (for a proof see Lemma 1 of [2]). In particular if $F$ is one of the prime fields $\mathbb{Z}_p$ or $\mathbb{Q}$ then the dimension of $V$ is $2^{\aleph_0}$.

Now let $V_n = \bigoplus_{i=n}^{\infty} F u_i + \prod_{j=n}^{\infty} F w_j$. We consider then the series

$$\mathcal{L} = \{V_i : i \in \mathbb{N}\} \cup \{0\}.$$ 

Let $R$ be the subring of $\text{End}(V)$ consisting of those endomorphisms that map $V_i$ into $V_{i+1}$ for all $i \in \mathbb{N}$. Then $G = 1 + R$ is the stabiliser of $\mathcal{L}$.

Let $x : V \to V$ be the endomorphism that maps $u_i$ to $w_{i+1}$ for $i \in \mathbb{N}$ and maps $W$ to zero. Notice that $g = 1 + x$ is in $G$ and that $g$ stabilises no finite subseries of $\mathcal{L}$. It remains to see that $g \in HP(G)$. The following lemma will be a key result in establishing this.

**Lemma 2.1** Let $y \in R$. There exists a positive integer $m$ such that

$$w_j y \in W$$

for all $j \geq m$.

**Proof** We argue by contradiction and suppose that there is no such bound $m$. Pick some element $w_{k_1}$ such that $w_{k_1} y \not\in W$, say

$$w_{k_1} y = \alpha_{k_1+1} u_{k_1+1} + \cdots + \alpha_{k_1+m_1} u_{k_1+m_1} + w_1$$
Hirch-Plotkin radical of stability groups

where $\alpha_{k_1+m_1}^m \neq 0$ and $w_1 \in W \cap V_{k_1+1}$. Now pick $k_2 > k_1 + m_1$ such that $w_{k_2}y \not\in W$, say

$$w_{k_2}y = \alpha_{k_2+1}^1 u_{k_2+1} + \cdots + \alpha_{k_2+m_2}^2 u_{k_2+m_2} + w_2$$

where $\alpha_{k_2+m_2}^m \neq 0$ and $w_2 \in W \cap V_{k_2+1}$. Continuing in this manner we get a strictly ascending sequence of positive integers

$$k_1 < k_1 + m_1 < k_2 < k_2 + m_2 < \ldots < k_j < k_j + m_j < \ldots$$

and a sequence of elements $w_1, w_2, \ldots$ with $w_j \in W \cap V_{k_j+1}$ and where

$$w_{k_j}y = \alpha_{k_j+1}^1 u_{k_j+1} + \cdots + \alpha_{k_j+m_j}^j u_{k_j+m_j} + w_j$$

with $\alpha_{k_j+m_j}^j \neq 0$. Now let $u \in U$ and $w \in W$ be such that

$$(w_{k_1} + w_{k_2} + \cdots) y = u + w.$$ 

Suppose that $u \in Fu_1 + \cdots + Fu_{k_j+m_j-1}$. By our construction above we must have that, modulo $\bigoplus_{i=k_j+m_j}^\infty Fu_i$,

$$\alpha_{k_1+1}^1 u_{k_1+1} + \cdots + \alpha_{k_1+m_j}^1 u_{k_1+m_j} + \cdots + \alpha_{k_j+1}^1 u_{k_j+1} \cdots + \alpha_{k_j+m_j}^j u_{k_j+m_j} = u.$$ 

But this is absurd as $\alpha_{k_j+m_j}^j \neq 0$. $\square$

**Corollary 2.2** Let $y \in R$. There exists a positive integer $m$ (such that $wy \in W$) for all $w \in W \cap V_m$.

**Proof** Let $m$ be as in Lemma 2.1. We argue by contradiction and suppose that we have (for some $\alpha_j \in F$, $j = m, \ldots$)

$$(\alpha_m w_m + \alpha_{m+1} w_{m+1} + \cdots) y = u + w$$

for some $u \in U$ and $w \in W$ where $u \neq 0$. Let $k > m$ be a large enough positive integer such that $u \not\in V_k$. Calculating modulo $V_k$ we get

$$(\alpha_m w_m + \cdots + \alpha_{k-1} w_{k-1} + V_k) y = u + w + V_k$$
Hirsch-Plotkin radical of stability groups

where \( u \not\in W + V_k \). It follows that \( (\alpha_m w_m + \cdots + \alpha_{k-1} w_{k-1})y \not\in W \) that contradicts Lemma 2.1. \( \square \)

We want to use these results to show that \( \langle g \rangle^G \) is locally nilpotent. Let \( S = Fx + Rx + xR + RxR \) be the ideal in \( R \) generated by \( x \). Notice that

\[
\langle g \rangle^G \leq 1 + S.
\]

It thus suffices to show that \( 1 + S \) is locally nilpotent. Take some finitely many elements \( x_1, \ldots, x_n \) from \( 1 + S \), where

\[
x_i = 1 + f_i x + r_i x + s_i t_i + x t_i l_i
\]

with \( r_i, s_i, t_i, l_i \in R \) and \( f_i \in F \). By Corollary 2.2, we know that there exists a positive integer \( m \) such that \( r_i, s_i, t_i, l_i \) leave \( V_m \cap W \) invariant for \( i = 1, \ldots, n \). Let \( T \) be the subring of \( R \) generated by \( x, r_i x, s_i t_i, t_i l_i, i = 1, \ldots, n \). To show that \( \langle x_1, \ldots, x_n \rangle \) is nilpotent, it suffices to show that \( T \) is nilpotent as a ring. In fact we show that \( T^{m+2} = 0 \). First notice that \( VT^m \leq V_m \). Now \( x \) maps \( V_m \) into \( V_m \cap W \) and, by our choice of \( m \), all the \( r_i, s_i, t_i, l_i \) leave \( V_m \cap W \) invariant. It follows that \( VT^{m+1} \leq V_m \cap W \). Now \( x \) maps \( V_m \cap W \) to zero and thus, by our choice of \( m \) again, we see that \( VT^{m+2} = 0 \). This shows that \( T \) is nilpotent. It follows that \( 1 + T \) is nilpotent and thus \( 1 + S \) is locally nilpotent. Hence \( \langle g \rangle^G \) is locally nilpotent.

**Remark.** When the underlying field \( F \) is one of the prime fields, we have an example of cardinality \( 2^{\aleph_0} \). Our example does however not address possible cardinals between \( \aleph_0 \) and \( 2^{\aleph_0} \). Thus we do not know what happens in this case in set theories where the continuum hypothesis doesn’t hold.

Although we do not have in general that \( F(\mathcal{L}) = HP(\mathcal{L}) \), the Hirsch-Plotkin radical of \( S(\mathcal{L}) \) satisfies a strong local nilpotence property. As we mentioned in the introduction, all the elements of \( HP(\mathcal{L}) \) act nilpotently on \( V \) through the commutator action. Thus for each \( h \in HP(\mathcal{L}) \) there exists a positive integer \( n = n(h) \) such that \( [v, n h] = 0 \) for all \( v \in V \). From this one can easily derive the following generalisation.

**Theorem 2.3** For each finitely generated subgroup \( H = \langle h_1, \ldots, h_n \rangle \) of \( HP(\mathcal{L}) \), there exists a positive integer \( d = d(H) \) such that \( [V, d H] = 0 \).
Hirsch-Plotkin radical of stability groups

Proof We know from [1] that for each word \( h = w(h_1, \ldots, h_n) \) in \( h_1, \ldots, h_n \) there exist a positive integer \( n = n(h) \) such that \([v, n, h] = 0\) for all \( v \in V \). Suppose that \( H \) is nilpotent of class \( c \) and let \( x \in VH \) and \( y = vh \) for some \( v \in V \). Then \([x, c, y] \in V \) and thus \([x, c+n(h), y] = 0\). This shows that for any \( v \in V \) the subgroup \( \langle v, H \rangle \) is a finitely generated Engel group. Furthermore if we let \( F = \langle x, x_1, \ldots, x_n \rangle \) be the largest group with \( \langle x \rangle^F \) abelian, where \( F/\langle x \rangle^F \) is isomorphic to \( H \) and where \( F \) satisfies the extra relations \([x, m(w(h_1, \ldots, h_n)), y] = 0\) for all words \( y = w(x_1, \ldots, x_n) \) in \( x_1, \ldots, x_n \), then \( F \) is a finitely generated Engel group and thus nilpotent by a classic result of Gruenberg [3]. Suppose that the class of \( F \) is \( d \). Then any \( \langle v, H \rangle \) is a homomorphic image of \( F \) and thus nilpotent of class at most \( d \) for all \( v \in V \). Hence \([v, d, H] = 0\) for all \( v \in V \). \( \Box \)

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References


