Complex Methods for Bernoulli Free-Boundary Problems In-The-Large

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Abstract

A Bernoulli free-boundary problem is one of finding domains in the plane on which a harmonic function simultaneously satisfies linear homogeneous Dirichlet and inhomogeneous Neumann boundary conditions. The boundary of such a domain is called the free boundary because it is not prescribed *a priori* and, despite appearances, the problem of determining free boundaries is nonlinear. Existence, multiplicity or uniqueness, and smoothness of free boundaries are important questions.

This paper examines an equivalence between Bernoulli free-boundary problems and equations for one real-valued function of one real variable. The equivalence is global in the sense that it involves neither a restriction on the amplitudes of solutions, nor on their smoothness. Since the equations in question have the gradient structure of an Euler-Lagrange equation, a Morse index can be assigned to free boundaries.

A significant observation is that these equations can be re-formulated as nonlinear Riemann-Hilbert problems and the theory of complex Hardy spaces in the unit disc is an important ingredient in the investigation.

Non-existence and regularity results, and bounds on solutions in terms of their Morse indices, are obtained. Interesting questions, about precisely how irregular a Bernoulli free boundary can be, are formulated; some of them remain open.

Contents

1. Introduction

We pursue in various directions the observation that the study of Bernoulli free-boundary problems from harmonic-function theory is equivalent to the study of a class of nonlinear pseudo-differential operator equations for periodic functions of one real variable and we examine in some detail its significance for the behaviour and properties of free boundaries.

The equations in question have gradient structure which can be used to define a Morse index for free boundaries. Moreover they can be formulated as nonlinear Riemann-Hilbert problems on the unit circle and the analysis makes extensive use of classical Hardy-space methods. Although they are non-local, because they involve the conjugation operator $\mathcal{C}$ which is a singular integral operator, they have a great deal of structure some of which we will uncover here. (Subject to a sign convention, $\mathcal{C}$ is the Hilbert transform on the unit circle; see the Appendix.)

This is not a perturbation theory and there are no *a priori* assumptions on the geometry of free boundaries, or, equivalently, on the amplitudes or smoothness of solutions of the equations under consideration. Indeed, the classical example from hydrodynamics [35], in which a Bernoulli free boundary is Lipschitz continuous but not C^1, and far from a straight line, is covered by the general theory.

We believe that this may be the first systematic account of these free-boundary problems. Our approach yields regularity theory for Bernoulli free boundaries and offers the opportunity to address questions about existence or non-existence, and the properties of solutions, using global analytical, variational and topological methods on a class of equations for functions of one real variable. We introduce a notion of non-singular solutions and show that non-singular solutions give rise to real-analytic free boundaries. Then we show that a set of non-singular solutions with bounded Morse index does not contain a singular solution in its closure. So the Morse indices of non-singular solutions grow without bound as they approach a singular solution.

We also examine how Bernoulli free-boundary problems, singular or not, arise in pairs, via a nonlinear duality connected with the Riemann-Hilbert theory (see [10] for a special case).

The prototypical free-boundary problem of the type under consideration is the one which arises from Bernoulli's theorem in the study of Stokes waves in hydrodynamics – this is how problems of this type get their name. In addition to placing recent results from that theory in a wider context, we have obtained some new insights and developed the technical competence and the motivation to address questions that are new even in the Stokes-wave setting [10, 33, 34, 37, 39]. The result of Section 3.3 below, which shows that a solution of our equation with denumerably many stagnation points gives rise to a Jordan curve in the plane, is a case in point. The equations under investigation generalize Babenko's equation [5], and the relation between this theory and a generalized Nekrasov equation [25] is explained. For other classes of Bernoulli problems, see [1, 2].

The rest of the paper is in three parts. Part 2 begins with a description of general Bernoulli free boundary problems and a discussion of the equivalent nonlinear equations. We are particularly concerned to ensure that the theory throughout covers the case when the boundaries, and therefore the solutions of these equations, are not smooth. This effort leads us to study solutions in the real Hardy class of functions on the unit circle and, in turn, to a Riemann-Hilbert formulation of the problem. As corollary of this line of enquiry a duality between two classes of problems, which we describe as of type (I) and of type

(II), is uncovered. This means that only type-(I) problems need be studied in the sequel.

Part 3 is devoted to the theory of type-(I) problems. We will see that a major question to be answered for type-(I) problems is how to ensure that solutions to the nonlinear equations lead to free boundaries that are non-self-intersecting. Even for smooth solutions this is a subtle issue [39]. For non-smooth free boundaries we obtain the same result, but only under the (weak) hypothesis that the set of points where the boundary is not smooth is a closed, countable set. The set of points where the boundary is not smooth is shown to be closed with zero measure, and we give estimates on its Minkowski dimension, but we do not yet know whether or not it is always countable.

We examine the Morse index of solutions at some length and show that a sequence of non-singular solutions (Definition 2.4) with bounded Morse index cannot converge to a solution that is singular. In this sense, singular solutions can be thought of as having infinite Morse index.

An elementary example is used to illustrate many aspects of the theory in Parts 2 and 3. This example is informative because general questions about it can be answered by explicit calculation, and its construction raises the prospect of finding other simple examples to illustrate different features of the general theory.

The proofs of all these results are given under the same headings in Part 4 and the notation is collected in an Appendix.

2. Bernoulli Free Boundaries

We begin with a Dirichlet boundary-value problem. Let Ω denote the domain below $\mathcal{S}$ in the (X, Y)-plane where

$$\mathcal{S} := \{(u(s), v(s)) : \ s \in \mathbb{R}\}, \tag{2.1a}$$

$$(u, v) \text{ is injective and absolutely continuous,} \tag{2.1b}$$

$$u'(s)^2 + v'(s)^2 > 0 \text{ for almost all } s, \tag{2.1c}$$

$$s \mapsto (u(s) - s, v(s)) \text{ is } 2\pi\text{–periodic,} \tag{2.1d}$$

and consider the problem of finding ψ with

$$\psi \in C(\overline{\Omega}) \cap C^2(\Omega), \tag{2.1e}$$

$$\Delta\psi = 0 \text{ in } \Omega, \tag{2.1f}$$

$$\psi \text{ is } 2\pi\text{–periodic in } X, \tag{2.1g}$$

$$\nabla\psi(X, Y) \to (0, 1) \text{ as } Y \to -\infty \ \text{ uniformly in } X, \tag{2.1h}$$

$$\psi \equiv 0 \text{ on } \mathcal{S}. \tag{2.1i}$$

By classical theory a solution always exists and, by Lemma 4.2, $\psi < 0$ and $\nabla\psi$ is nowhere zero on Ω. A Bernoulli free-boundary problem is one of determining those curves $\mathcal{S}$ with the property that the solution of this Dirichlet problem satisfies an additional inhomogeneous Neumann condition of the form

$$\frac{\partial \psi}{\partial n} = h(Y) \text{ almost everywhere on } \mathcal{S},$$

where h is given and n denotes the outward normal to Ω at points of $\mathcal{S}$.

As the outward normal derivative of ψ on $\mathcal{S}$ is non-negative, by the maximum principle, $h \geq 0$ on $\mathcal{S}$ is necessary for the existence of solutions. Because formally the tangential derivative of ψ is zero almost everywhere it is convenient to reformulate the Neumann condition as

$$|\nabla \psi(X,Y)|^2 = \lambda(Y) \quad \text{almost everywhere on } \mathcal{S}, \tag{2.1j}$$

where $\lambda = h^2$. We will consider only two types of functions λ:

(I) λ is continuous on $\mathcal{R}(v)$,

(II) λ is integrable and $1/\lambda$ is continuous on $\mathcal{R}(v)$,

where $\mathcal{R}(v)$ is the range of v, a compact interval. (A function λ can be of both types on $\mathcal{R}(v)$.) Moreover we suppose that λ is real-analytic on the open set of full measure where it is non-zero and finite. The real-analyticity hypothesis is made for technical convenience and our results have analogues for other classes of λ [26]. Note that λ may be unbounded.

An example: steady hydrodynamic waves

Problem such as (2.1) have their origins in steady hydrodynamic-wave theory in which $\lambda(Y) = 1-2gY$, where $g > 0$ is the acceleration due to gravity, ψ is the stream function, $(\psi_Y, -\psi_X)$ is the steady velocity field and the Dirichlet and Neumann boundary conditions (2.1i) and (2.1j) mean that $\mathcal{S}$ is a streamline at which the pressure in the flow is a constant. Starting with the work of Levi-Civita [21] and Nekrasov [25] in the 1920s, there has developed a substantial existence theory for some (but not all, see, for example, [6, 7, 36]) positive g and Corollary 2.9 (a) below shows that there are no non-trivial solutions when $g \leq 0$.

In hydrodynamics, a point on $\mathcal{S}$ where the velocity is zero is called a stagnation point. The existence of a wave with a stagnation point was predicted by Stokes [35] and established in [3, 28]. Lewy [22] showed that except at a stagnation point the profile of a Stokes waves is a real-analytic curve but it is not known whether a Stokes wave can have more than one stagnation point per (minimal) wavelength, and 'can there be uncountably many stagnation points?' is also an open question. On the positive side, the existence of a Stokes wave with a convex profile between stagnation points has recently been established [30].

2.1. The General Case

We begin the study of (2.1) by explaining the weak sense in which the trace of $\nabla\psi$ is expected to satisfy (2.1j).

Suppose we have a solution of (2.1). From Lemmas 4.2 and 4.3 we see that ψ has a harmonic conjugate $-\varphi$ in Ω for which $\varphi(X,Y)-X$ is 2π-periodic in X for $(X,Y)\in\Omega$. Moreover $\varphi+i\psi$ is a conformal bijection from Ω onto the lower half $(x+iy)$-plane $\mathbb{C}_-$ which has a homeomorphic extension that maps $\overline{\Omega}$ onto the closed half plane. Let $Z:\overline{\mathbb{C}_-}\to\overline{\Omega}$ denote the inverse of $\varphi+i\psi$, let

$$\Pi_y=\{(X,Y)\in\overline{\Omega}:\psi(X,Y)=y\},\quad \Upsilon=\{(X,Y)\in\overline{\Omega}:\varphi(X,Y)\in(-\pi,\pi)\},$$

and let $S_y=\Pi_y\cap\Upsilon,\quad y\le 0$. Note that $S_0=\mathcal{S}\cap\Upsilon$. According to the sentence that follows (4.9) in the proof of Theorem 2.5, for each fixed $y\le 0$, the mapping

$$x\mapsto Z(x+iy)=:u(y;x)+iv(y;x),\quad x\in(-\pi,\pi),$$

gives an absolutely continuous parameterization of the curve S_y. Let

$$(u'(y;x)^2+v'(y;x)^2)^{1/2}\,dx=ds_y$$

denote the element of arc-length along S_y, where $'$ denotes differentiation with respect to x. As in (2.1), we use (u,v) and ds instead of $(u(0;\cdot),v(0,\cdot)$ and ds_0 for the parametrization of S_0.

The Neumann boundary condition (2.1j) is to be satisfied as follows:

$$\left.\begin{array}{c}|\nabla\psi(u(y;x),v(y;x))|^2-\lambda(v(x))\to 0 \text{ almost everywhere,}\\ \int_{S_y}|\nabla\psi|\log^+|\nabla\psi|\,ds_y\to\int_{S_0}\sqrt{\lambda}\,\log^+\sqrt{\lambda}\,ds,\end{array}\right\}\tag{2.1j$'$}$$

as $y\nearrow 0$, where $\log^+\tau=\max\{\log\tau,\,0\}$, $\tau>0$. Note that (2.1j$'$) is satisfied if

$$|\nabla\psi(X,Y)|^2\to\lambda(Y)\text{ and }|\nabla\psi|\text{ is bounded as }(X,Y)\to\mathcal{S}\text{ in }\Omega,\tag{2.2}$$

and that (2.1j$'$) implies that

$$|\nabla\psi(u(y;\cdot),v(y;\cdot))|\log^+|\nabla\psi(u(y;\cdot),v(y;\cdot))|\to\sqrt{\lambda}\log^+\sqrt{\lambda}\text{ in }L^1_{2\pi}\text{ as }y\to 0.$$

Definition 2.1. *For a solution of* (2.1), *a point* $(X_0,Y_0)\in\mathcal{S}$ *is a stagnation point if* $\lambda(Y_0)=0$ *for type* (I), *or if* $\lambda(Y)\to\infty$ *as* $Y\to Y_0$ *for type* (II).

The description of $\mathcal{S}$ in (2.1) allows for the possibility of non-smooth $\mathcal{S}$ with many stagnation points. With less generality the treatment could be made simpler at the cost of being incomplete. Note that if λ is of type (I) on $\mathcal{R}(v)$ and there are no stagnation points, then λ is also of type (II) on $\mathcal{R}(v)$, and *vice versa*. As a consequence of results in Section 2.6, type-(I) and type-(II) problems can be paired in a form of duality, whether or not there are stagnation points.

Here is a summary of Theorems 2.6 and 2.7, which relate the free-boundary problem (2.1) to an operator equation for functions on the unit circle, for the two types of problem:

(I) There is a one-to-one correspondence between solutions $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ of (2.4) and solutions of (2.1) with $|\nabla\psi|$ bounded.

(II) There is a one-to-one correspondence between solutions $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ of (2.4) with $\lambda(w)w' \in \mathcal{H}^1_{\mathbb{R}}$ and solutions of (2.1) and (2.6).

2.2. Formulation as a Single Equation

We assume throughout that λ is of at least one of the two types in the previous section. It is shown in Theorem 2.5 that every solution of (2.1) give rise to solutions $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ of the equation

$$\lambda(w)\{w'^2 + (1 + \mathcal{C}w')^2\} = 1, \tag{2.3a}$$

in which $\mathcal{C}$ is the conjugation operator from harmonic analysis; see the Appendix. The extent to which this equation is equivalent to

$$\lambda(w)\big(1 + \mathcal{C}w'\big) + \mathcal{C}\big(\lambda(w)w'\big) = 1, \tag{2.3b}$$

which is a condition that must be satisfied by critical points of the functional $\mathcal{J}$ in Section 3.5, is less obvious. Even less obvious is the fact that a solution of the free-boundary problem (2.1) can be obtained from solutions of (2.3b) with at most countably many stagnation points (Definition 2.4) provided $\lambda \geq 0$ is of type (I) and satisfies (3.1). Under this hypothesis a solution of (2.3) defines a *non-self-intersecting* curve $\mathcal{S}$ by the formula (2.5) and (2.1) is satisfied in the region below $\mathcal{S}$.

We will see that if λ is increasing on $\mathcal{R}(v)$ there are no non-trivial solutions. In Section 2.6 it is shown that every solution of a problem of type (I) yields a solution of a problem of type (II), and *vice versa*. After this duality, which generalizes [10], has been observed, we focus on problems of type (I).

2.3. Equations

The following theorems explain how the two equations (2.3a) and (2.3b) are related. Standard notation and background material are collected in the Appendix. In brief, $\mathcal{H}^p_{\mathbb{C}}$, $0 < p \leq \infty$, is the usual Hardy class of holomorphic functions in the unit disc D; for $p \in [1, \infty]$, $\mathcal{H}^p_{\mathbb{R}}$ is the usual real Hardy class of real-valued 2π-periodic functions on $\mathbb{R}$ and $\mathcal{H}^{1,p}_{\mathbb{R}}$ is the class of absolutely continuous 2π-periodic functions u with weak derivative $u' \in \mathcal{H}^p_{\mathbb{R}}$. The Nevanlinna-Smirnov class of holomorphic functions on the unit disc is denoted by N^+ and U^* denotes the boundary values of a function $U \in N^+$. The outer function $\mathcal{O}(g)$ is defined in (A.3).

THEOREM 2.2. *For* $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ *let* $W \in \mathcal{H}^1_{\mathbb{C}}$ *be such that* $W^* = w' + i(1 + \mathcal{C}w')$.

(a) If w *satisfies* (2.3b) *then* $\lambda(w)w' \in \mathcal{H}^1_{\mathbb{R}}$ *if and only if* $\lambda(w)|W^*| \in L^1_{2\pi}$.

(b) Suppose that w *satisfies* (2.3b) *and that* $\lambda(w)w' \in \mathcal{H}^1_{\mathbb{R}}$. *Then* w *satisfies* (2.3a) *if and only if* $\lambda(w) \geq 0$. *In this case* $\log \lambda(w) \in L^1_{2\pi}$ *and* $W = i\mathcal{O}(1/\sqrt{\lambda(w)})$.

(c) Suppose w *satisfies* (2.3a), $1/W \in N^+$ *and* $\lambda(w)|W^*| \in L^1_{2\pi}$. *Then* w *satisfies* (2.3b).

Remark 4.6 expands on the hypothesis $\lambda(w)w' \in \mathcal{H}^1_{\mathbb{R}}$ used in the above theorem. The special case for type-(I) λ is the following generalization of [33, 37, 39].

COROLLARY 2.3. *Suppose that* λ *is of type* (I) *and* $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$. *Let* $W \in \mathcal{H}^1_{\mathbb{C}}$ *be such that* $W^* = w' + i(1 + \mathcal{C}w')$. *Then the following are equivalent.*

(i) w *satisfies* (2.3b) *and* $\lambda(w) \geq 0$;

(ii) $\log \lambda(w) \in L^1_{2\pi}$ *and* $W = i\mathcal{O}(1/\sqrt{\lambda(w)})$;

(iii) w *satisfies* (2.3a) *and* $1/W \in N^+$.

Note that formally (up to integrability of the integrands involved), every solution of (2.3b) satisfies the integrated form of (2.3a):

$$\int_{-\pi}^{\pi} \lambda(w)\big(w'^2 + (1 + \mathcal{C}w')^2\big) - 1\, dt = 0.$$

To see this, multiply (2.3b) by $1 + \mathcal{C}w'$ and integrate on $[-\pi, \pi]$. Since the integrand may not have one sign, this does not mean that (2.3a) follows from (2.3b). (In fact this question is closely related to the integrability of the integrands. See Theorem 3.2 (a) and (c).) The following is in line with Definition 2.1.

DEFINITION 2.4. *When* w *satisfies* (2.3) *and* $\lambda(w(t_0)) = 0$ *for type* (I), *or* $\lambda(w(t)) \to \infty$ *as* $t \to t_0$ *for type* (II), t_0 *is called a stagnation point of* w. *A solution* w *with stagnation points is called singular; otherwise* w *is non-singular. The set of all stagnation points of* w *is denoted by* $\mathcal{N}(w)$.

From the definition of the two types of λ and the continuity of $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$, it follows that $\mathcal{N}(w)$ is closed. For type (I) problems we will see, in Theorem 3.1, that it has zero measure and, in Section 3.2, that it has fractal dimension less than 1. The real-analyticity of λ away from its zeros implies that a non-singular solution is real-analytic. (This follows by the method of [37, Appendix].) For type (II) problems the same results follow by duality (Section 2.6). We finish this section with a simple example to which we will return to illustrate various aspects of the theory.

2.3.1. Example of (2.3) with Explicit Solutions

For $b \geq 0$ define

$$\lambda_b(w) = \frac{1}{1+b^2+2w}, \quad \text{for } w \text{ with } \ 1+b^2+2w \geq 0.$$

Let $w(t) = b\sin t$. Note that $0 < \lambda_b(w(t)) < \infty$ except when $t = -\pi/2$ mod 2π and $b = 1$. Since an explicit calculation yields

$$\big(w'(t)^2 + (1+\mathcal{C}w'(t))^2\big)\lambda_b(w(t)) \equiv 1,$$

w satisfies (2.3a) with $\lambda = \lambda_b$. Now consider the holomorphic function W defined on the unit disc by $W(z) = i + bz$ so that

$$W^*(t) = b\cos t + i(1+b\sin t) = w'(t) + i(1+\mathcal{C}w'(t)).$$

When $b \leq 1$, $1/W \in N^+$ and $\lambda_b(w)|W^*| \in L^1_{2\pi}$. Hence, by Theorem 2.2 (c), w satisfies (2.3b) with $\lambda = \lambda_b$. In this example the profile of $\mathcal{S}$ (see (2.5)) is

$$\mathcal{S} = \{-t + b\cos t, b\sin t) : t \in \mathbb{R}\},$$

which is the locus of a point on a circle of radius b rolling on a horizontal line with unit velocity. Although w is a real-analytic function for all $b > 0$, when $b = 1$ there is a stagnation point at $t = -\pi/2$ corresponding to an downward-pointing cusp on $\mathcal{S}$.

When $b > 1$, equation (2.3a) is still satisfied and λ_b is positive and finite on $\mathcal{R}(w)$. However, W has a zero in the unit disc and hence (2.3b) is not satisfied when $b > 1$, because of Theorem 2.2 (b) or Corollary 2.3 (ii). Because of the results of the next section, this example shows that a non-singular solution of (2.3a) alone may correspond neither to a solution of (2.3b) nor of (2.1).

This example can be checked by direct calculation as follows. For $z \in D$ (the open unit disc in $\mathbb{C}$) and let $f(z) = bz/(1-ibz)$ so that

$$f^*(t) = \frac{b\cos t}{1+b^2+2b\sin t} + i\frac{b^2+b\sin t}{1+b^2+2b\sin t}.$$

Therefore, when $|b| \leq 1$

$$\mathcal{C}(\lambda_b(w)w') = \mathcal{C}\frac{b\cos t}{1+b^2+2b\sin t} = \frac{b^2+b\sin t}{1+b^2+2b\sin t},$$

and (2.3b) follows.

2.4. Equivalence

The first observation relates (2.1) to holomorphic functions on the unit disc.

THEOREM 2.5. *Let u, v, ψ be a solution of* (2.1) *and let φ be a harmonic conjugate of $-\psi$. Then $\varphi + i\psi$ is an injective conformal mapping of Ω onto the open lower half-plane which has an extension as a homeomorphism from $\overline{\Omega}$ onto the closed lower half-plane. Let Z be its inverse (a homeomorphism of the closed lower half-plane onto $\overline{\Omega}$). Then $t \mapsto Z(iy - t)$ is an absolutely continuous parametrization of the curve $\{(X, Y) : \psi(X, Y) = y\}$ for all $y \leq 0$. Let*

$$w(t) = \mathrm{Im}\ Z(-t), \quad t \in \mathbb{R}.$$

Then $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ satisfies (2.3a) *and $1/W \in N^+$ where $W \in \mathcal{H}^1_{\mathbb{C}}$ is such that $W^* = w' + i(1 + \mathcal{C}w')$.*

The main results on equivalence are the following.

THEOREM 2.6. *Suppose λ in* (2.1) *is of type* (I). *Then, in addition to the conclusions of Theorem 2.5,*

$$w \textit{ satisfies } (2.3\text{b}); \tag{2.4a}$$

$$\lambda(w) \geq 0; \tag{2.4b}$$

$$t \mapsto (-(t + \mathcal{C}w(t)), w(t)) \textit{ injective on } \mathbb{R}. \tag{2.4c}$$

Conversely, suppose that $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ satisfies (2.4). *Let*

$$(u(t), v(t)) = (-t - \mathcal{C}w(t), w(t)), \quad t \in \mathbb{R},$$

$$\mathcal{S} = \{(-(t + \mathcal{C}w(t)), w(t)) : \ t \in \mathbb{R}\}, \tag{2.5}$$

and let Ω be the open domain below $\mathcal{S}$. There exists a conformal mapping ω of Ω onto $\mathbb{C}^-$ such that $\mathcal{S}$, ψ, u, v give a solution of (2.1), *where $\psi = \mathrm{Im}\,\omega$, and $|\nabla\psi|$ is bounded on Ω.*

THEOREM 2.7. *Suppose that λ in* (2.1) *is of type* (II) *and, in* (2.1), *that $\lambda(v)$ is integrable over one period of $\mathcal{S}$ with respect to arc-length:*

$$\int_{S_0} \lambda(v)\, ds = \int_{-\pi}^{\pi} \lambda(v(t))\big(u'(t)^2 + v'(t)^2\big)^{1/2}\, dt < \infty. \tag{2.6}$$

Then, in addition to the conclusions of Theorem 2.5, w satisfies (2.4) *and $\lambda(w)w' \in \mathcal{H}^1_{\mathbb{R}}$.*

Conversely, suppose that $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ is a solution of (2.4) *and $\lambda(w)w' \in \mathcal{H}^1_{\mathbb{R}}$. Then $\mathcal{S}$, ψ, u, v, defined as above, give a solution of* (2.1) *that satisfies* (2.6).

2.5. Inequalities

Here we give simple identities and inequalities for λ of both types which lead to criteria for non-existence of solutions. Suppose that $u \in \mathcal{H}^{1,1}_{\mathbb{R}}$ with $\lambda(u)u' \in L^1_{2\pi}$ is such that, for almost all t, λ is differentiable at $u(t)$. Let

$$\mathfrak{L}(u)(t) = \lambda(u(t))\mathcal{C}u'(t) - \mathcal{C}\big(\lambda(u)u'\big)(t), \quad t \in \mathbb{R}. \tag{2.7}$$

The proof of [38, Prop. 3] shows that, if λ is non-decreasing, then $\mathfrak{L}(u)(t) \geq 0$ almost everywhere and $\mathfrak{L}(u)$ is zero on a set of positive measure if and only if u is a constant in which case $\mathfrak{L}(u) \equiv 0$.

Theorem 2.8. *Suppose that $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ is a solution of* (2.3) *and $\lambda(w)w' \in \mathcal{H}^1_{\mathbb{R}}$. Then*

$$\mathfrak{L}(w) + \lambda(w)\{w'^2 + (\mathcal{C}w')^2\} = 0, \tag{2.8}$$

$$\mathfrak{L}(w) \leq 0 \textit{ almost everywhere}, \tag{2.9}$$

$$\int_{-\pi}^{\pi} \big(1 - \lambda(w)\big)dt \leq 0 \quad \textit{and} \quad \int_{-\pi}^{\pi} \lambda(w)\mathcal{C}w'\,dt \leq 0, \tag{2.10}$$

$$0 \leq \int_{-\pi}^{\pi} \frac{1}{\lambda(w)} - 1\,dt. \tag{2.11}$$

For a related identity, see (3.4).

Corollary 2.9. *Suppose that $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ is a solution of* (2.3).
(a) If λ is non-decreasing on $\mathcal{R}(w)$ then w is a constant and $\lambda(w) = 1$.
(b) If $\lambda(w) \geq 1$, or $\lambda(w) \leq 1$, on the whole interval $[-\pi, \pi]$, then w is constant and $\lambda(w) = 1$.

Proof. (a) If λ is non-decreasing on $\mathcal{R}(w)$, then $\mathfrak{L}(w) \geq 0$ almost everywhere on $[-\pi, \pi]$ by the proof of [38, Prop. 3]. It follows from (2.3a) that $\lambda(w) > 0$ almost everywhere and hence, by (2.8), $w'^2 + (\mathcal{C}w')^2 = 0$ almost everywhere. Therefore w is constant and (2.3b) implies $\lambda(w) = 1$. Part (b) is immediate from (2.10), (2.11) and part (a). □

2.6. Duality

Next we explain how type-(I) and type-(II) problems are related. It simplifies matters, and there is no loss of generality if we assume that $w(0) = 0$. (If not, let $a = w(0)$, replace w by $w - a$ and replace the nonlinearity λ by λ_a, where $\lambda_a(x) = \lambda(x + a)$.)

Theorem 2.10. *(a) Suppose that*
(i) λ is of type (I) on $\mathcal{R}(w)$ and $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ is a solution of (2.3) *with $w(0) = 0$. Let $\widetilde{w}(t) = -\int_0^t \lambda(w(x))w'(x)dx$. Then*

(ii) for some $\widetilde{\lambda}$ *of type* (II) *on* $\mathcal{R}(\widetilde{w})$, $\widetilde{w} \in \mathcal{H}^{1,1}_{\mathbb{R}}$ *is a solution of* (2.3) *with* $\widetilde{\lambda}$ *instead of* λ, $\widetilde{\lambda}(\widetilde{w})\widetilde{w}' \in \mathcal{H}^1_{\mathbb{R}}$ *and* $\widetilde{w}(0) = 0$.

(b) Conversely, if $\widetilde{w}$ *and* $\widetilde{\lambda}$ *are as in (ii), let* $w(t) = -\int_0^t \widetilde{\lambda}(\widetilde{w}(x))\widetilde{w}'(x)dx$. *Then* w *satisfies (i) for some* λ.

In both cases

$$\lambda(w(t))\widetilde{\lambda}(\widetilde{w}(t)) \equiv 1. \tag{2.12}$$

2.6.1. Example of (2.3) with Explicit Solutions: Duality

In the notation of Section 2.3.1 let

$$\Lambda_b(w) = \frac{1}{2}\log\Big(\frac{1+b^2+2w}{1+b^2}\Big),$$

which is the primitive of λ_b with $\Lambda_b(0) = 0$. Let $\widetilde{\lambda}_b(u) = \widetilde{\Lambda}_b'(-u)$, where $\widetilde{\Lambda}_b$ is the inverse of Λ_b. Then $\widetilde{\lambda}_b(u) = (1+b^2)e^{-2u}$. As a consequence of the duality theory developed in Section 4.4

$$\widetilde{w}_b(t) = -\frac{1}{2}\log\Big(\frac{1+b^2+2b\sin t}{1+b^2}\Big), \quad 0 \le b < 1, \tag{2.13}$$

is a non-singular solution of (2.3) with $\lambda = \widetilde{\lambda}_b$. Note that $w(0) = \widetilde{w}(0) = 0$. However as $b \to 1$,

$$\widetilde{w}_b(t) \to \widetilde{w}(t) = -\tfrac{1}{2}\log(1+\sin t),$$

which is unbounded and hence not in $\mathcal{H}^{1,1}_{\mathbb{R}}$. Although $\widetilde{w}$ is not in the class under consideration, it gives the parametrization of two unbounded smooth curves which form an infinitely high upward-pointing cusp at $t = -\pi/2$.

3. Type-(I) Problems

Because of the duality between type-(I) and type-(II) problems, from now on we consider only problems of type (I).

3.1. Regularity

Without making any assumptions about the sign of $\lambda(w)$ we first observe how $\lambda(w) \neq 0$ relates to the regularity of solutions w of (2.3b).

THEOREM 3.1. *Let* $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ *be a solution of* (2.3b). *Then* $\log|\lambda(w)| \in L^1_{2\pi}$, $\lambda(w) > 0$ *on a set of positive measure, and* w *is real-analytic on the open set of full measure where* $\lambda(w) \neq 0$.

Next we observe how the behaviour of λ at its zeros in $\mathcal{R}(w)$ affects the smoothness of w.

THEOREM 3.2. *Let $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ be a solution of* (2.3b). *Suppose that $\varrho > 0$ is such that for all $x_0 \in \mathcal{R}(w)$ with $\lambda(x_0) = 0$,*

$$|\lambda(x)| \le \text{constant } |x - x_0|^\varrho \text{ for all } x \in \mathcal{R}(w).$$

Let

$$p(\varrho) = \frac{\varrho + 2}{\varrho} \quad \text{and} \quad r(\varrho) = \frac{\varrho + 2}{\varrho + 1}.$$

(a) The following are equivalent:

(i) $w \in W^{1,p(\varrho)}_{2\pi}$;
(ii) w is real-analytic on $\mathbb{R}$;
(iii) $\lambda(w) > 0$ on $\mathbb{R}$.

(b) The function w is real-analytic if

$$\lambda(w) \ge 0 \text{ and } -(1 + \mathcal{C}w') + iw' = \big| -(1 + \mathcal{C}w') + iw' \big| e^{i\vartheta},$$
$$\text{where } \vartheta = \vartheta_1 + \vartheta_2 \text{ with } \vartheta_1 \text{ continuous and } \|\vartheta_2\|_\infty < \pi/(2p(\varrho)).$$

(c) If $w \in W^{1,r(\varrho)}_{2\pi}$ then $\lambda(w) \ge 0$.

(d) If $\varrho = 0$, which amounts to no additional hypothesis since λ is continuous and $\mathcal{R}(w)$ is compact, then $\lambda(w) \ge 0$ if $w \in W^{1,2}_{2\pi}$.

When λ is Lipschitz, $\varrho = 1$, $p(1) = 3$ and $r(1) = 3/2$.

It is not known whether there are solutions of (2.3b) which do not satisfy (2.3a) for which $\lambda(w)$ changes sign (see [33]). There are however solutions w of (2.3) for which $\lambda(w)$ has zeros. Here is a family of examples.

Examples [30] For $\varrho \ge 1$ there exists a constant $\hat{g} = \hat{g}(\varrho) > 0$ and a solution

$$(\hat{\lambda}, \hat{w}) \in (0, \infty) \times \{\cap_{p<p(\varrho)} W^{1,p}_{2\pi}\}$$

of (2.3) with $\hat{\lambda}(\hat{w}) = |1 - 2\hat{g}\hat{w}|^\varrho > 0$ almost everywhere, for which $\hat{w}$ is not Lipschitz continuous at a discrete set of points $t \in \mathbb{R}$ where $1 - 2\hat{g}\hat{w}(t) = 0$. Although $\hat{w}$ is not a Lipschitz continuous function, it is nevertheless true that $\mathcal{S}$ in (2.5) is a Lipschitz, but not a C^1, curve that gives a solution of (2.1). □

3.1.1. Example of (2.3) with Explicit Solutions: Regularity

Here we simply observe that, when $b = 1$, Example 2.3.1 is of type (II) with a stagnation point at $t = -\pi/2$ even though w is real-analytic. This contrasts with the conclusion of Theorem 3.2 for type-(I) stagnation points.

3.2. Dimension of the Set of Stagnation Points

It follows from Theorem 3.1 that $\mathcal{N}(w)$ in Definition 2.4 has measure 0. The following result implies that its dimension is not greater than 2/3 if λ is Lipschitz continuous. Note that Minkowski dimension is sometimes refereed to as box dimension [14, page 42] and the lower Minkowski dimension, $\dim_M$, bounds the Hausdorff dimension from above, see [14, (3.17)] or [23, 5.3].

THEOREM 3.3. *Let $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ be a solution of* (2.3) *where λ is such that*

$$c|x - x_0|^\varrho \le \lambda(x) \le C|x - x_0|^\varrho, \quad c,\, C,\, \varrho > 0,$$

for all x in a neighbourhood of x_0 in $\mathcal{R}(w)$ when $\lambda(x_0) = 0$. Let $q(\varrho) = (\varrho+2)/2$ be the conjugate of $p(\varrho)$ in Theorem 3.2. Then

$$\dim_M \mathcal{N}(w) \le 1/q(\varrho).$$

If $w \in W^{1,p}_{2\pi}$, $p > 1$, then

$$\dim_M \mathcal{N}(w) \le 1 - (p/p(\varrho)), \quad 1 < p < p(\varrho), \ \text{and} \mathcal{N}(w) = \emptyset \ \text{when } p \ge p(\varrho).$$

3.3. Jordan Curves

If the functional $\mathcal{J}$ in Section 3.5, or its Euler-Lagrange equation (2.3b), is to be used in the study of free-boundary problems, it will be important to know that (2.1) is equivalent to (2.3) without further qualification. This section concerns problems of type (I) when w has at most countably many stagnation points and shows, under quite general hypotheses on λ, that (2.4c) follows automatically from (2.3) when (3.1) holds. In that case, (2.3) gives a solution of (2.1) with at most countably many stagnation points for which $\mathcal{S}$ is the graph of a function. Suppose that

$$\lambda \ge 0, \quad \log\lambda \text{ is non-constant, concave, and } \lambda' \le 0 \text{ where } \lambda \ne 0 \text{ on } \mathcal{R}(w). \tag{3.1}$$

Let

$$\vartheta(t) = \mathcal{C}\big(\log\sqrt{\lambda(w)}\big)(t), \quad t \in \mathbb{R}. \tag{3.2}$$

THEOREM 3.4. *If $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ satisfies* (2.3)*, then*

$$\sqrt{\lambda(w)}\, w' = \sin\vartheta \quad \text{and} \quad \sqrt{\lambda(w)}(1 + \mathcal{C}w') = \cos\vartheta. \tag{3.3}$$

Suppose that (3.1) *is satisfied on $\mathcal{R}(w)$ and that $\mathcal{N}(w)$ is at most denumerable. Then $\vartheta(t) \in \big(-\pi/2, \pi/2\big)$, and hence $1 + \mathcal{C}w' > 0$, almost everywhere. For smooth functions $1 + \mathcal{C}w' > 0$ everywhere.*

The hypotheses of this theorem are valid when $\lambda(w) = 1 - 2gw$ for any $g > 0$. Unfortunately even in that case it is not known whether the requirement that $\mathcal{N}(w)$ be denumerable is necessary. In fact no examples are known in which $\mathcal{N}(w) \cap [0, 2\pi)$ contains more than one point when $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ satisfies (2.3).

REMARK 3.5. *The following relates* (2.7) *to* (3.2) *and complements Theorem 2.8. Let* $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ *be a solution of* (2.3)*. Then we prove at the end of Section 4.7 that*

$$\mathfrak{L}(w) = \lambda(w)^{\frac{1}{2}}\big\{2\cos\vartheta - \lambda(w)^{\frac{1}{2}} - \lambda(w)^{-\frac{1}{2}}\big\}. \tag{3.4}$$

3.3.1. Example of (2.3) with Explicit Solutions: Jordan Curves

We observe that λ_b defined in Section 2.3.1 is decreasing but (3.1) is not satisfied because $\log\lambda_b$ is not concave for $1+b^2+2w>0$. On the other hand, from Section 2.6.1, $\widetilde{\lambda}_b(\widetilde{w}) = (1+b^2)e^{-2\widetilde{w}}$ and so $\log\widetilde{\lambda}_b$ is concave and decreasing for $\widetilde{w} \in \mathbb{R}$. Thus $\cos\widetilde{\vartheta}$, and hence $\cos\vartheta$, is positive almost everywhere (see (3.9)), because of the abstract theory applied to the dual problem, even though it does not apply to the original problem.

REMARK 3.6. *In this example* $\log\lambda_b$ *is not concave, but* $\log\widetilde{\lambda}_b$ *is concave. With this in mind, it is worth pointing out in general that concavity of one of the functions* $\log\widetilde{\lambda}$ *and* $\log\lambda$ *is enough to draw the conclusion that* $\mathcal{S}$ *in* (2.5) *is non-self-intersecting when* $\mathcal{N}(w)$ *is countable.*

3.4. Nekrasov's Equation

Let $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ be a solution of (2.3) and suppose that ϑ is defined in (3.2). By the proof of (4.21), $\vartheta \in L^p_{2\pi}$ for all finite p, ϑ is infinitely differentiable on the complement of a closed set of measure zero and, from its definition, ϑ has zero mean on $[-\pi,\pi]$. As in Section 2.6 we may suppose, without loss of generality, that $w(0)=0$. Define

$$\chi(x) = \int_0^x \sqrt{\lambda(y)}\,dy, \quad x \in \mathcal{R}(w).$$

Since, by Theorem 3.4,

$$\sqrt{\lambda(w(t))}w'(t) = \sin\vartheta(t),$$

and $w(0)=0$, it follows that

$$\chi(w(t)) = \int_0^t \sin\vartheta(s)\,ds. \tag{3.5}$$

Every solution of (2.3a) has $\lambda(w(t)) \neq 0$ for almost all t, w is continuous and there is no interval in $\mathcal{R}(w)$ on which λ is identically zero. Therefore χ is strictly monotone on $\mathcal{R}(w)$ and

$$w(t) = \chi^{-1}\Big(\int_0^t \sin\vartheta(s)\,ds\Big).$$

From Corollary 2.3, $W(0) = i$ and

$$1 = |W(0)| = \left|\mathcal{O}\left(1/\sqrt{\lambda(w)}\right)(0)\right|.$$

Consequently $\log \sqrt{\lambda(w)}$ has zero mean on $[-\pi, \pi]$ and therefore (see (3.2)) $\mathcal{C}\vartheta = -\log \sqrt{\lambda(w)}$ and

$$\mathcal{C}\vartheta = -\frac{1}{2} \log \lambda\Big(\chi^{-1}\big(\int_0^t \sin \vartheta(s)\, ds\big)\Big). \tag{3.6}$$

In the water-wave example $\lambda(y) = 1 - 2gy$. Let w be a solution of (2.3). To obtain the normalization that $w(0) = 0$ as in Section 2.6, let $a = w(0)$, $w_a(t) = w(t) - a$ and put $\lambda_a(y) = 1 - 2g(y + a)$. Then, in an obvious notation,

$$\chi_a(x) = \frac{1}{3g}\Big((1 - 2ga)^{3/2} - (1 - 2g(x + a))^{3/2}\Big). \tag{3.7}$$

It follows, from (3.5), (3.7) and the definitions of λ_a and w_a, that

$$\sqrt{\lambda(w(t))}^{\,3} = (1 - 2gw(t))^{3/2} = (1 - 2gw(0))^{3/2} - 3g \int_0^t \sin \vartheta_a(s)\, ds$$

where

$$\vartheta_a = \mathcal{C} \log \sqrt{\lambda_a(w_a)} = \mathcal{C} \log \sqrt{\lambda(w)} = \vartheta.$$

Then

$$\mathcal{C}\vartheta_a = -\log \sqrt{\lambda(w)}$$

and

$$\mathcal{C}\vartheta = -\frac{1}{3} \log \Big(\nu - 3g \int_0^t \sin \vartheta(s)\, ds\Big),$$

where $\nu = (1 - 2gw(0))^{3/2}$. Let $\vartheta = -\theta$. Upon differentiating, where that is allowed, we obtain the Nekrasov equation [25] from Stokes-wave theory :

$$\big(\mathcal{C}\theta\big)' = \frac{g \sin \theta}{\nu + 3g \int_0^t \sin \theta(s) ds}.$$

Here θ is the function that appears in [7, page 255, equation (3.17)]. Thus (3.6) extends Nekrasov's equations from water-wave theory [24] to general Bernoulli free-boundary problems.

Now we show how to obtain solutions of (2.3) from solutions to the generalized Nekrasov equation.

Theorem 3.7. *Suppose that $\vartheta \in L^\infty_{2\pi}$ is a solution of* (3.6) *with zero mean such that*

$$\Big|\lambda\big(\chi^{-1}\big(\int_0^t \sin \vartheta(s)\, ds\big)\big)\Big|^{-1/2} \textit{ is integrable.} \tag{3.8}$$

Let

$$w(t) = \chi^{-1}\Big(\int_0^t \sin\vartheta(s)\,ds\Big).$$

Then $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ *is a solution of* (2.3) *with*

$$w' + i(1+\mathcal{C}w') = i\exp\big(-i(\vartheta + i\mathcal{C}\vartheta)\big).$$

REMARK 3.8. *Hypothesis* (3.8) *is not required if*

$$\big(\textit{ess sup}\,\vartheta - \textit{ess inf}\,\vartheta\big) < \pi,$$

since it is then a consequence of (3.6) *and* [40, I, VII (2.11)].

3.4.1. Nekrasov Duality

It follows from (2.12) and (3.2) that

$$2\widetilde{\vartheta}(t) = \mathcal{C}\big(\log\widetilde{\lambda}(\widetilde{w})\big)(t) = -\mathcal{C}\big(\log\lambda(w)\big)(t) = -2\vartheta(t). \tag{3.9}$$

Since $w(0) = \widetilde{w}(0) = 0$, in an obvious notation,

$$\begin{aligned}-\widetilde{\chi}(\widetilde{w}(t)) &= -\int_0^{\widetilde{w}(t)} \sqrt{\widetilde{\lambda}(y)}\,dy = -\int_0^t \sqrt{\lambda(\widetilde{w}(s))}\widetilde{w}'(s)\,ds\\ &= \int_0^t \sqrt{\lambda(w(s))}w'(s)\,ds = \chi(w(t)),\ \text{by (4.13)}.\end{aligned}$$

This shows that $-\widetilde{\chi}(\mathcal{R}(\widetilde{w})) = \chi(\mathcal{R}(w))$ and, from (4.13), for y in this set,

$$\lambda(\chi^{-1}(y)) = \frac{1}{\widetilde{\lambda}(\widetilde{\chi}^{-1}(-y))}.$$

It now follows from (3.9) that, in the generalized Nekrasov equation (3.6),

$$-\frac{1}{2}\log\lambda\Big(\chi^{-1}\Big(\int_0^t \sin\vartheta ds\Big)\Big) = \frac{1}{2}\log\widetilde{\lambda}\Big(\widetilde{\chi}^{-1}\Big(\int_0^t \sin\widetilde{\vartheta} ds\Big)\Big). \tag{3.10}$$

That (3.9) is a statement of the duality of Section 2.6 in terms of the Nekrasov's independent variable ϑ is illustrated by the following example.

3.4.2. Example of (2.3) with Explicit Solutions: Nekrasov Duality

We illustrate this with the example in Section 2.3.1. When $\lambda(y) = \lambda_b(y) = 1/(1+b^2+2y)$, $b \in (0,1)$, it is easy to see, from explicit calculations using the definitions, that

$$-\frac{1}{2}\log\big(\lambda(\chi^{-1}(y)\big) = \log(\sqrt{1+b^2}+y),$$

and, with $\widehat{\lambda}(y) = \widehat{\lambda}_b(y) = (1+b^2)e^{-2y}$,

$$-\frac{1}{2}\log\big(\widehat{\lambda}(\widehat{\chi}^{-1}(y)\big) = -\log(\sqrt{1+b^2} - y).$$

This illustrates the equality (3.10). Differentiation of (3.6) in these cases yields the generalized Nekrasov equation

$$\big(\mathcal{C}\vartheta\big)' = \frac{\sin\vartheta}{\sqrt{1+b^2} + \int_0^t \sin\vartheta(s)ds}.$$

and its dual

$$\big(\mathcal{C}\widetilde{\vartheta}\big)' = \frac{\sin\widetilde{\vartheta}}{\sqrt{1+b^2} - \int_0^t \sin\widetilde{\vartheta}(s)ds}.$$

Note that each of these equations is obtained from the other by changing the sign of the dependent variable, and in that sense they are different versions of the same equation. Nevertheless they correspond to different Bernoulli problems (2.1), with distinct functions λ given by λ_b and $\widetilde{\lambda}_b$. In general, changing the sign of ϑ in a Nekrasov equation leads, surprisingly, to a solution of the dual problem.

3.5. Morse Index of Non-singular Solutions

We develop a Morse index theory for non-singular type-(I) solutions of (2.3). But since non-singular type-(II) solutions are also of type (I), there is no need to make the distinction in this part of the theory. It is almost trivial to confirm that critical points w of $\mathcal{J}$ defined on $\mathcal{H}^{1,1}_{\mathbb{R}}$ by

$$\mathcal{J}(w) = \int_{-\pi}^{\pi} \big\{\Lambda(w)\big(1+\mathcal{C}w'\big) - w\big\}\,dt, \quad w \in \mathcal{H}^{1,1}_{\mathbb{R}},$$

where λ is of type (I) and Λ is a primitive of λ, satisfy (2.3b). Despite the simple and attractive form of $\mathcal{J}$, the global variational theory remains largely unexplored. For the hydrodynamic example there is some variational theory [8, 9]; by contrast the theory based on topological degree and real-analyticity is extensive [6, 7]. For the case of general λ, the latter approach has been developed to some extent in [26].

Here we are interested in the Hilbert space theory of $\mathcal{J}$ considered as a functional on $W^{1,2}_{2\pi}$. The corresponding Euler-Lagrange equation is (2.3b):

$$\lambda(w)(1+\mathcal{C}w') + \mathcal{C}(\lambda(w)w') - 1 = 0, \quad w \in W^{1,2}_{2\pi}.$$

From Theorem 3.2 (d) and Corollary 2.3, every critical point of $\mathcal{J}$ on $W^{1,2}_{2\pi}$ has $\lambda(w) \geq 0$ and satisfies (2.3). For a non-singular solution w, the hypotheses on λ ensure that λ is real-analytic on $\mathcal{R}(w)$ and the linearization of the left side of (2.3b) is

$$L_w u = \{\lambda'(w)(1+\mathcal{C}w')u + \lambda(w)\mathcal{C}u' + \mathcal{C}\big(\lambda'(w)w'u + \lambda(w)u'\big)\}. \tag{3.11}$$

Clearly $L_w : W^{1,2}_{2\pi} \to L^2_{2\pi}$ and, for $u \in W^{1,2}_{2\pi}$,

$$Q_w(u) = \langle L_w u, u\rangle_{L^2_{2\pi}} = \int_{-\pi}^{\pi} \big\{\lambda'(w)(1+\mathcal{C}w')u^2 \;+\; 2\lambda(w)u\mathcal{C}u'\big\}dt \qquad (3.12)$$

defines a quadratic form on $W^{1,2}_{2\pi}$. For a non-singular solution w of (2.3b) we define the Morse index

$$\mathcal{M}(w) = \sup\big\{\dim E \,:\, Q_w(u) < 0,\; u \in E\setminus\{0\}\big\}, \qquad (3.13)$$

where E denotes a linear subspace of $W^{1,2}_{2\pi}$.

Following Plotnikov [29], in Section 4.10 it is shown that the Morse index (3.13) of a non-singular solution w of (2.3) is given by the much more convenient formula

$$\mathcal{M}(w) = \sup\big\{\dim E \,:\, \mathcal{Q}_w(u) < 0,\; u \in E\setminus\{0\}\big\}, \qquad (3.14)$$

where E denotes a linear subspace of $W^{1,2}_{2\pi}$,

$$\mathcal{Q}_w(u) := \int_{-\pi}^{\pi} \{u\mathcal{C}u' + qu^2\}dt$$

and, with ϑ defined by (3.2), the potential

$$q = -\vartheta' + \frac{\lambda'(w)}{2\lambda(w)}(1+\mathcal{C}w') = -\vartheta' + \frac{1}{2}\lambda'(w)\lambda(w)^{-3/2}\cos\vartheta. \qquad (3.15)$$

Observe that if w, a solution of (2.3), is replaced by $w_a = w - a$ while simultaneously λ is replaced $\lambda_a = \lambda(\cdot + a)$, $a \in \mathbb{R}$ (as at the beginning of Section 2.6), then w_a is a solution of (2.3) with λ_a in place of λ, and the potential q, determined by w and λ, is the same as the potential q_a, determined by w_a and λ_a.

Counted in increasing order of size, the n^{th} distinct eigenvalue of $u \mapsto \mathcal{C}u'$ is $n-1$, $n \in \mathbb{N}$, with multiplicity 2 when $n \geq 2$ and multiplicity 1 when $n = 1$. This observation gives the Morse index of a non-singular solution when the potential $q \equiv q_c$, a constant (but see Theorem 3.13). In that case $\mathcal{M}(w) = 2n+1$ if and only if $-(n+1) \leq q_c < -n$, n a non-negative integer; otherwise the Morse index is zero. When q is not constant, it gives simple bounds on the Morse index as follows.

Lemma 3.9. *For any non-singular solution w of* (2.3) *with q not constant and n, m integers,*

$$q_{\min} \geq -n \Rightarrow \mathcal{M}(w) \leq 2n-1, \qquad (3.16)$$

$$q_{\max} \leq -m \Rightarrow \mathcal{M}(w) \geq 2m+1. \qquad (3.17)$$

Therefore, because of (4.35)*, if* (3.1) *holds there are no non-singular local minimizers w of $\mathcal{J}$. (From the remark preceding the lemma,* (3.16) *also holds for constant q.)*

REMARK 3.10. *It is easy to see that if* w *and* $\widetilde{w}$ *are solutions of dual problems with potentials* q *and* $\widetilde{q}$*, respectively, where* q *is given by* (3.15)*, then*

$$\begin{aligned}
\widetilde{q} &= -\widetilde{\vartheta}' + \frac{\widetilde{\lambda}'(\widetilde{w})}{2\widetilde{\lambda}(\widetilde{w})}(1+\mathcal{C}\widetilde{w}') = -\widetilde{\vartheta}' + \frac{1}{2}\widetilde{\lambda}'(\widetilde{w})\widetilde{\lambda}(\widetilde{w})^{-3/2}\cos\widetilde{\vartheta} \\
&= \vartheta' + \frac{1}{2}\lambda'(w)\lambda(w)^{-3/2}\cos\vartheta = \vartheta' + \frac{\lambda'(w)}{2\lambda(w)}(1+\mathcal{C}w') \qquad (3.18)\\
&= q + 2\vartheta'. \qquad (3.19)
\end{aligned}$$

This observation follows from (4.13) *and* (3.2) *since they imply* (3.9) *and that* $\lambda'(w(t))\lambda(w(t))^{-3/2} = \widetilde{\lambda}'(\widetilde{w}(t))\widetilde{\lambda}(\widetilde{w}(t))^{-3/2}$.

Define an operator R on 2π-periodic functions by $Rw(t) = w(-t)$. Then, by (A.1) and the definition of $\mathcal{C}$, $\mathcal{C}Rw = -R\mathcal{C}w$ and $\mathcal{C}(Rw)' = R(\mathcal{C}w')$. Hence w is a solution of (2.3) if and only if Rw is a solution of (2.3). Let ϑ_R be given by (3.2) with Rw instead of w. Then $\vartheta_R(t) = -\vartheta(-t) = -R\vartheta(t)$ and hence, in the same notation,

$$q_R(t) = -\vartheta'(-t) + \frac{1}{2}\frac{\lambda'(w(-t))}{\lambda(w(-t))^{3/2}}\cos\vartheta(-t) = q(-t) = Rq(t).$$

It follows from this and (3.14), or indeed from (3.11), that a non-singular solution w has the same Morse index as Rw. A more significant relation between Morse indices is the following.

THEOREM 3.11. *Suppose that* w *is a non-singular solution of* (2.3) *and let* $\widetilde{w}$ *denote the solution of the dual problem from Theorem 2.10. Then the Morse indices of* w *and* $\widetilde{w}$ *are equal.*

When λ is globally real-analytic, the linearized operator L_w is well-defined at any $W^{1,2}_{2\pi}$-solution w of (2.3b) irrespective of the smoothness of w. But the Morse index may be infinite if w is singular. The next result says that if the Morse indices of the elements of a set of non-singular solutions are bounded, then none of them is close to a singular solution. In fact, our result is for families of systems (2.3) with possibly different nonlinearities.

THEOREM 3.12. *Suppose that* (**H**) *from Section 4.11 is satisfied by a sequence* $\{w_k\}$ *of non-singular solutions of systems* (2.3) *with* $\{\mathcal{M}(w_k)\}$ *bounded. Then, for some* $\alpha > 0$,

$$\lambda_k(w_k(t)) \geq \alpha, \quad t \in [-\pi,\pi], \quad k \in \mathbb{N}.$$

3.5.1. Example of (2.3) with Explicit Solutions: Morse Index

We return to the example of Section 2.3.1 where, by (3.3),

$$w(t) = b\sin t, \quad w'(t) = b\cos t, \quad 1 + \mathcal{C}w'(t) = 1 + b\sin(t),$$
$$\tan\theta = \frac{w'}{1+\mathcal{C}w'} = \frac{b\cos t}{1+b\sin t}.$$

Therefore,

$$\vartheta'(t) = -(b\sin t + b^2)\lambda_b(w)$$

where $\lambda_b(w) = (1+b^2+2w)^{-1}$. Note also that $\lambda_b'(w) + 2\lambda_b^2(w) = 0$, whence

$$\frac{1}{2}\frac{\lambda_b'(w)}{\lambda_b(w)}\big(1+\mathcal{C}w'\big) = -\lambda_b(w)(1+b\sin t).$$

When substituted in (3.15) and (3.19) this gives

$$q \equiv \lambda_b(w)(b^2-1) \text{ and } \widetilde{q} \equiv -1.$$

This shows that in non-trivial examples q can be constant. But Theorem 3.13 shows that there is essentially only one case where this occurs.

It also follows, for the dual problem, that the Morse index of every solution (2.13), $0 \le b < 1$, of (2.3) when $\lambda = \widetilde{\lambda}_b$, is 1. Theorem 3.11 now implies that the Morse index of the solution $b\sin t$ of (2.3) with $\lambda = \lambda_b$, $b \in (0,1)$, is also 1. Since $q < 0$, $\|q\|_{L^1_{2\pi}} = 2\pi$ for all $b \in (0,1)$, by (3.19), but $\|q\|_{L^\infty_{2\pi}}$ is unbounded as $b \nearrow 1$.

Theorem 3.13. *Suppose that w is a non-constant, non-singular solution of (2.3) and that the corresponding potential q is a constant, $\alpha/2$ say. Then $\alpha < 0$ and there exists $\beta > 0$ such that $\lambda(w) = \beta e^{\alpha w}$.*

4. Proofs of Main Results

Standard results and definitions used below without citation are summarized in the Appendix.

4.1. Equations: proofs of Theorem 2.2 and Corollary 2.3

Proof Theorem 2.2

(a) This is immediate from equation (2.3b) and the definition of $\mathcal{H}^1_{\mathbb{R}}$.

(b) Let $u = \lambda(w)(1+\mathcal{C}w')$ where w satisfies (2.3b) with $\lambda(w)w' \in \mathcal{H}^1_{\mathbb{R}}$. Then $u = 1 - \mathcal{C}(\lambda(w)w') \in \mathcal{H}^1_{\mathbb{R}}$. Since $\lambda(w)w'$ has zero mean on $[-\pi,\pi]$ (see Remark 4.6), this can be written in complex notation as

$$u + i\mathcal{C}u = \lambda(w)\big(1+\mathcal{C}w' + iw'\big). \tag{4.1}$$

Let $W, U \in \mathcal{H}^1_{\mathbb{C}}$ be such that $U^* = -i(u + i\mathcal{C}u)$ and $W^* = w' + i(1 + \mathcal{C}w')$. Then (4.1) is equivalent to the following nonlinear Riemann-Hilbert problem

$$U^* = \lambda(w)\overline{W^*}, \quad U, W \in \mathcal{H}^1_{\mathbb{C}}. \tag{4.2}$$

LEMMA 4.1. *Suppose that* $\Phi, \Psi \in \mathcal{H}^1_{\mathbb{C}}$, $\Psi \not\equiv 0$, $a \in L^\infty_{2\pi}$, $a \geq 0$, *and* $\Psi^* = a\overline{\Phi^*}$. *Then* $\log a \in L^1_{2\pi}$, Φ *and* Ψ *are outer functions and*

$$a|\Phi^*|^2 = \Psi(0)\Phi(0) \quad \textit{almost everywhere.}$$

The same result holds if $1/a \in L^\infty_{2\pi}$.

Proof. It is clear that $\Psi \not\equiv 0$ implies $\Phi \not\equiv 0$ and $a \not\equiv 0$. Moreover, $\log|a| \in L^1_{2\pi}$, since $\log|\Psi^*|, \ \log|\Phi^*| \in L^1_{2\pi}$. Let $F := \Psi/\mathcal{O}(\sqrt{a})$, $G := \Phi\mathcal{O}(\sqrt{a})$. Then $F^* = \overline{G^*}$ almost everywhere, and $\Phi, \Psi \in \mathcal{H}^1_{\mathbb{C}}$, $\mathcal{O}(\sqrt{a}) \in \mathcal{H}^\infty_{\mathbb{C}}$ and $1/\mathcal{O}(\sqrt{a}) \in N^+$ imply $G \in \mathcal{H}^1_{\mathbb{C}}$ and $F \in N^+$. Since $F^* = \overline{G^*}$, it follows from Smirnov's theorem that $F \in \mathcal{H}^1_{\mathbb{C}}$. Hence $F \equiv \text{const} =: c \neq 0$, $\ G \equiv \overline{c}$ by Carleman's theorem (see, e.g., [20, 2°, E, Ch. III]). Thus $\Psi = c\mathcal{O}(\sqrt{a})$ and $\Phi = \overline{c}/\mathcal{O}(\sqrt{a}) = \overline{c}\,\mathcal{O}(\sqrt{1/a})$ are outer functions and

$$a|\Phi^*|^2 = \Psi^*\Phi^* = c\mathcal{O}^*(\sqrt{a})\,\overline{c}/\mathcal{O}^*(\sqrt{a}) = c\overline{c} = \Psi(0)\Phi(0) \quad \text{almost everywhere.}$$

To deal with the case when $1/a \in L^\infty_{2\pi}$, note that now $\Phi^* = (1/a)\overline{\Psi^*}$ and the previous result applies. □

To complete the proof of (b) suppose that (2.3b) holds, $\lambda(w)w' \in \mathcal{H}^1_{\mathbb{R}}$ and $\lambda(w) \geq 0$. Let $U, W \in \mathcal{H}^1_{\mathbb{C}}$ be as in (4.2). Since

$$\begin{aligned} U(0)W(0) &= \left(\frac{-i}{2\pi}\int_0^{2\pi} u(t)dt\right)\left(\frac{1}{2\pi}\int_0^{2\pi} w'(t)dt + i\right) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\Big(1 - \mathcal{C}(\lambda(w)w')(t)\Big)dt = 1, \end{aligned} \tag{4.3}$$

$U, W \not\equiv 0$ and, since λ is of type (I) or (II), Lemma 4.1 implies that W is an outer function and $\lambda(w)|W^*|^2 = 1$ almost everywhere. Thus (2.3a) holds. In fact, from the proof of Lemma 4.1, $W = c\mathcal{O}(1/\sqrt{\lambda(w)}$ for some $c \in \mathbb{C}$ and, since $W^* = w' + i(1 + \mathcal{C}w')$ it follows that $c = i$. Conversely $\lambda(w) \geq 0$ if w satisfies (2.3a).

(c) Suppose w satisfies (2.3a), $1/W \in N^+$ and $\lambda(w)|W^*| \in L^1_{2\pi}$. Then

$$1/W^* = \lambda(w)\overline{W^*} \in L^1_{2\pi}, \tag{4.4}$$

which implies that $1/W \in \mathcal{H}^1_{\mathbb{C}}$, since $1/W \in N^+$. Therefore there exist $u \in \mathcal{H}^1_{\mathbb{R}}$ and $\gamma \in \mathbb{R}$ such that $1/W^* = u + i(\gamma + \mathcal{C}u)$. Since $W(0) = i$ (cf. the proof of part (b)), $1/W(0) = -i$, and hence $\gamma = -1$. So $1/W^* = u + i(-1 + \mathcal{C}u)$, and (4.4) implies that $u = \lambda(w)w'$ and $\mathcal{C}u = 1 - \lambda(w)(1 + \mathcal{C}w')$. Thus (2.3b) is satisfied. □

Proof of Corollary 2.3

If λ is of type (I), then hypothesis $\lambda(w)w' \in \mathcal{H}^1_{\mathbb{R}}$ in part (b) and $\lambda(w)|W^*| \in L^1_{2\pi}$ in part (c) of the theorem follow automatically from the fact that $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ [18, 38]. That (i) implies (ii) follows from Theorem 2.2 (b). That (ii) implies (iii) is immediate from the properties of an outer function and the observation that $1/W = -i\mathcal{O}(\sqrt{\lambda(w)}) \in \mathcal{H}^\infty_{\mathbb{R}} \subset N^+$. That (iii) implies (i) is a consequence of Theorem 2.2 (c). □

4.2. Equivalence: proofs of Theorems 2.5, 2.6 and 2.7

We begin with technical observations in a little more generality than is required for the proofs. Let $\mathbb{C}_- = \{z = x + iy : x \in \mathbb{R},\, y < 0\}$, the open lower half plane.

LEMMA 4.2. *Let $\mathcal{S} = \{(u(t), v(t)) : t \in \mathbb{R}\}$ be a non-self-intersecting continuous curve such that $u(t) \to \pm\infty$ as $t \to \pm\infty$ and let Ω denote the domain below $\mathcal{S}$ in the (X, Y)-plane. Suppose $\psi \not\equiv 0$ is harmonic and bounded above in Ω, continuous in $\Omega \cup \mathcal{S}$ and $\psi \equiv 0$ on $\mathcal{S}$. Let φ be a harmonic conjugate of $-\psi$.*
Then $\varphi + i\psi$ is a conformal bijection from Ω onto the lower half-plane $\mathbb{C}_-$ which maps ∞ onto ∞. Moreover $\varphi + i\psi$ can be extended as a homeomorphism of the closure $\Omega \cup \mathcal{S}$ of Ω onto the closure $\mathbb{R} \cup \mathbb{C}_-$ of $\mathbb{C}_-$.

Proof. It follows from the maximum principle that $\psi \le 0$ in Ω. Indeed, let $\zeta \in \mathbb{C}$ be such that the distance from ζ to $\Omega \cup \mathcal{S}$ is greater than 1. Then for any $\varepsilon > 0$, the function $\psi_\varepsilon(X, Y) := \psi(X, Y) - \varepsilon \log |X + iY - \zeta|$ is harmonic in Ω, continuous in $\Omega \cup \mathcal{S}$, $\psi_\varepsilon \le 0$ on $\mathcal{S}$, and $\psi_\varepsilon(X, Y) \to -\infty$ as $X^2 + Y^2 \to +\infty$. Hence $\psi_\varepsilon \le 0$ in Ω by the maximum principle. Therefore $\psi(X, Y) \le \varepsilon \log |X + iY - \zeta|$ for all $(X, Y) \in \Omega$ and all $\varepsilon > 0$ which yields $\psi \le 0$ in Ω.

Let F be a conformal bijection of $\mathbb{C}_-$ onto Ω which maps ∞ onto ∞. Then F has an extension as a homeomorphism of $\mathbb{C}_- \cup \mathbb{R}$ onto $\Omega \cup \mathcal{S}$ (see [16, Ch. II, Section 3, Theorem 4]). Since $\psi \circ F$ is harmonic in $\mathbb{C}_-$, continuous in $\mathbb{C}_- \cup \mathbb{R}$ and non-positive, there exist a constant $c \ge 0$ and a Borel measure $\mu \ge 0$ on $\mathbb{R}$ such that (see [20, A, Ch. VI] or [4, Theorem 7.26])

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1 + t^2} < \infty \quad \text{and} \quad \psi \circ F(x + iy) = cy + \frac{1}{\pi} \int_{\mathbb{R}} \frac{y\, d\mu(t)}{(x - t)^2 + y^2}, \quad y < 0. \tag{4.5}$$

Let f be an arbitrary continuous function on $\mathbb{R}$ with bounded support $B \subset \mathbb{R}$ and note that

$$\frac{1}{\pi} \int_{\mathbb{R}} \Big| \frac{f(x)\, y}{(x - t)^2 + y^2} \Big| \, dx \le \text{const} \, \frac{1}{1 + t^2},$$

uniformly for $y \in [-1,0]$. Since $\psi \circ F(x+iy) \to 0$ uniformly for $x \in B$ as $y \to 0$,

$$\int_{\mathbb{R}} f(x)\psi \circ F(x+iy)dx \to 0 \quad \text{as} \quad y \nearrow 0.$$

From (4.5), Fubini's theorem and the dominated convergence theorem,

$$\begin{aligned} 0 &= \lim_{y \nearrow 0} \int_{\mathbb{R}} f(x) \frac{1}{\pi} \int_{\mathbb{R}} \frac{y\, d\mu(t)}{(x-t)^2+y^2} dx \\ &= \lim_{y \nearrow 0} \int_{\mathbb{R}} \left(\frac{1}{\pi} \int_{\mathbb{R}} f(x) \frac{y\, dx}{(x-t)^2+y^2} \right) d\mu(t) = \int_{\mathbb{R}} f(t) d\mu(t). \end{aligned}$$

Consequently $\mu \equiv 0$ and $\psi \circ F(x+iy) \equiv cy$ for some $c \in \mathbb{R}$. Since $\psi \not\equiv 0$ and $(\varphi + i\psi) \circ F$ is analytic on $\mathbb{C}_-$,

$$(\varphi + i\psi) \circ F(x+iy) \equiv c(x+iy) + c_1, \quad c \neq 0, \quad c_1 \in \mathbb{R}.$$

Thus the inverse of F is given by $F^{-1} = c^{-1}(\varphi - c_1 + i\psi)$. Therefore both $c^{-1}(\varphi - c_1 + i\psi)$ and $\varphi + i\psi$ are conformal mappings of Ω onto $\mathbb{C}_-$ which map ∞ onto ∞ and extend as homeomorphisms of $\Omega \cup \mathcal{S}$ onto $\mathbb{C}_- \cup \mathbb{R}$. □

Lemma 4.3. *Suppose the hypotheses of Lemma 4.2 and* (2.1d) *are satisfied. Then (a) ψ is 2π–periodic in X and there is a constant $\varpi > 0$ such that*

$$\varphi(X+2\pi, Y) = \varphi(X,Y) + \varpi \quad \textit{for all } (X,Y) \in \Omega \cup \mathcal{S}.$$

(b) If (2.1h) *is also satisfied, then $\varpi = 2\pi$ in part (a).*

Proof. (a) According to Lemma 4.2, $\omega := \varphi + i\psi$ is a bijective conformal mapping of Ω onto $\mathbb{C}_-$ which maps ∞ onto ∞. It follows from (2.1d) that $(X + 2\pi m, Y) \in \Omega$ for all $m \in \mathbb{Z}$ if $(X,Y) \in \Omega$. It is clear that $\omega_1(X+iY) := \omega(X+2\pi+iY)$ is also a bijective conformal mapping of Ω onto $\mathbb{C}_-$ which maps ∞ onto ∞. Hence $\omega_1 \circ \omega^{-1}$ is a conformal automorphism of $\mathbb{C}_-$ which maps ∞ onto ∞. Therefore there exist $\alpha > 0$ and $\varpi \in \mathbb{R}$ such that $\omega_1 \circ \omega^{-1}(z) = \alpha z + \varpi$ for all $z \in \mathbb{C}_- \cup \mathbb{R}$. In other words,

$$\omega(X+2\pi+iY) = \alpha\omega(X+iY) + \varpi \quad \text{for all } (X,Y) \in \Omega \cup \mathcal{S}. \tag{4.6}$$

To show that $\alpha = 1$, suppose first that $0 < \alpha < 1$ and let $(X_0, Y_0) \in \Omega \cup \mathcal{S}$. It follows from (4.6), by induction, that

$$\begin{aligned} \omega(X_0 + 2m\pi + iY_0) &= \varpi \sum_{k=0}^{m-1} \alpha^k + \omega(X_0 + iY_0)\alpha^m \\ &= \varpi \frac{1-\alpha^m}{1-\alpha} + \omega(X_0+iY_0)\alpha^m, \quad \text{for all } m \in \mathbb{N}. \end{aligned}$$

Hence $\omega(X_0+2m\pi+iY_0)\to\varpi/(1-\alpha)$ as $m\to\infty$, which contradicts $\omega(\infty)=\infty$. Now suppose $\alpha>1$. Then (4.6) implies

$$\omega(X-2\pi+iY)=\omega(X+iY)/\alpha-\varpi/\alpha,\quad \text{for all } (X,Y)\in\Omega\cup\mathcal{S},$$

and so $\omega(X_0-2m\pi+iY_0)\to-\varpi/(\alpha-1)$ as $m\to\infty$, which contradicts $\omega(\infty)=\infty$. Thus $\alpha=1$ and

$$\omega(X+2\pi+iY)=\omega(X+iY)+\varpi \text{ for all } (X,Y)\in\Omega\cup\mathcal{S}.$$

Since the conformal mapping ω is one-to-one and preserves the positive directions of the boundaries of Ω and $\mathbb{C}_-$ [11, Section 137], $\varpi>0$ and *(a)* is proved.

(b) The Cauchy–Riemann equations and (2.1h) imply that $\partial_X\varphi(X,Y)\to 1$ as $Y\to-\infty$ uniformly in X. Hence

$$\begin{aligned}\varpi&=\lim_{Y\to-\infty}(\varphi(2\pi,Y)-\varphi(0,Y))\\&=\lim_{Y\to-\infty}\int_0^{2\pi}\partial_X\varphi(X,Y)dX=\int_0^{2\pi}1\,dt=2\pi.\end{aligned}$$

□

REMARK 4.4. *(a) It is easy to see that if $\psi\in C(\Omega\cup\mathcal{S})\cap C^2(\Omega)$ satisfies* (2.1g) *and* (2.1h)*, then it is bounded above. Conversely, Lemma 4.3(a) implies that if the conditions of Lemma 4.2 and* (2.1d) *are satisfied, then ψ is 2π–periodic in X.*

(b) If it is assumed that ψ satisfies (2.1g)*, then the property of φ in Lemma 4.3 (a) is more easily established. Indeed, if ψ is 2π–periodic in X, by the Cauchy–Riemann equations, so are the partial derivatives of φ. Therefore the partial derivatives of $\varphi(X+2\pi,Y)-\varphi(X,Y)$ are identically zero and $\varphi(X+2\pi,Y)-\varphi(X,Y)$ is a constant.*

Proof of Theorem 2.5

That $\varphi+i\psi$ leads to a homeomorphism from $\overline{\Omega}$ onto the closed half plane is the result of Lemma 4.2. To prove the rest of the theorem we use notation from the proof of Lemma 4.3. Since, by Lemma 4.3, $\omega(X+iY)-(X+iY)$ is 2π–periodic in X and (2.1h) holds, there exists a constant $C>0$ such that

$$\frac{1}{2}|X+iY|-C\le|\omega(X+iY)|\le 2|X+iY|+C \text{ for all } (X,Y)\in\Omega\cup\mathcal{S}.$$

Consequently

$$\frac{1}{2}|Z(z)|-C\le|z|\le 2|Z(z)|+C \text{ for all } z=x+iy\in\mathbb{C}_-\cup\mathbb{R}, \tag{4.7}$$

where $Z : \mathbb{C}_- \to \Omega$ is the conformal mapping which is the inverse of ω. Since, by Lemma 4.2, Z has a continuous extension onto $\mathbb{R} \cup \mathbb{C}_-$, Lemma 4.3 implies that $Z(z) - z$ is a 2π–periodic function of z in $\mathbb{R} \cup \mathbb{C}_-$. Moreover, the analytic function $V : D \setminus [-1, 0] \to \mathbb{C}$ (D is the open unit disc with centre 0 in $\mathbb{C}$) defined by

$$V(\zeta) = Z(i \log \zeta) - i \log \zeta, \tag{4.8}$$

has an analytic extension to $D \setminus \{0\}$. It follows from (4.7) that

$$|V(\zeta)| \leq 3|\log \zeta| + \text{constant} \quad \text{for all } \zeta \in D \setminus \{0\}.$$

Hence 0 is a removable singularity of V which therefore has an analytic extension to all of D. It is clear from Lemma 4.2 that V is continuous in $\overline{D}$. Let

$$w(t) := \text{Im } V^*(t) = \lim_{r \nearrow 1} V(re^{it}),$$

for all $t \in \mathbb{R}$. Then

$$-(\gamma + \mathcal{C}w(t)) + iw(t) = V^*(t) = Z(-t) + t, \quad t \in \mathbb{R},$$

where $\gamma \in \mathbb{R}$ is a constant. In particular, $w(t) = \text{Im } Z(-t)$, $t \in \mathbb{R}$. Since $\tau \mapsto Z(\tau)$, $\tau \in \mathbb{R}$, is a parametrization of the locally rectifiable curve $\mathcal{S}$ (see (2.1b)), V^* has bounded variation over $[0, 2\pi]$. Therefore $V' \in \mathcal{H}^1_{\mathbb{C}}$ and

$$\frac{d}{dt} V^*(t) = i \lim_{r \nearrow 1} r\, e^{it} V'(re^{it}) \;\; \text{almost everywhere} \tag{4.9}$$

(see [13, Section 3.4]). Thus $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ and $x \mapsto Z(x, y)$ is an absolutely continuous parametrization of the curve $\{(X, Y) : \psi(X, Y) \equiv y\}$ for all $y \leq 0$. Moreover,

$$\begin{aligned} -\mathcal{C}w'(t) + iw'(t) &= ie^{it} \lim_{r \nearrow 1} \Big(\frac{i}{\zeta} \big(Z'(i \log \zeta) - 1 \big) \Big) \Big|_{\zeta = re^{it}} \\ &= 1 - \lim_{y \nearrow 0} Z'(-t + iy), \end{aligned}$$

whence

$$-(1 + \mathcal{C}w'(t)) + iw'(t) = -\lim_{y \nearrow 0} Z'(-t + iy) \;\; \text{almost everywhere.} \tag{4.10}$$

The formula for the derivative of an inverse function and the Cauchy–Riemann equations imply

$$|Z'(-t + iy)|^{-2} = |\omega'(Z(-t + iy))|^2 = |\nabla \psi(Z(-t + iy))|^2.$$

Now (4.10) and (2.1j$'$) imply that $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ satisfies (2.3a) and (2.4b), and (2.4c) follows from (2.1b). Let $W \in \mathcal{H}^1_{\mathbb{C}}$ be the function which is analytic in D

such that $W^* = w' + i(1+\mathcal{C}w')$. Then $W(0) = i$ and it follows from (4.8)–(4.9) that

$$W(\zeta) = \zeta V'(\zeta) + i = \zeta \frac{i}{\zeta}(Z'(i\log\zeta) - 1) + i = iZ'(i\log\zeta), \quad \text{for all } \zeta \in D\setminus\{0\}.$$

(This formula for W in terms of Z follows directly from (4.10) if we invoke the Lusin–Privalov uniqueness theorem [20, D, Ch. III].) Since Z is a conformal mapping, $W(\zeta) \neq 0$ for all $\zeta \in D$ and

$$\frac{1}{|W(\zeta)|} = \frac{1}{|Z'(i\log\zeta)|} = |\omega'(Z(i\log\zeta))| = |\nabla\psi(Z(i\log\zeta))|. \tag{4.11}$$

Therefore, with $y = \log r$ for $r \in (0,1)$,

$$\begin{aligned}
\int_{-\pi}^{\pi} \log^+ \frac{1}{|W(re^{it})|}\, dt &= \int_{-\pi}^{\pi} \log^+ |\nabla\psi(Z(-t+iy))| dt \\
&= \int_{-\pi}^{\pi} |Z'(-t+iy)||\nabla\psi(Z(-t+iy)| \log^+ |\nabla\psi(Z(-t+iy))| dt \\
&= \int_{S_y} |\nabla\psi| \log^+ |\nabla\psi|\, ds_y \to \int_{S_0} \sqrt{\lambda} \log^+ \sqrt{\lambda}\, ds \text{ as } r \nearrow 1 \\
&= \int_{S_0} \frac{\log^+ |W^*|^{-1}}{|W^*|} ds = \int_{-\pi}^{\pi} \log^+ \frac{1}{|W^*(t)|}\, dt,
\end{aligned}$$

by (2.1j$'$), since w satisfies (2.3a). Therefore $1/W \in N^+$ by (A.2). This completes the proof of Theorem 2.5. $\square$

Proof of Theorem 2.6

If λ is of type (I), it is immediate from Corollary 2.3 and Theorem 2.5 that w is a solution of (2.3b). Hence (2.4) is satisfied.

To prove the converse suppose that w satisfies (2.4). Since w and $\mathcal{C}w$ are absolutely continuous, there exists a function V which is continuous on the closed unit disc $\overline{D}$, analytic in D and absolutely continuous on its boundary, with $V^*(t) = i(w(t) + i\mathcal{C}w(t))$, $t \in \mathbb{R}$. Let Z be defined in $\mathbb{C}_- \cup \mathbb{R}$ by

$$Z(x+iy) = x + iy + V(e^{-i(x+iy)}), \quad x \in \mathbb{R},\ y \leq 0. \tag{4.12}$$

It is clear that $Z(z) - z$ is a bounded continuous 2π–periodic function of z in $\mathbb{C}_- \cup \mathbb{R}$ which is analytic in $\mathbb{C}_-$ and locally absolutely continuous on $\mathbb{R}$. Further,

$$Z(x+iy) - (x+iy) \to V(0), \quad Z'(x+iy) \to 1,$$

as $y \to -\infty$, uniformly in x. Now (4.12) with $y = 0$ and $x = -t$ gives

$$Z(-t) = -t + V^*(t) = -(t + \mathcal{C}w(t)) + iw(t).$$

According to (2.4c), Z is injective on $\mathbb{R}$. Let

$$\mathcal{S} = \{Z(x) :\ x \in \mathbb{R}\} = \{Z(-t) :\ t \in \mathbb{R}\} = \{(-(t + \mathcal{C}w(t)), w(t)) :\ t \in \mathbb{R}\}\,.$$

It follows from [6, Lemma 2.2] that Z is a conformal bijection from $\mathbb{C}_-$ onto the region Ω below $\mathcal{S}$ and a homeomorphism of $\mathbb{C}_- \cup \mathbb{R}$ onto $\Omega \cup \mathcal{S}$. Let $\omega = \varphi + i\psi : \Omega \to \mathbb{C}_-$ be the inverse of this conformal mapping. Then $\omega(X+iY) - (X+iY)$ is 2π–periodic in X, bounded and continuous in $\Omega \cup \mathcal{S}$. It is clear that $\psi \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfies (2.1f), (2.1g), and (2.1i). The properties of Z and the formula for the derivative of the inverse function imply

$$\omega'(X+iY) \to 1 \quad \text{as } Y \to -\infty \quad \text{uniformly in } X,$$

and hence ψ satisfies (2.1h).

Since $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ satisfies (2.3b) with $\lambda(w) \geq 0$, it also satisfies (2.3a), by Corollary 2.3. Let $W \in \mathcal{H}^1_{\mathbb{C}}$ be such that $W^* = w' + i(1 + \mathcal{C}w')$. According to Corollary 2.3, W is an outer function. Hence it follows from (2.3a) that, almost everywhere,

$$\frac{1}{|W^*|} = \Big(w'^2 + (1 + \mathcal{C}w')^2\Big)^{-1/2} = \sqrt{\lambda(w)}$$

and, since λ is of type (I), $1/W \in H^\infty_{\mathbb{C}}$. Since (4.8) and (4.12) are equivalent, it follows, as for (4.11), that

$$|\nabla\psi(Z(x+iy))| = \frac{1}{|W(e^{-i(x+iy)})|}, \quad x \in \mathbb{R},\ y < 0.$$

Thus (2.2) holds and so (2.1j$'$) is satisfied. It remains to observe that (2.1b) and (2.1d) follow from the definition of $\mathcal{S}$, while (2.1c) is a consequence of (2.3a). This completes the proof of Theorem 2.6. □

Proof of Theorem 2.7

We indicate briefly the adjustments needed in the proof of Theorem 2.6. They amount to the use of Theorem 2.2 and hypothesis (2.6) here where there we used Corollary 2.3. When λ is of type (II), under the additional hypothesis that (2.6) holds, we find that $\lambda(w)|W^*| \in L^1_{2\pi}$. Now it follows from Theorem 2.2(c) that w, defined in Theorem 2.5, is a solution of (2.3b). Hence (2.4) holds and, by Theorem 2.2 (a), $\lambda(w)w' \in \mathcal{H}^1_{\mathbb{R}}$.

For the converse when λ is of type (II), $\lambda(w)w' \in \mathcal{H}^1_{\mathbb{R}}$ for solutions of (2.3b) is equivalent to $\lambda(w)|W^*| \in L^1_{2\pi}$. Therefore (2.6) holds in this case. Note that Theorem 2.2 (b) implies that W is an outer function and hence, as $r \nearrow 1$,

$$\int_{|z|=r} \log^+ |W| \to \int_{-\pi}^{\pi} \log^+ |W^*|\, dt.$$

Now (2.1j$'$) follows from a calculation similar to the one following (4.11). □

REMARK 4.5. *For any w, the example $\lambda(w) = 1 - 2gw$ from hydrodynamics is of type (I) on $\mathcal{R}(w)$ and* (2.2) *is natural from a physical viewpoint because it means that the fluid velocity is bounded irrespective of whether or not there is a stagnation point. Mathematically, this simplifies the proof of Theorem 2.6 by implying that W is an outer function, which allows application of Corollary 2.3 to show that w is a solution of* (2.3b). *In general, for functions λ of type (I) this simplification is available and there is a one-to-one correspondence between solutions of* (2.4) *and solutions of* (2.1) *and* (2.2).

Lemma 4.2 implies that $|\nabla\psi|$ is the modulus of the derivative of a conformal mapping of Ω onto $\mathbb{C}_-$. It follows from (2.1g), (2.1h) *and* (2.1j) *that $|\nabla\psi(X,Y)|$ is 2π–periodic in X, $|\nabla\psi(X,Y)| \to 1$ as $Y \to -\infty$ uniformly in X, and $|\nabla\psi|$ has a limit almost everywhere on $\mathcal{S}$, which is a 2π–periodic function. In the case when this limit is bounded one might think that the maximum modulus principle should imply the boundedness of $\nabla\psi$ in Ω. This matter is not so easy, and is related to the notion of a Smirnov domain.*

Let Ω_0 be the bounded inner domain of a closed rectifiable Jordan curve and let Z_0 be a conformal mapping of D onto Ω_0 . Then $Z_0' \in \mathcal{H}^1_{\mathbb{C}}$ (see, for example, [31, Theorem 6.8]) and Ω_0 is called a Smirnov domain [31, 7.1] if Z_0' is an outer function. By a result by Keldy̌s–Lavrentiev [31, 7.3] there exists a non–Smirnov domain Ω_0 with $|(Z_0')^*| = 1$ almost everywhere. In this case Z_0' is not an outer function and hence $1/|Z_0'|$ is not bounded. Consequently the modulus of the derivative of the inverse conformal mapping $\omega_0 : \Omega_0 \to D$ is not bounded, even though on $\partial\Omega_0$ it has non-tangential limit equal to 1 almost everywhere. See also [13, Notes on Ch. 10].

4.3. Inequalities: proof of Theorem 2.8

To obtain (2.8), note from (2.3) that

$$\begin{aligned}0 &\equiv \lambda(w)\{w'^2 + (1+\mathcal{C}w')^2\} - \{\lambda(w)(1+\mathcal{C}w') + \mathcal{C}(\lambda(w)w')\}\\ &= \lambda(w)\{w'^2 + (\mathcal{C}w')^2\} - \mathcal{C}(\lambda(w)w') + \lambda(w)\mathcal{C}w'\\ &= \lambda(w)\{w'^2 + (\mathcal{C}w')^2\} + \mathfrak{L}(w).\end{aligned}$$

Since, according to (2.3a), $\lambda(w) \geq 0$, (2.8) implies (2.9). The second inequality in (2.10) follows from (2.9) because $\lambda(w)\mathcal{C}w' = \mathfrak{L}(w) + \mathcal{C}(\lambda(w)w')$, while the first follows from integration of (2.3b) in the light of the second. Finally (2.11) follows by re-writing (2.3a) and integrating over the set $E_\epsilon = \{t : \epsilon^{-1} \geq \lambda(w(t) \geq \epsilon\}$, $\epsilon > 0$, to obtain, from Jensen's inequality, that

$$\Big(\frac{1}{|E_\epsilon|}\int_{E_\epsilon} 1 + \mathcal{C}w'\,dt\Big)^2 + \frac{1}{|E_\epsilon|}\int_{E_\epsilon} w'^2\,dt \leq \frac{1}{|E_\epsilon|}\int_{E_\epsilon}\frac{1}{\lambda(w)}\,dt.$$

When $\epsilon \searrow 0$, the result follows since the first term on the left tends to 1. □

4.4. Duality

In general, the composition of two absolutely continuous functions need not be absolutely continuous, see [17, Exercise (18.37)]. However, in the context of type-(I) and type-(II) functions λ, the following naïve remark is what we need.

REMARK 4.6. *Let w be absolutely continuous and 2π-periodic, and let Λ be a primitive of $-\lambda$ on $\mathcal{R}(w)$. When λ is of type (I), Λ is continuously differentiable. Therefore $\Lambda(w)$ is absolutely continuous and 2π-periodic, and $\lambda(w)w'$ is integrable with zero mean on $[-\pi,\pi]$.*

Suppose that λ is of type (II) on $\mathcal{R}(w)$ for some $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ and $\lambda(w)w' \in L^1_{2\pi}$. Let

$$\widetilde{w}(t) = -\int_0^t \lambda(w(s))w'(s)ds \text{ and } \Lambda(v) = -\int_0^v \lambda(u)du, \quad v \in \mathcal{R}(w).$$

To see that $\widetilde{w}(t) = \Lambda(w(t))$, let λ_k be a sequence of functions with

$$0 \le \lambda_k(u) \le \min\{k, \lambda(u)\} \text{ and } \lambda_k(u) \to \lambda(u) \text{ for almost all } u \in \mathcal{R}(w).$$

Since λ is integrable on $\mathcal{R}(w)$ and $\lambda(w)w' \in L^1_{2\pi}$, two applications of the dominated convergence theorem, with the observation that the composition of a Lipschitz continuous function and an absolutely continuous function is absolutely continuous and satisfies the chain rule (see Remark A.3), yields the required result:

$$\Lambda(w(t)) = -\lim_{k\to\infty}\int_0^{w(t)} \lambda_k(u)du = -\lim_{k\to\infty}\int_0^t \lambda_k(w(s))w'(s)ds = \widetilde{w}(t).$$

This shows that if λ is of type (I), or of type (II) and $\lambda(w)w' \in \mathcal{H}^1_{\mathbb{R}}$, then it is automatic that $\Lambda(w) \in \mathcal{H}^{1,1}_{\mathbb{R}}$ and $(\Lambda(w))' = -\lambda(w)w'$ has zero mean.

Proof of Theorem 2.10

Suppose that (i) holds. Let

$$\Lambda(u) = -\int_0^u \lambda(v)dv, \quad u \in \mathcal{R}(w).$$

Then Λ is continuously differentiable and $\Lambda(w) \in \mathcal{H}^{1,1}_{\mathbb{R}}$, by the main result of [18, 38], since $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$. So $\Lambda(w)$ is absolutely continuous and

$$\Lambda(w(t)) = -\int_0^t \lambda(w(s))w'(s)ds = \widetilde{w}(t).$$

Since, by (2.3), $\lambda(w) \ge 0$, it follows that Λ is strictly decreasing and continuous. Therefore its inverse, $\widetilde{\Lambda}$, is also strictly decreasing and continuous, and hence

$\widetilde{\Lambda}$ has a classical derivative almost everywhere on $\mathcal{R}(\widetilde{w})$. Let $\widetilde{\lambda}(u) = -\widetilde{\Lambda}'(u)$ at points $u \in \mathcal{R}(\widetilde{w})$ where it exists. Note that $\widetilde{\lambda} \geq 0$ and, although $\widetilde{\Lambda}$ need not be absolutely continuous, $\widetilde{\lambda}$ is integrable (see [19, Theorem 1, Chapter VI, Section 4]). Moreover, by the analytic version of the inverse function theorem, $\widetilde{\Lambda}$ is analytic at all points v where $v = \Lambda(u)$ and $\lambda(u) \neq 0$ and, at all such points v, $\widetilde{\lambda}(v) = 1/\lambda(u)$. Thus $\widetilde{\lambda}$ is of type (II) on $\mathcal{R}(\widetilde{w})$. According to Corollary 2.3 (ii), $\lambda(w(t)) \neq 0$ almost everywhere and therefore

$$\widetilde{\lambda}(\widetilde{w}(t)) = 1/\lambda(w(t)) \text{ for almost all } t \in \mathbb{R}. \tag{4.13a}$$

When this observation, and the fact that

$$\widetilde{w}'(t) = -\lambda(w(t))w'(t) \text{ for almost all } t \in \mathbb{R}, \tag{4.13b}$$

are substituted into (2.3b), we find that $\widetilde{w}$ also satisfies (2.3b), but with $\widetilde{\lambda}$ in place of λ.

That $\widetilde{w}$ satisfies (2.3a) now follows from Theorem 2.2 (b) since $\widetilde{\lambda}(\widetilde{w}) \geq 0$ almost everywhere. Hence (i) implies (ii).

If (ii) holds let

$$w(t) = -\int_0^t \widetilde{\lambda}(\widetilde{w}(s))\widetilde{w}'(s)ds \text{ and } \widetilde{\Lambda}(v) = -\int_0^v \widetilde{\lambda}(u)du, \quad v \in \mathcal{R}(\widetilde{w}).$$

Then w has period 2π and $\widetilde{\Lambda}$ is strictly decreasing and continuous. That $w(t) = \widetilde{\Lambda}(\widetilde{w}(t))$ follows from Remark 4.6. Now let Λ be the inverse of $\widetilde{\Lambda}$ and let $\lambda = -\Lambda'$, which is defined almost everywhere. As in the first part,

$$\lambda(u) = 1/\widetilde{\lambda}(v) \text{ when } \Lambda(u) = v \text{ and } |\widetilde{\lambda}(v)| \neq \infty.$$

Hence λ is of type (I) and $\lambda(w(t)) \geq 0$ almost everywhere. Since w is 2π-periodic and, from the definition of w, $w' = -\widetilde{\lambda}(\widetilde{w})\widetilde{w}' \in \mathcal{H}^1_{\mathbb{R}}$, by (ii), it is immediate that $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$. Since $|\widetilde{\lambda}(\widetilde{w})| < \infty$ almost everywhere it follows that λ, w satisfy (2.3b), and hence (2.3a), by Corollary 2.3. This completes the proof. □

4.5. Regularity: proofs of Theorems 3.1 and 3.2

We need an extension, to systems of equations rather than to a single equation, of a special case of Lewy's theorem [22].

Lemma 4.7. *Let* $D_r = \{x + iy : x^2 + y^2 < r^2,\ y > 0\} \subset \mathbb{C}$. *Suppose* $F = U + iV : D_r \to \mathbb{C}^m$ *is an analytic function such that* $U, V \in C^1(\overline{D_r}; \mathbb{R}^m)$ *and*

$$\frac{\partial U}{\partial y}(x,0) = \mathcal{A}(x, U(x,0), V(x,0)), \quad |x| < r, \tag{4.14}$$

where $\mathcal{A}$, an analytic $\mathbb{C}^m$-valued function of all its arguments in a neighbourhood of $(0, U(0,0), V(0,0))$ in $\mathbb{C}^{2m+1}$, takes values in $\mathbb{R}^m$ on $\mathbb{R}^{2m+1}$. Then there exists a disk $\widetilde{D}$ centred at $(0,0)$, and an analytic function $\widetilde{U} + i\widetilde{V} : \widetilde{D} \to \mathbb{C}^m$ such that

$$U + iV = \widetilde{U} + i\widetilde{V} \text{ on } D_r \cap \widetilde{D}.$$

Proof. The proof is almost identical to that of Lewy's theorem (see [22] or [36, Section 4]). In fact there is a slight simplification because the right-hand side of (4.14) does not depend on $U_x(x,0)$, and hence there is no need to use the implicit function theorem. $\square$

Proof of Theorem 3.1

From the first line of the proof of Lemma 4.1, (4.2) implies that $\log|\lambda(w)| \in L^1_{2\pi}$.

Suppose $\lambda(w) \le 0$ almost everywhere. Then it follows from (4.2) and Lemma 4.1 that $U(0)W(0) \le 0$. On the other hand, $U(0)W(0) = 1$ by (4.3). This contradiction shows that $\lambda(w) > 0$ on a set of positive measure.

Let I be an arbitrary open interval upon which the continuous function $\lambda(w)$ does not have zeros. It follows from (4.2) that the functions

$$F := U/\mathcal{O}\big(\sqrt{|\lambda(w)|}\big) \in N^+ \text{ and } G := W\mathcal{O}\big(\sqrt{|\lambda(w)|}\big) \in \mathcal{H}^1_{\mathbb{C}}$$

satisfy the equality $F^* = s\overline{G^*}$ almost everywhere on I, where $s := \operatorname{sgn}\{\lambda(w)\}$ is a constant ± 1 on I. As in the proof of Lemma 4.1 it follows from Carleman's theorem that F and G are analytic on I, as also is $FG = UW$. Hence $U^*W^* = \lambda(w)|W^*|^2$ is real-analytic on I.

Now re-write equation (2.3b) as

$$2\lambda(w)\mathcal{C}w' = \lambda(w)\mathcal{C}w' - \mathcal{C}\big(\lambda(w)w'\big) + 1 - \lambda(w) = \mathfrak{L}(w) + 1 - \lambda(w),$$

where $\mathfrak{L}(w)$ is defined by (2.7). A simple bootstrap argument, using the regularising properties of $\mathfrak{L}$ as in the proof of [37, Theorem A.1], yields that $w \in C^{1,\alpha}(I)$, $\alpha > 0$ (see also [26, Theorem 2.6]).

Further,

$$w' + i(1 + \mathcal{C}w') = W^* = \frac{G^*}{\mathcal{O}(\sqrt{|\lambda(w)|})^*} = \frac{G^*}{\sqrt{|\lambda(w)|}} \exp\Big(i\mathcal{C}\big(\log\frac{1}{\sqrt{|\lambda(w)|}}\big)\Big).$$

Let $g = -\log\sqrt{|\lambda(w)|}$. Then $g' = -\lambda'(w)w'/(2\lambda(w))$ and

$$\begin{aligned} w' &= e^g(\operatorname{Re} G^* \cos(\mathcal{C}g) - \operatorname{Im} G^* \sin(\mathcal{C}g)), \\ g' &= -\frac{s}{2}\,\lambda'(w)e^{3g}(\operatorname{Re} G^* \cos(\mathcal{C}g) - \operatorname{Im} G^* \sin(\mathcal{C}g)), \end{aligned}$$

where, as above, $s = \operatorname{sgn}\{\lambda(w)\}$. Since w is a $C^{1,\alpha}$-function on I, so is g, and hence $\mathcal{C}g$ (cf. [37, Appendix]). Since G is analytic on I, it follows from Lemma 4.7 that w and g are real-analytic on I. $\square$

To prove Theorem 3.2 we begin with a technical lemma.

LEMMA 4.8. *Let* $\alpha \in [0,1]$ *and* $a : \mathbb{R} \to \mathbb{R}$ *be a continuous* 2π*–periodic function such that*

$$|a(t)| \le \mathrm{const}\, \rho(t)^\alpha \quad \textit{for all} \quad t \in \mathbb{R},$$

where $\rho(t) := \mathrm{dist}(t, \mathcal{N}_a)$*,* $\mathcal{N}_a := \{s \in \mathbb{R} | \ a(s) = 0\}$*. Suppose that* a *changes sign. Then the Riemann-Hilbert problem*

$$\Phi^* = a\overline{\Psi^*}, \quad \Phi, \Psi \in \mathcal{H}^p_{\mathbb{C}}, \ 1 \le p \le \infty, \tag{4.15}$$

does not have solutions with $\Psi \not\equiv 0$ *if*

$$p \ge \frac{2}{1+\alpha}\,. \tag{4.16}$$

Proof. Suppose (4.16) is satisfied and (Φ, Ψ) is a solution of (4.15) with $\Psi \not\equiv 0$. Since $a \not\equiv 0$, we have $\Phi \not\equiv 0$ and $\log|a| = \log|\Phi^*| - \log|\Psi^*| \in L^1_{2\pi}$.

Let

$$E^\pm = \{t \in \mathbb{R} : \ \pm a(t) \ge 0\}.$$

The function $\rho(t)^\alpha$ (the composition of $\tau \mapsto \tau^\alpha$ and the Lipschitz continuous function $t \mapsto \rho(t)$) belongs to $C^\alpha_{2\pi}$. Let

$$a_0(t) := \rho(t)^\alpha s(t),$$

where $s(t) := \mathrm{sgn}\{a(t)\}$. Then a_0 changes sign and $a_0 \in C^\alpha_{2\pi}$ because, for $t_1, t_2 \in E^\pm$, since $\alpha \in [0,1]$,

$$|a_0(t_1) - a_0(t_2)| = |\rho(t_1)^\alpha - \rho(t_2)^\alpha| \le \mathrm{const}\ |t_1 - t_2|^\alpha.$$

If $t_1 \in E^+$ and $t_2 \in E^-$, then there exists $t_0 \in \mathcal{N}_a$ between t_1 and t_2, and

$$\begin{aligned} |a_0(t_1) - a_0(t_2)| &\le |a_0(t_1) - a_0(t_0)| + |a_0(t_2) - a_0(t_0)| \\ &\le \mathrm{const}\ (|t_1 - t_0|^\alpha + |t_2 - t_0|^\alpha) \le \mathrm{const}\ |t_1 - t_2|^\alpha. \end{aligned}$$

It is clear that $a = a_0 h$, where $h(t) := |a(t)|/\rho(t)^\alpha \ge 0$, $h \in L^\infty_{2\pi}$ and $\log h \in L^1_{2\pi}$. Let

$$F := \frac{\Phi}{\mathcal{O}\big(\sqrt{h}\big)}\,, \quad G := \mathcal{O}\big(\sqrt{h}\big)\Psi.$$

Then

$$F^* = a_0\overline{G^*}. \tag{4.17}$$

Since $\mathcal{O}\big(\sqrt{h}\big) \in \mathcal{H}^\infty_{\mathbb{C}}$ is an outer function, $G \in \mathcal{H}^p_{\mathbb{C}}$ and $F \in N^+$. Hence it follows from Smirnov's theorem and (4.17) that $F \in \mathcal{H}^p_{\mathbb{C}}$. Therefore $F \equiv G \equiv 0$ by [33, Theorem 2.2]. So $\Phi \equiv \Psi \equiv 0$. This contradiction proves the lemma. $\square$

Remark 4.9. *The reference to the M. Riesz theorem in the penultimate sentence of [33, Proof of Theorem 2.2] should be changed by the following argument, in the notation of [33], to include the case* $\alpha = 1$.

> *Hence if* $\Phi,\ \Psi \in \mathcal{H}^p_{\mathbb{C}}$ *is a solution of (2.1) then* $\operatorname{Im}\Psi^* \in L^{2/(1-\alpha)}_{2\pi}$. *Since* $(-i\Phi, i\Psi)$ *is obviously a solution of (2.1) as well, the same argument gives* $a\eta = \operatorname{Re}\Psi^* = \operatorname{Im}(i\Psi^*) \in L^{2/(1-\alpha)}$.

The proof continues as in [33].

Proof of Theorem 3.2

(a) If $\lambda(w) > 0$ on $\mathbb{R}$ then w is real-analytic on $\mathbb{R}$ by Theorem 3.1. Thus (iii) implies (i) and (ii). Now suppose that (i) holds. Then $w \in C^{1/q(\varrho)}_{2\pi}$, $p(\varrho)^{-1} + q(\varrho)^{-1} = 1$, by Hölder's inequality,

$$|\lambda(w(t)) - \lambda(w(s))| \leq \text{const } |t-s|^{\varrho/q(\varrho)},$$

and hence the hypotheses of Lemma 4.8 hold with $\Psi = U \in \mathcal{H}^p_{\mathbb{C}},\ \Phi = W \in \mathcal{H}^p_{\mathbb{C}}$, from (4.2), and $p = p(\varrho),\ \alpha = \min\{1, \varrho/q(\varrho)\},\ a(t) = \lambda(w(t))$. Therefore $\lambda(w) \geq 0$, by Lemma 4.8, and (2.3a) holds, by Corollary 2.3. Now if $\lambda(w(t_0)) = 0$, then $|\lambda(w(t))| \leq$ const $|t - t_0|^{\varrho/q(\varrho)}$. Since $\varrho p(\varrho)/(2q(\varrho)) = 1$ and (2.3a) holds, and since $w',\ \mathcal{C}w' \in L^{p(\varrho)}_{2\pi}$, it follows that $\lambda(w)$ is nowhere zero. This proves that (i) implies (iii) and (ii). Since (ii) implies (i), the proof of (a) is complete.
(b) In the light of (4.2) and part (a), the proof of part (b) is identical to that of [33, Theorem 1.7 (d)].
(c) As in the proof of part (a), if $w \in W^{1,p}_{2\pi}$, then $|\lambda(w(t))| \leq$ constant $|t-t_0|^\alpha$ when $\lambda(w(t_0)) = 0$, where $\alpha = \varrho(1-p^{-1})$. Since $p \geq 2/(1+\alpha)$ is equivalent to $p \geq r(\varrho)$, (4.2), (4.3) and Lemma 4.8 imply that $\lambda(w)$ does not change sign. Hence $\lambda(w) \geq 0$ by Theorem 3.1.
(d) One proof of (d) is almost identical to that of [33, Prop. 2.1] or [37, Theorem 5.1]. Here we give a different proof in Section 4.10.1. □

4.6. Dimension of the Set of Stagnation Points: proof of Theorem 3.3

Let $t \in [0, 2\pi] \setminus \mathcal{N}(w)$ and let t_0 be a closest point of $\mathcal{N}(w)$ to t. Then the equality $\sqrt{\lambda(w)}\, w' = \sin\vartheta$ (see (4.20)) and the hypotheses of the theorem imply

$$\begin{aligned}\lambda(w(t))^{(\varrho+2)/2\varrho} &\le \text{const}\,|w(t)-w(t_0)|^{(\varrho+2)/2}\\ &= \text{const}\left|\int_{w(t_0)}^{w(t)} |x-w(t_0)|^{\varrho/2}\,dx\right| \le \text{const}\left|\int_{w(t_0)}^{w(t)} \sqrt{\lambda(x)}\,dx\right|\\ &\le \text{const}\left|\int_{t_0}^{t} \sqrt{\lambda(w(s))}|w'(s)|\,ds\right| \le \text{const}\,|t-t_0|,\end{aligned}$$

where the constant changes from line to line. Therefore

$$\sqrt{\lambda(w)(t)} \le \text{const}\ \big(\text{dist}(t,\mathcal{N}(w))\big)^{\varrho/(2+\varrho)} \quad \text{for all } t\in[0,2\pi]\setminus\mathcal{N}(w). \tag{4.18}$$

Let μ_ε be the Lebesgue measure of the ε–neighbourhood $\mathcal{N}_\varepsilon$ of $\mathcal{N}(w)$. Since $\log\lambda(w)\in L^1_{2\pi}$, the Lebesgue measure of $\cap_{\varepsilon>0}\mathcal{N}_\varepsilon=\mathcal{N}(w)$ is 0. Therefore, for $w\in W^{1,p}_{2\pi}$, and with $p=1$ if $w\in\mathcal{H}^{1,1}_{\mathbb{R}}$, (2.3a) implies that

$$\int_{N_\varepsilon}\left(\frac{1}{\sqrt{\lambda(w)(t)}}\right)^p dt \to 0 \ \text{ as } \ \varepsilon\searrow 0.$$

On the other hand, if the lower Minkowski dimension of $\mathcal{N}(w)$ is greater than $1-\varrho p/(2+\varrho)$, then there exists $\varepsilon_0>0$ such that (see [14, Prop. 3.2 and remark following proof] or [23, 5.5])

$$\mu_\varepsilon \ge \varepsilon^{1-(1-\varrho p/(2+\varrho))} = \varepsilon^{\varrho p/(2+\varrho)}, \quad \text{for all } \varepsilon\in(0,\varepsilon_0], \tag{4.19}$$

and (4.18) and (4.19) imply that

$$\int_{N_\varepsilon}\left(\frac{1}{\sqrt{\lambda(w)(t)}}\right)^p dt \ge \text{const}\,\varepsilon^{-\varrho p/(2+\varrho)}\mu_\varepsilon \ge \text{const}\ >0.$$

This is a contradiction. Since

$$1-\frac{\varrho p}{2+\varrho} = 1-\frac{p}{p(\varrho)}$$

this completes the proof when $w\in\mathcal{H}^{1,1}_{\mathbb{R}}\cup W^{1,p}_{2\pi}$. That $\mathcal{N}(w)=\emptyset$ when $p\ge p(\varrho)$ is a result of Theorem 3.2. □

4.7. Jordan Curves: proofs of Theorem 3.4 and (3.4)

Suppose that $w\in\mathcal{H}^{1,1}_{\mathbb{R}}$ is a solution of (2.3b) with $\lambda(w)\ge 0$ and recall that $\mathcal{N}(w)$ denotes the zero set of $\lambda(w)$. We know that $\log\lambda(w)\in L^1_{2\pi}$ and w is real analytic on $\mathbb{R}\setminus\mathcal{N}(w)$. By Corollary 2.3,

$$W = i\mathcal{O}\big(1/\sqrt{\lambda(w)}\big),\ \mathcal{O}\big(1/\sqrt{\lambda(w)}\big)(0)=1 \text{ and } |W^*| = 1/\sqrt{\lambda(w)}.$$

It follows that

$$W^* = w' + i(1 + \mathcal{C}w') = i\,\mathcal{O}^*\big(1/\sqrt{\lambda(w)}\big) = i\frac{e^{-i\vartheta}}{\sqrt{\lambda(w)}}\,, \tag{4.20}$$

where $\vartheta = \mathcal{C}\log\sqrt{\lambda(w)}$, as in (3.2). Then $\vartheta \in C^\infty\big(\mathbb{R}\setminus\mathcal{N}(w)\big)$ and (3.3) holds. Since $W \in \mathcal{H}^1_{\mathbb{C}}$, it follows from (4.20) that $1/\sqrt{\lambda(w)} \in L^1_{2\pi}$ and, since $\lambda(w)$ is bounded, $\log\lambda(w) \in L^p_{2\pi}$ for all $p < \infty$. Therefore

$$\vartheta \in L^p_{2\pi} \text{ for all } p < \infty. \tag{4.21}$$

Suppose that $t_0 \in \mathcal{N}(w)$. Then for all $\delta > 0$

$$\lambda'(w)|W^*|^3 \notin L^1((t_0-\delta, t_0)), \quad \lambda'(w)|W^*|^3 \notin L^1((t_0, t_0+\delta)). \tag{4.22}$$

Indeed, let $(c, d) \subset (t_0 - \delta, t_0)$ be such that $(c, d) \cap \mathcal{N}(w) = \emptyset$ and $d \in \mathcal{N}(w)$. Since

$$|\lambda'(w)||W^*|^3 = \frac{|\lambda'(w)|}{\lambda(w)}\,|W^*| \geq \frac{|\lambda'(w)|}{\lambda(w)}\,|w'|,$$

we have

$$\begin{aligned}\int_c^{d-\varepsilon} |\lambda'(w(t))||W^*(t)|^3 dt &\geq \left|\int_c^{d-\varepsilon} \frac{\lambda'(w(t))}{\lambda(w(t))}\,w'(t)dt\right|\\ &\geq \left|\ln\frac{\lambda(w(d-\varepsilon))}{\lambda(w(c))}\right| \to +\infty \quad \text{as } \varepsilon \searrow 0.\end{aligned}$$

The second inequality in (4.22) is proved similarly.

Since, by (3.1), λ is continuous and non-increasing, the set $\{x \in \mathcal{R}(w) : \lambda(x) = 0\}$ is a closed interval, which may be empty or may consist of only one point. Since w is continuous and $\log\lambda(w) \in L^1_{2\pi}$, λ has at most one zero in the closed interval $\mathcal{R}(w)$.

Fix $t \in \mathbb{R}\setminus\mathcal{N}(w)$ and let $w_m \in C^\infty_{2\pi}$, $m \in \mathbb{N}$, be such that $\mathcal{R}(w_m) \subset \mathcal{R}(w)$, $\lambda(w_m) > 0$, $w_m = w$ in a neighbourhood of t, with

$$\|w_m - w\|_{L^1_{2\pi}} \to 0 \text{ and } \big\|\log|\lambda(w_m)| - \log|\lambda(w)|\big\|_{L^1_{2\pi}} \to 0 \text{ as } m \to \infty.$$

Let $f(x) = \log\lambda(x)$, $x \in \mathcal{R}(w)$. By (3.1), f is non-increasing and is not constant on $\mathcal{R}(w)$. Note that if f is affine then $\lambda'/\lambda < 0$ on $\mathcal{R}(w)$, since f is not constant. Now with t fixed as above,

$$\begin{aligned}
&\frac{\lambda'(w_m(t))}{\lambda(w_m(t))}\mathcal{C}w_m'(t) - \Big(\mathcal{C}\log\lambda(w_m)\Big)'(t)\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f'(w_m(t))w_m'(s) - f'(w_m(s))w_m'(s)}{2\tan\frac{1}{2}(t-s)}ds\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{-\dfrac{d}{ds}\displaystyle\int_{w_m(t)}^{w_m(s)} f'(x) - f'(w_m(t))\,dx}{2\tan\frac{1}{2}(t-s)}ds\\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{\displaystyle\int_{w_m(t)}^{w_m(s)} f'(x) - f'(w_m(t))\,dx}{\sin^2\frac{1}{2}(t-s)}ds\\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{\log\big(\lambda(w_m(s))/\lambda(w_m(t))\big) - (w_m(s)-w_m(t))\big(\lambda'(w_m(t))/\lambda(w_m(t))\big)}{\sin^2\frac{1}{2}(t-s)}ds\\
&\to \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{\log\lambda(w(s)) - \log\lambda(w(t)) - (w(s)-w(t))\big(\lambda'(w(t))/\lambda(w(t))\big)}{\sin^2\frac{1}{2}(t-s)}ds
\end{aligned}$$

as $m\to\infty$. It is easy to see that

$$\mathcal{C}w_m'(t)\to\mathcal{C}w'(t) \text{ and } \big(\mathcal{C}\log\lambda(w_m)\big)'(t)\to\big(\mathcal{C}\log\lambda(w)\big)'(t).$$

Since $\log\lambda$ is concave on $\mathcal{R}(w)$ we have proved that, for all $t\in\mathbb{R}\setminus\mathcal{N}(w)$,

$$\frac{\lambda'(w(t))}{\lambda(w(t))}\mathcal{C}w'(t) - \Big(\mathcal{C}\log\lambda(w)(t)\Big)' \le 0,$$

or, equivalently, from (3.3) and (4.20),

$$\frac{\lambda'(w(t))}{\lambda(w(t))}\left(\frac{\cos\vartheta}{\sqrt{\lambda(w)}} - 1\right) - 2\vartheta' \le 0,$$

and the inequality is strict if $\log\lambda$ is not affine on $\mathcal{R}(w)$. On the other hand if $\log\lambda$ is affine on $\mathcal{R}(w)$ then, as observed earlier, (3.1) implies that $\lambda' < 0$ on $\mathcal{R}(w)$. Therefore, when (3.1) holds,

$$\vartheta' - \frac{1}{2}\lambda'(w)|W^*|^3\cos\vartheta > 0 \text{ on } \mathbb{R}\setminus\mathcal{N}(w). \tag{4.23}$$

This is a simple extension of the inequality used in [39], and the argument there leads immediately to the result required here when w, and hence ϑ, are C^1-functions. The treatment of less smooth w is more complicated and the following estimate with its corollary will be one of the keys.

LEMMA 4.10. *Let $\Psi\in\mathcal{H}^p_{\mathbb{C}}$, $p<\infty$. Then, for any $t_0\in\mathbb{R}$,*

$$|\Psi(re^{it_0})| = o\big((1-r)^{-1/p}\big) \text{ as } r\nearrow 1.$$

Proof. If $\Psi \in \mathcal{H}^p_{\mathbb{C}}$ there exists a Blaschke product B, and a function $\widetilde{\Psi} \in \mathcal{H}^1_{\mathbb{C}}$ with no zeros in the open unit disc, such that $\Psi = \widetilde{\Psi}^{1/p} B$. Hence it suffices to prove the required result when $\Psi \in \mathcal{H}^1_{\mathbb{C}}$.

Let $\varepsilon > 0$. Since $\Psi^* \in L^1_{2\pi}$, there exists $\delta > 0$ such that

$$\|\Psi^*\|_{L^1([t_0-\delta, t_0+\delta])} \leq 2\pi\varepsilon.$$

The Cauchy integral formula implies

$$\begin{aligned}
|\Psi(re^{it_0})| &= \left| \frac{1}{2\pi} \int_{t_0-\pi}^{t_0+\pi} \frac{\Psi^*(t)}{e^{it} - re^{it_0}}\, e^{it} dt \right| \\
&\leq \frac{1}{2\pi} \int_{t_0-\delta}^{t_0+\delta} \frac{|\Psi^*(t)|}{|e^{it} - re^{it_0}|}\, dt + \frac{1}{2\pi} \int_{\delta \leq |t-t_0| \leq \pi} \frac{|\Psi^*(t)|}{|e^{it} - re^{it_0}|}\, dt \\
&\leq \frac{1}{2\pi} \|\Psi^*\|_{L^1([t_0-\delta, t_0+\delta])} (1-r)^{-1} + \frac{1}{2\pi} \frac{\|\Psi^*\|_{L^1_{2\pi}}}{\sin\delta} \\
&\leq \varepsilon(1-r)^{-1} + C_\varepsilon, \quad 0 \leq r < 1,
\end{aligned}$$

where C_ε is a constant. The result now follows. □

COROLLARY 4.11. *Suppose that, for some $p < \infty$, $\Psi = \exp(-i\Phi) \in \mathcal{H}^p_{\mathbb{C}}$ where $\Phi \in N$, $\Phi^* = \vartheta + i\mathcal{C}\vartheta$ with $\vartheta \in L^1_{2\pi}$ and, for some $\delta > 0$,*

$$\vartheta(-t) - \vartheta(t) \geq \kappa\pi, \quad 0 < t < \delta.$$

Then $\kappa < 1/p$. In particular if, for some $t_0 \in \mathbb{R}$,

$$\vartheta(t_0 + t) \leq M^+\pi \quad \textit{and} \quad \vartheta(t_0 - t) \geq m^-\pi, \quad t \in (0, \delta),$$

for some $\delta > 0$, then $m^- < M^+ + 1/p$.

Proof. It is immediate from its definition (see [15, Ch. III, Sect. 1]) that

$$\begin{aligned}
|\Psi(r)| = \exp &\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2r\sin(-t)}{1 - 2r\cos t + r^2}\, \vartheta(t) dt \right) \\
&= \exp \left(\frac{1}{2\pi} \int_{-\delta}^{\delta} \frac{2r\sin(-t)}{1 - 2r\cos t + r^2}\, \vartheta(t) dt \right) \\
&\times \exp \left(\frac{1}{2\pi} \int_{(-\pi,-\delta)\cup(\delta,\pi)} \frac{2r\sin(-t)}{1 - 2r\cos t + r^2}\, \vartheta(t) dt \right) \\
&\geq C \exp \left(\frac{1}{\pi} \int_0^{\delta} \frac{(\vartheta(-t) - \vartheta(t)) r \sin t}{1 - 2r\cos t + r^2} dt \right) \\
&\geq C \exp \left(\kappa \int_0^{\delta} \frac{r\sin t}{1 - 2r\cos t + r^2} dt \right).
\end{aligned}$$

But

$$\int_0^\delta \frac{r\sin t}{1-2r\cos t+r^2}\,dt = \operatorname{Im}\left(\int_0^\delta \frac{-e^{it}}{e^{it}-r}\,dt\right)$$
$$= \log\left|\frac{e^{i\delta}-r}{1-r}\right| \geq c+\log\big(|1-r|^{-1}\big)$$

as $r \nearrow 1$, where c is a constant. Therefore

$$|\Psi(r)| \geq \text{constant } (1-r)^{-\kappa}.$$

That $\kappa < 1/p$ follows now from Lemma 4.10, since $\Psi \in \mathcal{H}^p_{\mathbb{C}}$. The particular case follows from this by translating the origin of t to t_0. □

Recall, from Section 3.4 (see also Section 4.8), that $\log\sqrt{\lambda(w)}$ and ϑ, defined in (3.2), have zero mean and, from (4.21), $\vartheta \in L^p_{2\pi}$ for all $p < \infty$. Moreover,

$$e^{-i\Phi} = \mathcal{O}\big(1/\sqrt{\lambda(w)}\big) \in \mathcal{H}^1_{\mathbb{C}} \text{ and } e^{i\Phi} = \mathcal{O}\big(\sqrt{\lambda(w)}\big) \in \mathcal{H}^\infty_{\mathbb{C}}, \tag{4.24}$$

where $\Phi \in \mathcal{H}^1_{\mathbb{C}}$ is such that $\Phi^* = \vartheta + i\mathcal{C}\vartheta$.

Lemma 4.12. *Suppose that for $t \in (0,\delta)$, $\delta > 0$,*

$$m^+\pi \leq \vartheta(t_0+t) \leq M^+\pi \quad \textit{and} \quad m^-\pi \leq \vartheta(t_0-t) \leq M^-\pi.$$

Then

$$m^+ \leq M^- \textit{ and } m^- < M^+ + 1.$$

Proof. This follows from Corollary 4.11 using (4.24). □

Lemma 4.13. *Suppose that $c \in (a,b) \subset \mathbb{R}\setminus\mathcal{N}(w)$ and let $\ell \in \mathbb{Z}$ be such that $\vartheta(c) \in \big[(\ell-\frac12)\pi, (\ell+\frac12)\pi\big]$. Then*

$$\vartheta(t) > (\ell-\tfrac12)\pi \textit{ for } t \in (c,b) \quad \textit{and} \quad \vartheta(t) < (\ell+\tfrac12)\pi \textit{ for } t \in (a,c). \tag{4.25}$$

Proof. Suppose that the set $E = \{t \in (c,b) : \vartheta(t) \leq (\ell-\frac12)\pi\}$ is non-empty and let $t_0 \in [c,b)$ denote its infimum. Then $\vartheta'(t_0) \leq 0$ and $\cos\vartheta(t_0) = 0$. Since this contradicts (4.23), the set E must be empty, which proves the first inequality in (4.25). The second one is proved similarly. □

Lemma 4.14. *Suppose that $a, b \in \mathcal{N}(w)$ and $(a,b) \subset \mathbb{R}\setminus\mathcal{N}(w)$. Then either $\vartheta(t) \to +\infty$ as $t \nearrow b$ or there exists $\delta > 0$ such that*

$$\cos\vartheta(t) > 0 \textit{ for } t \in (b-\delta, b).$$

Similarly, either $\vartheta(t) \to -\infty$ as $t \searrow a$ or there exists $\delta > 0$ such that

$$\cos\vartheta(t) > 0 \textit{ for } t \in (a, a+\delta).$$

Proof. Take an arbitrary $c \in (a,b)$. Suppose that $\vartheta(t) \not\to +\infty$ as $t \nearrow b$. Then ϑ is bounded above on (c,b) by Lemma 4.13. Let $\sup_{t\in[c,b)} \vartheta(t) = M < \infty$.

Suppose there exists $d \in [c,b)$ such that $\vartheta(d) = M$. Then $\vartheta'(d) = 0$ if $d \neq c$ and $\vartheta'(d) \leq 0$ if $d = c$. In both cases (4.23) implies that $\cos\vartheta(d) > 0$ and $\vartheta(d) \in ((2\ell - \frac{1}{2})\pi, (2\ell + \frac{1}{2})\pi)$ for some $\ell \in \mathbb{Z}$. By Lemma 4.13, $\vartheta(t) > (2\ell - \frac{1}{2})\pi$ for any $t \in (d,b)$ and $\vartheta(t) \leq M = \vartheta(d) < (2\ell + \frac{1}{2})\pi$. Hence $\cos\vartheta(t) > 0$ for any $t \in (d,b)$ in this case.

Next suppose that $\vartheta < M$ on $[c,b)$ and that ϑ' has a sequence of zeros tending monotonically upwards to b. There exists a sequence d_j with $\vartheta'(d_j) = 0$, $d_j \nearrow b$ and $\vartheta(d_j) \to M$ as $j \to \infty$. By (4.23), $\cos\vartheta(d_j) > 0$ and therefore $M \in ((2\ell - \frac{1}{2})\pi, (2\ell + \frac{1}{2})\pi]$ for some $\ell \in \mathbb{Z}$ and $\vartheta(d_j) \in ((2\ell - \frac{1}{2})\pi, (2\ell + \frac{1}{2})\pi)$ for sufficiently large j. Since M is not attained on (c,b) it follows from Lemma 4.13 that $\vartheta(t) \in ((2\ell - \frac{1}{2})\pi, (2\ell + \frac{1}{2})\pi)$ and $\cos\vartheta(t) > 0$ for any $t \in (d_j, b)$ in this case.

Finally suppose $\vartheta < M$ on $[c,b)$ and ϑ' has no zeros on (d,b) for some $d \in (c,b)$. Then ϑ is increasing on (d,b) and $M = \lim_{t\nearrow b} \vartheta(t) =: \vartheta(b-)$. Suppose $\cos\vartheta(b-) < 0$. Then there exist ε, $\delta > 0$ such that $\cos\vartheta(t) < -\varepsilon$, for all $t \in [b-\delta, b)$. It follows from (4.23) that $\vartheta' > \frac{\varepsilon}{2}|\lambda'||W^*|^3$ on $[b-\delta, b)$. Therefore (4.22) implies

$$M > \vartheta(t) = \vartheta(b-\delta) + \int_{b-\delta}^{t} \vartheta'(x)dx \to \infty \quad \text{as} \quad t \nearrow b,$$

which is a contradiction. Hence $\cos\vartheta(b-) \geq 0$ in this case.

Since $\lambda \geq 0$ is non-increasing and $\lambda(w(b)) = 0$, w attains its maximum at b. Since, by (3.3), $\sin\vartheta = \sqrt{\lambda(w)}\, w'$ and $\lim_{t\nearrow b} \vartheta(t) = \vartheta(b-)$, it follows that $\sin\vartheta(b-) \geq 0$. Hence $\vartheta(b-) \in [2\ell\pi, (2\ell + \frac{1}{2})\pi]$ for some $\ell \in \mathbb{Z}$ and there exists $d \in [c,b)$ such that $\vartheta(t) \in ((2\ell - \frac{1}{2})\pi, (2\ell + \frac{1}{2})\pi)$ and $\cos\vartheta(t) > 0$ for any $t \in (d,b)$. This proves the existence of δ such that $\cos\vartheta(t) > 0$ for $t \in (b-\delta, b)$.

The second statement is proved similarly. The main difference is that instead of M one needs to consider $m := \inf_{t\in(a,c]} \vartheta(t)$. □

Let $g : \mathbb{R} \to \mathbb{C}$ be measurable. Then we call $\zeta \in \mathbb{C}$ a *right cluster value* of g at $t \in \mathbb{R}$ (written $\zeta \in g\{t+\}$) if

$$\operatorname{meas}\big((t, t+\varepsilon) \cap g^{-1}(B)\big) > 0$$

for every open ball $B \subset \mathbb{C}$ with centre ζ and every $\varepsilon > 0$, and ζ is a *left cluster value* ($\zeta \in g\{t-\}$) if the same is true with $(t, t+\varepsilon)$ replaced by $(t-\varepsilon, t)$. Cluster values are the same for functions that are equal almost everywhere. Let

$$g\{t\} = g\{t-\} \cup g\{t+\}.$$

For any $t \in \mathbb{R} \setminus \mathcal{N}(w)$ there exists $\ell(t) \in \mathbb{Z}$ such that

$$\vartheta(t) \in \big((\ell(t) - \tfrac{1}{2})\pi, (\ell(t) + \tfrac{1}{2})\pi\big].$$

Lemma 4.13 says that ℓ is non-decreasing on any interval $(a,b) \subset \mathbb{R} \setminus \mathcal{N}(w)$. Since $\mathcal{N}(w)$ has measure zero, the function ℓ is defined almost everywhere on $\mathbb{R}$ and is clearly 2π–periodic and integer-valued.

From now on in this Section we suppose that $\mathcal{N}(w)$ is at most countable.

LEMMA 4.15. *For $[\alpha,\beta] \subset \mathbb{R}$, suppose $\ell \in L^\infty[\alpha,\beta]$, $L_+ := \max_{\alpha \le t \le \beta} \ell(t)$ and $c \in [\alpha,\beta] \setminus \mathcal{N}(w)$ is such that $\ell(c) = L_+$. Then $\ell(t) = L_+$ for any $t \in [c,\beta] \setminus \mathcal{N}(w)$.*

Proof. First let L_+ be odd so that $\cos\vartheta(c) \le 0$. Let $d = \min(\mathcal{N}(w) \cap (c,\beta])$, assuming that this set is non-empty. Then $d > c$ and $\cos\vartheta(d-s) > 0$ for all $s > 0$ sufficiently small, by Lemma 4.14. Therefore, since $\cos\vartheta(c) \le 0$, $\vartheta(d-s) > (L_+ + \frac{1}{2}\pi)$ for all $s > 0$ sufficiently small, by Lemma 4.13. Since this contradicts the definition of L_+, no such d exists, $\mathcal{N}(w) \cap (c,\beta]$ is empty and the result holds by Lemma 4.13

Now suppose that L_+ is even so that $\cos\vartheta(c) \ge 0$. Let

$$\mathcal{N}(w)_+ := \big\{\tau \in \mathcal{N}(w) \cap [\alpha,\beta] \;:\; L_+ \in \ell\{\tau\} \ne \{L_+\}\big\}.$$

At points of $\mathcal{N}(w)_+$, ℓ has at least two distinct cluster values one of which is L_+. Since ℓ is integer-valued and bounded, $\mathcal{N}(w)_+$ is closed. We now show that $\mathcal{N}(w)_+$ is a perfect set. Suppose that t_0 is an isolated point of $\mathcal{N}(w)_+$ and that $(t_-,t_+) \cap \mathcal{N}(w)_+ = \{t_0\}$.

Suppose that $L_+ \in \ell\{t_0+\}$. Then there exists a sequence $\{d_n\} \subset (t_0,t_+) \setminus \mathcal{N}(w)$ such that $\ell(d_n) = L_+$ and $d_n \searrow t_0$ as $n \to \infty$. Fix n and let

$$\underline{d} = \sup\big\{\tau \in (d_n,t_+) \setminus \mathcal{N}(w) \;:\; \ell\{t\} = \{L_+\} \;\text{ for all } t \in [d_n,\tau]\big\}.$$

The set on the right is not empty because it contains d_{n-1}. If $\underline{d} < t_+$ then $L_+ \in \ell\{\underline{d}-\}$ and there exists $\underline{\ell}$ with $L_+ \ne \underline{\ell}$ with $\{L_+,\underline{\ell}\} \subset \ell\{\underline{d}\}$ which implies that $\underline{d} \in \mathcal{N}(w)_+$. This contradicts the fact that $t_0 < \underline{d} < t_+$. Since $d_n \searrow t_0$ as $n \to \infty$ it follows that $\ell(t) = L_+$ for all $t \in (t_0,t_+) \setminus \mathcal{N}(w)$ if $L_+ \in \ell\{t_0+\}$.

If $L_+ > \ell \in \ell\{t_0-\}$ then (by a similar argument) $\ell(t) < L_+$ for all $t \in (t_-,t_0) \setminus \mathcal{N}(w)$ and therefore $L_+ \notin \ell\{t_0-\}$.

If $\max(\ell\{t_0\} \setminus \{L_+\}) = \ell_+$, since $t_0 \in \mathcal{N}(w)_+$ there are only two possibilities:

(i) $\ell\{t_0+\} = \{L_+\}$ and $\max \ell\{t_0-\} \le \ell_+$

(ii) $\ell\{t_0-\} = \{L_+\}$ and $\max \ell\{t_0+\} \le \ell_+$,

It follows from Lemmas 4.13 and 4.14 that ℓ_+ is even. This contradict Lemma 4.12, since L_+ and ℓ_+ are distinct even integers.

This shows that the closed set $\mathcal{N}(w)_+$ does not have isolated points and so is a perfect set. Therefore $\mathcal{N}(w)_+$ is either uncountable or empty. Since $\mathcal{N}(w)_+ \subset \mathcal{N}(w)$, which is at most countable, $\mathcal{N}(w)_+ = \emptyset$.

Now a repeat of the main argument in the proof shows that $\ell(t) = L_+$ for any $t \in [c, \beta] \setminus \mathcal{N}(w)$. □

Corollary 4.16. *Let $\ell \in L^\infty[\alpha, \beta]$. Then ℓ is non-decreasing on $[\alpha, \beta] \setminus \mathcal{N}(w)$,*

$$t_1, t_2 \in [\alpha, \beta] \setminus \mathcal{N}(w),\ t_1 < t_2 \quad \Longrightarrow \quad \ell(t_1) \leq \ell(t_2).$$

Proof. It is sufficient to apply Lemma 4.15 to the interval $[t_1, t_2]$. □

Lemma 4.17. *ℓ is bounded.*

Proof. Let

$$\mathcal{N}(w)_\infty := \{\tau \in \mathcal{N}(w) : \ \ell \text{ is unbounded in every neighbourhood of } \tau\}.$$

It is clear that $\mathcal{N}(w)_\infty$ is closed. Let t_0 be an isolated point of $\mathcal{N}(w)_\infty$ and let $t_- < t_0 < t_+$ be such that $[t_-, t_+] \cap \mathcal{N}(w)_\infty = \{t_0\}$.

It is easy to see that, for any $d \in (t_0, t_+)$, ℓ is bounded on $[d, t_+]$ and if ℓ is not bounded on $(t_0, t_+]$, it follows from Corollary 4.16 that $\ell(t) \searrow -\infty$ as $t \searrow t_0$. Similarly, if ℓ is not bounded on $[t_-, t_0)$, then $\ell(t) \nearrow +\infty$ as $t \nearrow t_0$. Both cases contradict Lemma 4.12.

Hence $\mathcal{N}(w)_\infty$ has no isolated points and, as in the case of $\mathcal{N}(w)_+$ (see the proof of Lemma 4.15), $\mathcal{N}(w)_\infty$ is empty. Thus ℓ is bounded, since it is periodic. □

Proof of Theorem 3.4

Since ℓ is bounded, there exists $d \in \mathbb{R} \setminus \mathcal{N}(w)$ such that

$$\ell(d) = L_+ := \max_{t \in \mathbb{R} \setminus \mathcal{N}(w)} \ell(t) < +\infty.$$

Then Lemma 4.15 with $[\alpha, \beta] = [d, d + 2\pi]$ implies that $\ell(t) = L_+$ for any $t \in [d, d + 2\pi] \setminus \mathcal{N}(w)$. Since ℓ is 2π–periodic, $\ell(t) = L_+$ for any $t \in \mathbb{R} \setminus \mathcal{N}(w)$. Since ϑ has zero mean on $[-\pi, \pi]$ (see Section 3.4) it follows that $L^+ = 0$ and $\cos \vartheta(t) > 0$ for any $t \in \mathbb{R} \setminus \mathcal{N}(w)$. □

Proof of the Identity (3.4)

Every solution w of (2.3) has $\lambda(w) > 0$ almost everywhere and

$$\begin{aligned} 1 + \mathcal{C}w' &= \frac{1}{\lambda(w)}\Big(1 - \mathcal{C}\big(\lambda(w)w'\big)\Big) = \frac{1}{\lambda(w)}\big(1 - \lambda(w)\mathcal{C}w'\big) + \frac{\mathfrak{L}(w)}{\lambda(w)} \\ &= 1 + \frac{1}{\lambda(w)} - (1 + \mathcal{C}w') + \frac{\mathfrak{L}(w)}{\lambda(w)}. \end{aligned}$$

From this and (3.3),

$$\cos\vartheta = \lambda(w)^{\frac{1}{2}}\big(1+\mathcal{C}w'\big) = \frac{1}{2}\Big\{\lambda(w)^{\frac{1}{2}} + \frac{1}{\lambda(w)^{\frac{1}{2}}} + \frac{\mathfrak{L}(w)}{\lambda(w)^{\frac{1}{2}}}\Big\}.$$

This may be rearranged to give formula (3.4) for $\mathfrak{L}(w)$. □

4.8. Nekrasov's Equation: proof of Theorem 3.7

Let Φ be such that $\Phi^* = \vartheta + i\mathcal{C}\vartheta$ and let $W = i\exp(-i\Phi)$. Note that $W(0) = i$ and that $W \in \mathcal{H}^p_{\mathbb{C}}$ for some $p > 0$ since ϑ is bounded (see [40, vol. I, ch. VII, (2.11)], [15, Corollaries 2.5 and 2.6, Ch. III]). Since, by (3.6),

$$\int_{-\pi}^{\pi} |W^*(t)| dt = \int_{-\pi}^{\pi} \Big\{\big|\lambda\big(\chi^{-1}\Big(\int_0^t \sin\vartheta(s)\,ds\Big)\big)\big|\Big\}^{-1/2} dt < \infty,$$

by hypothesis, it follows that $W \in \mathcal{H}^1_{\mathbb{C}}$. Also

$$\log|W(0)| = 0 = \int_{-\pi}^{\pi} \mathcal{C}\vartheta(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|W^*(t)|\,dt,$$

from which it follows that W is an outer function (see remarks in Appendix after definition (A.3)) and there exists $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ such that

$$w' + i(1+\mathcal{C}w') = W^*.$$

(So far w is determined only up to an additive constant.) Now let

$$u(t) = \chi^{-1}\Big(\int_0^t \sin\vartheta(s)\,ds\Big).$$

In the light of equation (3.6), it follows from the definitions of u and w that

$$\lambda(u(t))^{1/2} w'(t) = \sin\vartheta(t),$$

and $\lambda(u(t)) \neq 0$ almost everywhere because of (3.8). Now differentiation of the formula for u gives

$$\sin\vartheta(t) = \chi'(u(t))u'(t) = \lambda(u(t))^{1/2}u'(t).$$

Hence $u' = w'$ almost everywhere and we may now fix w by putting $w = u$. Thus

$$w' + i(1+\mathcal{C}w') = W^* = i\exp(-i\Phi^*) = \frac{i\exp(-i\vartheta)}{\sqrt{\lambda(w)}}. \tag{4.26}$$

We have shown that $W \in \mathcal{H}^1_{\mathbb{C}}$ is an outer function and $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ satisfies (2.3a). Hence (2.3b) holds by Corollary 2.3. □

4.9. Morse Indices

We begin our study of the Morse index of non-singular solutions of (2.3) defined in (3.13) with a proof that solutions in duality have the same Morse index.

4.9.1. Duality: Proof of Theorem 3.11

Suppose that w is a non-singular solution of (2.3) (where we have assumed, without loss of generality, that $w(0) = 0$, see Section 2.6) and that $\widetilde{w}$ is the non-singular solution of the dual problem given by Theorem 2.10. Since $\lambda(w)$ is nowhere zero, λ is a real-valued function on a neighbourhood of $\mathcal{R}(w)$. As in Section 4.4, let Λ denote the primitive of $-\lambda$ with $\Lambda(0) = 0$. Let $\widetilde{\Lambda}$ be the inverse of Λ and let $\widetilde{\lambda}(u) = -\widetilde{\Lambda}'(u)$, . Then $\widetilde{w}(t) = \Lambda(w(t))$. Now define nonlinear operators H and $\widetilde{H}$ on neighbourhoods of w and $\widetilde{w}$, respectively, in $W^{1,2}_{2\pi}$ by

$$\begin{aligned} H(v) &= \lambda(v)\big(1 + \mathcal{C}v'\big) + \mathcal{C}\big(\lambda(v)v'\big) - 1, \\ \widetilde{H}(v) &= \widetilde{\lambda}(v)\big(1 + \mathcal{C}v'\big) + \mathcal{C}\big(\widetilde{\lambda}(v)v'\big) - 1. \end{aligned}$$

Let v be a function in $W^{1,2}_{2\pi}$ with range $\mathcal{R}(v)$ in the neighbourhood of $\mathcal{R}(w)$ in which λ is analytic. From the identities in Section 4.4,

$$H(v) = -\lambda(v)\widetilde{H}(\Lambda(v)). \tag{4.27}$$

By (2.3b), $H(w) \equiv 0 \equiv \widetilde{H}(\widetilde{w})$. Taking the Fréchet derivative of both sides of (4.27) *at a solution w of* (2.3) in the direction u leads to the identity

$$dH[w]u = -\lambda(w)d\widetilde{H}[\Lambda(w)] \circ d\Lambda[w]u = \lambda(w)d\widetilde{H}[\Lambda(w)]\lambda(w)u,$$

since the Fréchet derivative of the mapping $v \mapsto \Lambda(v)$ at w is $u \mapsto -\lambda(w)u$. Therefore, for all $u \in W^{1,2}_{2\pi}$,

$$\langle dH[w]u, u\rangle_{L^2_{2\pi}} = \langle d\widetilde{H}[\widetilde{w}]\lambda(w)u, \lambda(w)u\rangle_{L^2_{2\pi}}.$$

Since $\lambda(w)$ is nowhere zero, $d\Lambda[w]$ is a homeomorphism on $W^{1,2}_{2\pi}$ and therefore the Morse index of $\widetilde{w}$ as a solution of $\widetilde{H}(\widetilde{w}) = 0$ and the Morse index of w as a solution of $H(w) = 0$, in $W^{1,2}_{2\pi}$, are equal. This proves Theorem 3.11. □

4.10. Plotnikov's Transformation

Here we generalize Plotnikov's method [29], which transforms the quadratic form Q_w given by (3.12) at a non-singular solution into (4.33). For $u \in L^1_{2\pi}$ let

$$[u] = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\,dt, \qquad \mathcal{E}u = u + i\mathcal{C}u,$$

and, for $w \in W^{1,2}_{2\pi}$, let $W \in \mathcal{H}^2_{\mathbb{C}}$ be such that

$$W^* = i + \mathcal{E}w', \quad W(0) = i.$$

Then $|W^*| \neq 0$ almost everywhere on $[-\pi, \pi]$.

Lemma 4.18. *(a) For any $w \in W^{1,2}_{2\pi}$ and $v \in \mathcal{H}^\infty_{\mathbb{R}}$, there exists $u \in L^2_{2\pi}$ with*

$$v = \operatorname{Im}\Big(\frac{\mathcal{E}u}{W^*}\Big) \text{ and } \int_{-\pi}^{\pi} v\,dt = -\int_{-\pi}^{\pi} u\,dt.$$

(b) For $w \in \mathcal{H}^{1,\infty}_{\mathbb{R}}$, the range of the map

$$u \mapsto \operatorname{Im}\Big(\frac{\mathcal{E}u}{W^*}\Big), \quad u \in L^2_{2\pi},$$

is $L^2_{2\pi}$ and $[v] = -[u]$ in that case also. If, in addition, $1/W$ is bounded, this mapping is a homeomorphism on $L^2_{2\pi}$.

Proof. (a) Suppose that $v \in \mathcal{H}^\infty_{\mathbb{R}}$, so that $\mathcal{E}v = V^*$ of some $V \in \mathcal{H}^\infty_{\mathbb{C}}$ with $V(0) = [v]$. Let $U = iWV \in \mathcal{H}^2_{\mathbb{C}}$. Then $U(0) = -[v] \in \mathbb{R}$ and $U^* = u + i\mathcal{C}u$ for some $u \in L^2_{2\pi}$ with $[u] = [-v]$. Hence

$$iW^*\mathcal{E}v = \mathcal{E}u \text{ and } v = \operatorname{Im}\Big(\frac{\mathcal{E}u}{W^*}\Big).$$

For part (b) simply note that $v \in L^2_{2\pi}$ and $w \in \mathcal{H}^{1,\infty}_{\mathbb{R}}$ implies that, in the proof of (a), $W \in \mathcal{H}^\infty_{\mathbb{R}}$ and $V, U \in \mathcal{H}^2_{\mathbb{C}}$. It is clear that if $1/W$ is bounded then the mapping is a bijection. $\square$

The following simple application is motivated by the proof of [8, Prop. 1].

4.10.1. Another proof of Theorem 3.2 (d)

For $w \in W^{1,2}_{2\pi}$ and $u \in L^2_{2\pi}$, let

$$I(w,u) = \int_{-\pi}^{\pi} u\{\lambda(w)(1 + \mathcal{C}w') + \mathcal{C}(\lambda(w)w') - 1\}dt.$$

When $w \in W^{1,2}_{2\pi}$ is a solution of (2.3b), $I(w,u) = 0$ for all $u \in L^2_{2\pi}$. This can be written

$$\begin{aligned} 0 = -I(w,u) &= \int_{-\pi}^{\pi} \lambda(w)\big\{w'\mathcal{C}u - u(1 + \mathcal{C}w')\big\} + u\,dt \\ &= \int_{-\pi}^{\pi} \Big(\lambda(w)|W^*|^2 \operatorname{Im}\Big(\frac{\mathcal{E}u}{W^*}\Big) + u\Big)\,dt. \end{aligned}$$

Let $v \in L^\infty_{2\pi}$ be arbitrary and choose $u \in L^2_{2\pi}$ as in Lemma 4.18 (a). It follows that

$$I(w,u) = \int_{-\pi}^{\pi} \big\{|W^*|^2\big(|W^*|^{-2} - \lambda(w)\big)\big\}v\,dt = 0. \tag{4.28}$$

Since $|W^*|^2\big(|W^*|^{-2}-\lambda(w)\big)$ is integrable when $w \in W^{1,2}_{2\pi}$, and since $v \in L^\infty_{2\pi}$ is arbitrary, w satisfies (2.3a) and $\lambda(w) \geq 0$. $\square$

For definiteness, let $\widehat{w}$ denote a non-singular solution of (2.3) and $\widehat{W} \in \mathcal{H}^{1,\infty}_{\mathbb{R}}$ the corresponding holomorphic function on the disc with $i + \mathcal{E}\widehat{w}' = \widehat{W}^*$. Since $\lambda(\widehat{w})$ is bounded, $1/|\widehat{W}^*|$ is bounded by (2.3a). Let $\mathcal{W}$ be a neighbourhood of $\widehat{w}$ in $\mathcal{H}^{1,\infty}_{\mathbb{R}}$ with the property that $1/|W^*| \in L^\infty_{2\pi}$ for all $w \in \mathcal{W}$. Let

$$b_0(w) = |W^*|^{-2} - \lambda(w),$$

note that $b_0(\widehat{w}) = 0$ at a solution of (2.3a) and, from (4.28),

$$I(w,u) = \int_{-\pi}^{\pi} |W^*|^2\, b_0(w)\, \mathrm{Im}\Big(\frac{\mathcal{E}u}{W^*}\Big) dt. \tag{4.29}$$

We denote the t-derivatives of the real-analytic functions $\widehat{w}$ and $\widehat{W}^*$ by $\widehat{w}'$ and $\widehat{W}^{*\prime}$. Since $b_0(\widehat{w}) \equiv 0$, differentiation with respect to t, followed by multiplication by $-|\widehat{W}^*|^2$, gives

$$0 = \frac{2\mathrm{Re}\,(\overline{\widehat{W}^*}\widehat{W}^{*\prime})}{|\widehat{W}^*|^2} + \lambda'(\widehat{w})\widehat{w}'|\widehat{W}^*|^2 = \mathrm{Re}\Big\{\frac{2\widehat{W}^{*\prime}}{\widehat{W}^*} + \lambda'(\widehat{w})|\widehat{W}^*|^2\widehat{W}^*\Big\}. \tag{4.30}$$

Hence

$$\Big\{\frac{\widehat{W}^{*\prime}}{\widehat{W}^*} + \frac{1}{2}\lambda'(\widehat{w})|\widehat{W}^*|^2\widehat{W}^*\Big\} = i\widehat{q} \text{ almost everywhere,}$$

where

$$\widehat{q} = \mathrm{Im}\Big\{\frac{\widehat{W}^{*\prime}}{\widehat{W}^*} + \frac{\lambda'(\widehat{w})}{2\lambda(\widehat{w})}\widehat{W}^*\Big\} \tag{4.31}$$

is a real-valued, real-analytic function of a real variable. Therefore the Gateaux derivative of $b_0 : \mathcal{W} \to L^\infty_{2\pi}$ at $\widehat{w}$ in the direction $u \in \mathcal{H}^{1,\infty}_{\mathbb{R}}$ is

$$\begin{aligned}
db_0[\widehat{w}](u) &= -\lambda'(\widehat{w})u - \frac{2(\widehat{w}'u' + (1+\mathcal{C}\widehat{w}')\mathcal{C}u')}{|\widehat{W}^*|^4}\\
&= -|\widehat{W}^*|^{-2}\mathrm{Re}\Big\{\lambda'(\widehat{w})|\widehat{W}^*|^2\mathcal{E}u + 2\big(\frac{\mathcal{E}u'}{\widehat{W}^*}\big)\Big\}\\
&= -|\widehat{W}^*|^{-2}\mathrm{Re}\Big\{\Big(\lambda'(\widehat{w})|\widehat{W}^*|^2\widehat{W}^* + 2\frac{\widehat{W}^{*\prime}}{\widehat{W}^*}\Big)\big(\frac{\mathcal{E}u}{\widehat{W}^*}\big) + 2\big(\frac{\mathcal{E}u}{\widehat{W}^*}\big)'\Big\}\\
&= -|\widehat{W}^*|^{-2}\mathrm{Re}\Big\{2\big(\frac{\mathcal{E}u}{\widehat{W}^*}\big)' + 2i\widehat{q}\Big(\frac{\mathcal{E}u}{\widehat{W}^*}\Big)\Big\}\\
&= -|\widehat{W}^*|^{-2}\Big\{2\,\mathrm{Re}\,\big(\frac{\mathcal{E}u}{\widehat{W}^*}\big)' - 2\widehat{q}\,\mathrm{Im}\,\Big(\frac{\mathcal{E}u}{\widehat{W}^*}\Big)\Big\} \in L^\infty_{2\pi}.
\end{aligned}$$

Since $\mathcal{E}u/\widehat{W}^* \in L^\infty_{2\pi}$ for $u \in \mathcal{H}^{1,\infty}_{\mathbb{R}}$, it follows from (4.29) that the quadratic form $\partial_w I(\widehat{w},u)u$, $u \in \mathcal{H}^{1,\infty}_{\mathbb{R}}$, gives a new formula for $Q_{\widehat{w}}(u)$ in (3.12):

$$Q_{\widehat{w}}(u) = 2\int_{-\pi}^{\pi}\Big\{-\,\mathrm{Im}\,\big(\frac{\mathcal{E}u}{\widehat{W}^*}\big)\mathrm{Re}\,\big(\frac{\mathcal{E}u}{\widehat{W}^*}\big)' + \widehat{q}\Big(\mathrm{Im}\,\big(\frac{\mathcal{E}u}{\widehat{W}^*}\big)\Big)^2\Big\}dt. \tag{4.32}$$

This leads to the following result which, in the Stokes-wave problem when $\Lambda(w) = w - gw^2$, is due to Plotnikov [29].

Lemma 4.19. *Suppose that $\widehat{w}$ is a non-singular solution of* (2.3). *Then, for $u \in W^{1,2}_{2\pi}$,*

$$Q_{\widehat{w}}(u) = 2\int_{-\pi}^{\pi}\Big\{-\ \mathrm{Im}\,\big(\frac{\mathcal{E}u}{\widehat{W^*}}\big)\mathrm{Re}\,\big(\frac{\mathcal{E}u}{\widehat{W^*}}\big)' + \widehat{q}\,\Big(\mathrm{Im}\,\big(\frac{\mathcal{E}u}{\widehat{W^*}}\big)\Big)^2\Big\}dt.$$

Proof. Since $\mathcal{H}^{1,\infty}_{\mathbb{R}}$ is dense in $W^{1,2}_{2\pi}$, the result follows from (3.12) and (4.29), by taking limits in both sides of (4.32). □

Suppose that $\widehat{w}$ is a non-singular solution of (2.3) and define $P : W^{1,2}_{2\pi} \to W^{1,2}_{2\pi}$ by

$$Pu = \mathrm{Im}\,\big(\frac{\mathcal{E}u}{\widehat{W^*}}\big).$$

From Lemma 4.18 (b), P is a homeomorphism on $W^{1,2}_{2\pi}$. We refer to P as *Plotnikov's transformation* which reduces the quadratic form $Q_{\widehat{w}}$ to

$$\widehat{Q}_{\widehat{w}}(\psi) = 2\int_{-\pi}^{\pi}\{\psi\mathcal{C}\psi' + \widehat{q}\psi^2\}dt, \quad \psi = Pu, \tag{4.33}$$

where $\widehat{q}$ is given by (4.31). Since $Q_{\widehat{w}}$ is negative on a subspace $E\setminus\{0\}$ if and only if $\widehat{Q}_{\widehat{w}}$ is negative on $PE\setminus\{0\}$, we can use $\widehat{Q}_{\widehat{w}}$ to calculate the Morse index of a smooth solution $\widehat{w}$. The linear operator on $W^{1,2}_{2\pi}$ which corresponds to the quadratic form $\widehat{Q}_{\widehat{w}}$,

$$\mathcal{C}\psi' + \widehat{q}\,\psi$$

is much simpler than the corresponding operator (3.11) for $Q_{\widehat{w}}$.

4.10.2. Sign of the Plotnikov Potential

To simplify notation, let W, W^* and the potential q be defined in terms of a non-singular solution $w \in W^{1,2}_{2\pi}$ of equation (2.3) with the $\widehat{}$ notation omitted, and note that the Morse index (3.13) of w is given by

$$\mathcal{M}(w) = \sup\,\big\{\dim E\,:\,\mathcal{Q}_w(u) < 0,\ u \in E\setminus\{0\}\big\}, \tag{4.34}$$

where E denotes a linear subspace of $W^{1,2}_{2\pi}$ and

$$\mathcal{Q}_w(u) := \int_{-\pi}^{\pi}\{u\mathcal{C}u' + qu^2\}dt.$$

By the formula (4.26) for W^* we have (recall that $\lambda(w) \geq 0$)

$$W^{*'} = -i\vartheta' W^* - \frac{i}{2}\frac{\lambda'(w)w'e^{-i\vartheta}}{\lambda(w)^{3/2}} = -\Big(i\vartheta' + \frac{1}{2}\frac{\lambda'(w)}{\lambda(w)}w'\Big)W^*.$$

Therefore
$$\frac{W^{*\prime}}{W^*} = -i\vartheta' - \frac{\lambda'(w)}{2\lambda(w)}w'$$
and, from (4.31) without the $\widehat{\ }$,
$$\begin{aligned} q &= -\vartheta' + \frac{\lambda'(w)}{2\lambda(w)}(1+\mathcal{C}w') = -\vartheta' + \frac{1}{2}\lambda'(w)\lambda(w)^{-3/2}\cos\vartheta \\ &= -\Big(\vartheta' - \frac{1}{2}\lambda'(w)|W^*|^3\cos\vartheta\Big) < 0, \end{aligned} \tag{4.35}$$
by (4.23), when (3.1) holds on $\mathcal{R}(w)$.

4.10.3. Constant Plotnikov Potentials: Proof of Theorem 3.13

The observation following (3.15) means that we may assume, without loss of generality, that $w(0) = 0$ and invoke the duality theory of Sections 2.6 and 4.4. Recall the relations, (3.15), (3.18) and (3.19), between the potentials for primary and dual problems and note that if w is a non-constant solution of (2.3b), then neither $\lambda(w)$ nor $\widetilde{\lambda}(\widetilde{w})$ is a constant, where $\widetilde{w}$ is the solution of the dual problem.

For simplicity with notation, let $\widetilde{q} \equiv \frac{1}{2}\alpha$, a constant, and let q denote the potential for the dual problem. Then, from (3.2) and (3.18),
$$\alpha \equiv 2\widetilde{q} = \mathcal{C}\Big(\frac{\lambda'(w)w'}{\lambda(w)}\Big) + \frac{\lambda'(w)}{\lambda(w)}\big(1+\mathcal{C}w'\big).$$

Suppose first that $\alpha = 0$ and let $\widehat{\lambda}(w) = \lambda'(w)/(\lambda(w))$. Then
$$\widehat{\lambda}(w)w' + i\mathcal{C}(\widehat{\lambda}(w)w') = \widehat{\lambda}(w)\big(w' - i(1+\mathcal{C}w')\big).$$
From Lemma 4.1 it follows, by the argument used to prove Theorem 2.2 (b), that $\lambda'(w)|w'|^2/\lambda(w)$ is zero almost everywhere. This is false since $\lambda(w)$ is not constant. Hence $\alpha \neq 0$.

Now let $\widehat{\lambda}(w) = \lambda'(w)/(\alpha\lambda(w))$. Then w satisfies (2.3b) with $\lambda = \widehat{\lambda}$, as well as with the original λ. Since w is non-singular, it follows from Corollary 2.3 that w satisfies (2.3a) with $\lambda(w)$ and with $\widehat{\lambda}(w)$. Therefore
$$0 \le \lambda(w(t)) = \widehat{\lambda}(w(t)) = \frac{\lambda'(w(t))}{\alpha\lambda(w(t))}.$$
Since $\lambda(w) > 0$, it follows from Corollary 2.9 (a) that $\lambda' \le 0$ and hence $\alpha < 0$ and
$$\lambda(w) = \lambda_\beta(w) := \frac{1}{\beta - \alpha w} \quad \text{for some } \beta > 0.$$
Since the problem dual to λ_β has $\lambda(w) = \widetilde{\lambda}_\beta(w) := \beta e^{\alpha w}$ the proof of Theorem 3.13 is complete. □

4.10.4. Simple Morse-Index Estimates: Proof of Lemma 3.9

The two implications follow from (4.34), the minimax characterization of the n^{th} eigenvalue [12, (4.5.1), page 88)] and the remark preceding the statement of the lemma. Note that, since q is not constant, for all $u \neq 0$,

$$\int_{-\pi}^{\pi} u\mathcal{C}u' + q_{\min}u^2\, dt < \int_{-\pi}^{\pi} u\mathcal{C}u' + qu^2\, dt < \int_{-\pi}^{\pi} u\mathcal{C}u' + q_{\max}u^2\, dt.$$

Since $q_{\min}$ is a constant, $\int_{-\pi}^{\pi} u\mathcal{C}u' + q_{\min}u^2\, dt < 0$ on a $(2n+1)$-dimensional space where n is the largest integer with $2n \leq \mathcal{M}(w)$. Since it is a constant, it follows that $q_{\min} < -n$. This proves the first implication. On the other hand, if $q_{\max} \leq -m$ and q is not constant, it follows that $\int_{-\pi}^{\pi} u\mathcal{C}u' + qu^2\, dt$ is negative on a $2m+1$ dimensional space. This proves the second.

Now if (3.1) holds, then $q < 0$ for any non-singular solution w of (2.3) and hence $\mathcal{M}(w) > 0$ by (3.17). □

4.11. Morse Index and Stagnation Points

Now we study the functional $\mathcal{J}$ from Section 3.5 in the Hilbert space $W^{1,2}_{2\pi}$, with a view to controlling non-singular solutions using bounds on Morse indices.

Definition 4.20. *When w is a non-singular solution of* (2.3) *let*

$$\nu(w) = \max_{a \in \mathcal{R}(w)} \frac{1}{\lambda(a)}.$$

The following estimates q in terms of λ and w .

Lemma 4.21. *Let* (3.1) *hold on $\mathcal{R}(w)$ where w is a non-singular solution of* (2.3). *Then*

$$\|q\|_{L^1_{2\pi}} \leq \pi \max_{\mathcal{R}(w)} \frac{|\lambda'|}{\lambda}.$$

Proof. It follows from (3.2) and formula (4.35) for q that

$$2q = \frac{\lambda'(w)}{\lambda(w)}(1 + \mathcal{C}w') - \mathcal{C}\Big(\frac{\lambda'(w)w'}{\lambda(w)}\Big) < 0. \tag{4.36}$$

Since, by Theorem 3.4, $1 + \mathcal{C}w' > 0$,

$$\|q\|_{L^1_{2\pi}} = \int_{-\pi}^{\pi} -q(t)dt = \int_{-\pi}^{\pi} \frac{|\lambda'(w)|}{2\lambda(w)}(1 + \mathcal{C}w')dt \leq \pi \max_{\mathcal{R}(w)} \frac{|\lambda'|}{\lambda}.$$

□

Conversely, $\nu(w)$ can be bounded in terms of $\|q\|_{L^1_{2\pi}}$, as the next result shows. Note that the hypotheses for the next result implies the hypotheses Theorems 3.2 and 3.3 for the same $\varrho > 0$.

LEMMA 4.22. *For a non-singular solution w of* (2.3) *satisfying* (3.1) *suppose, for some $\varrho > 0$ and positive constants $m_\varrho(w)$ and $M_\varrho(w)$, that*

$$\frac{m_\varrho(w)}{\lambda(a)^{1/\varrho}} \leq \frac{|\lambda'(a)|}{\lambda(a)} \leq \frac{M_\varrho(w)}{\lambda(a)^{1/\varrho}} \quad \textit{for} \quad a \in \mathcal{R}(w). \tag{4.37}$$

Then, for some constant C_ϱ,

$$\log\left(1 + 2^{\frac{\varrho}{2}}\nu(w)^{\frac{\varrho+2}{2\varrho}}\left(\frac{\pi M_\varrho(w) c_{\frac{\varrho+2}{\varrho}}}{\varrho}\right)^{\frac{\varrho+2}{2}}\right)$$
$$\leq C_\varrho\left(1 + c_{\frac{\varrho+2}{\varrho}}^{\frac{\varrho+2}{\varrho}}\right)\left\{\frac{2M_\varrho(w)c_{\frac{\varrho+2}{\varrho}}\|q\|_{L^1_{2\pi}}}{\varrho m_\varrho(w)}\right\}^{\frac{\varrho+2}{\varrho}} \tag{4.38}$$

where, c_p is the norm of $\mathcal{C}$ as an operator on $L^p_{2\pi}$, $1 < p < \infty$.

According to [27], $c_p = \tan(\pi/2p)$ if $p \in (1, 2]$ and $\cot(\pi/2p)$ if $p \in [2, \infty)$.

Proof. Since q is negative, (4.35) and (2.3a) give

$$\begin{aligned}\|q\|_{L^1_{2\pi}} &= \frac{1}{2}\int_{-\pi}^{\pi}\frac{|\lambda'(w)|}{\lambda(w)}(1 + \mathcal{C}w')dt \geq \frac{m_\varrho(w)}{2}\int_{-\pi}^{\pi}\frac{1 + \mathcal{C}w'}{\lambda(w)^{1/\varrho}}dt \\ &\geq \frac{m_\varrho(w)}{2}\int_{-\pi}^{\pi}(1 + \mathcal{C}w')^{\frac{\varrho+2}{\varrho}}dt \geq \pi m_\varrho(w),\end{aligned} \tag{4.39}$$

by Jensen's inequality. Let $\sigma = (\varrho + 2)/\varrho$. Then

$$\|w'\|^\sigma_{L^\sigma_{2\pi}} \leq c^\sigma_\sigma\int_{-\pi}^{\pi}\big(1 + \mathcal{C}w'\big)^\sigma dt \leq 2\frac{c^\sigma_\sigma}{m_\varrho(w)}\|q\|_{L^1_{2\pi}} \tag{4.40}$$

and

$$\int_{-\pi}^{\pi}\{|w'|^\sigma + |1 + \mathcal{C}w'|^\sigma\}\,dt \leq 2\frac{1 + c^\sigma_\sigma}{m_\varrho(w)}\|q\|_{L^1_{2\pi}}. \tag{4.41}$$

Relocating the origin if necessary suppose, without loss of generality, that $\nu(w) = 1/\lambda(w(0))$. Then

$$\begin{aligned}0 \leq \lambda(w(t))^{1/\varrho} &\leq \nu(w)^{-1/\varrho} + \frac{M_\varrho(w)}{\varrho}\int_0^t |w'(s)|ds \\ &\leq \nu(w)^{-1/\varrho} + \frac{M_\varrho(w)}{\varrho}\|w'\|_{L^\sigma_{2\pi}}\,|t|^{\frac{2}{\varrho+2}}.\end{aligned} \tag{4.42}$$

For any $r > 0$, let $K_r = \max\{1,\, 2^{r-1}\}$. Then $(|a| + |b|)^r \le K_r(|a|^r + |b|^r)$ for all $a,\, b \in \mathbb{R}$. Let $C\varrho = K_{\frac{\varrho+2}{2}} K_{\frac{\varrho+2}{2\varrho}}$. Then (2.3a) and (4.42) give

$$\begin{aligned} 1 &= \big\{\lambda(w(t))\big(w'(t)^2 + (1+\mathcal{C}w'(t))^2\big)\big\}^{\frac{\varrho+2}{2\varrho}} \\ &\le C_\varrho\Big\{\nu(w)^{-\frac{\varrho+2}{2\varrho}} + \Big(\frac{M_\varrho(w)}{\varrho}\|w'\|_{L^\sigma_{2\pi}}\Big)^{\frac{\varrho+2}{2}}|t|\Big\}\Big\{|w'(t)|^\sigma + |1+\mathcal{C}w'(t)|^\sigma\Big\}. \end{aligned} \tag{4.43}$$

Therefore, by (4.41), (4.43), (4.40) and (4.39), in turn,

$$\begin{aligned} 2\frac{1+c_\sigma^\sigma}{m_\varrho(w)}\|q\|_{L^1_{2\pi}} &\ge \int_{-\pi}^{\pi} \frac{dt}{C_\varrho\Big\{\nu(w)^{-\frac{\varrho+2}{2\varrho}} + \Big(\frac{M_\varrho(w)}{\varrho}\|w'\|_{L^\sigma_{2\pi}}\Big)^{\frac{\varrho+2}{2}}|t|\Big\}} \\ &\ge \int_0^{\pi} \frac{2\;dt}{C_\varrho\Big\{\nu(w)^{-\frac{\varrho+2}{2\varrho}} + \big(\frac{M_\varrho(w)c_\sigma}{\varrho}\big)^{\frac{\varrho+2}{2}}\big(2\|q\|_{L^1_{2\pi}}/m_\varrho(w)\big)^{\frac{\varrho}{2}}t\Big\}} \\ &= \frac{2\log\Big(1+\nu(w)^{\frac{\varrho+2}{2\varrho}}\big(\frac{M_\varrho(w)c_\sigma}{\varrho}\big)^{\frac{\varrho+2}{2}}\big(2\|q\|_{L^1_{2\pi}}/m_\varrho(w)\big)^{\frac{\varrho}{2}}\pi\Big)}{C_\varrho\big(\frac{M_\varrho(w)c_\sigma}{\varrho}\big)^{\frac{\varrho+2}{2}}\big(2\|q\|_{L^1_{2\pi}}/m_\varrho(w)\big)^{\frac{\varrho}{2}}} \\ &\ge \frac{2\log\Big(1+\nu(w)^{\frac{\varrho+2}{2\varrho}}\big(\frac{M_\varrho(w)c_\sigma}{\varrho}\big)^{\frac{\varrho+2}{2}}\big(2\pi\big)^{\frac{\varrho}{2}}\pi\Big)}{C_\varrho\big(\frac{M_\varrho(w)c_\sigma}{\varrho}\big)^{\frac{\varrho+2}{2}}\big(2\|q\|_{L^1_{2\pi}}/m_\varrho(w)\big)^{\frac{\varrho}{2}}}. \end{aligned}$$

Now multiplying through by the denominator on the right yields (4.38). □

Corollary 4.23. *Let w be a non-singular solution of* (2.3) *and* $K,\ \varrho > 0$ *such that*

$$\frac{1}{K} \le m_\varrho(w) \le M_\varrho(w) \le K. \tag{4.44}$$

Suppose $\|q\|_{L^1_{2\pi}} \le K_1$. *Then* $\nu(w) \le C(K, K_1, \varrho)$ *for some constant* $C(K, K_1, \varrho)$.

The rest of this section concerns the behaviour of non-singular solutions of (2.3) when a uniform bound on the Morse index is known. It can happen that such results are needed for families of equations, rather than for a single equation, for example, when λ in (2.3) depends on parameters. To cover this possibility, suppose that $\{\lambda_k\}$ is a sequence of functions and $\{w_k\}$ is a sequence of non-singular solutions of the corresponding equations

$$\lambda_k(w_k)\{{w_k'}^2 + (1+\mathcal{C}w_k')^2\} = 1, \tag{4.45a}$$

$$\lambda_k(w_k)(1+\mathcal{C}w_k') + \mathcal{C}(\lambda(w_k)w_k') - 1 = 0, \quad w_k \in W^{1,2}_{2\pi}. \tag{4.45b}$$

Let $\{q_k\}$ denote the corresponding sequence of potentials in (4.33) (without the$\widehat{\ }$) and let $\mathcal{M}_k$ denote the Morse index of the non-singular solution w_k of (4.45b), for each $k \in \mathbb{N}$. The aim is to show that when their Morse indices are bounded, the solutions w_k are not close to stagnation. The following are the main hypotheses.

$$\left.\begin{array}{c} w_k \text{ is a non-singular solution of (4.45) for each } k;\\ \lambda_k \text{ satisfies (3.1) on } \mathcal{R}(w_k) \text{ for each } k;\\ \text{for some } K \text{ and } \varrho, \text{ (4.44) is satisfied with } w = w_k \text{ for all } k \in \mathbb{N};\\ \lambda_k \text{ converges uniformly to } \lambda \text{ on } \mathcal{R}_0 := \cup_{k\in\mathbb{N}}\mathcal{R}(w_k)\\ \text{where } \lambda \text{ satisfies (3.1) and (4.44) for the same } \varrho \text{ and } K;\\ \lambda_k(a_k) \to 0 \text{ as } \ k \to \infty,\ a_k \in \mathcal{R}(w_k), \ \text{ implies } \{a_k\} \text{ is bounded;}\\ \text{for any compact interval } I \text{ on which } \lambda \text{ is bounded away from } 0,\\ \{\lambda_k\} \text{ converges to } \lambda \text{ in } C^{1,\alpha}(I), \text{ for some } \alpha \in (0,1). \end{array}\right\} \quad \textbf{(H)}$$

Let $\vartheta_k = \mathcal{C}\log\sqrt{\lambda_k(w_k)}$. Since (3.1) holds and there are no stagnation points, $\cos\vartheta_k > 0$ and $1 + \mathcal{C}w_k' > 0$, for all $k \in \mathbb{N}$, by Theorem 3.4.

THEOREM 4.24. *Suppose that* (**H**) *holds,* $\{\mathcal{M}_k\}$ *is bounded and* $\{\|q_k\|_{L^1_{2\pi}}\}$ *is unbounded. Then* $q_\infty \notin L^1_{2\pi}$ *where*

$$-q_\infty(x) = \liminf_{k\to\infty} -q_k(x). \tag{4.46}$$

Proof. The proof proceeds via a sequence of lemmas.

LEMMA 4.25. *Suppose that* (**H**) *holds and* $\{\|q_k\|_{L^1_{2\pi}}\}$ *is unbounded. There exists a subsequence of* $\{w_k\}$ *which converges uniformly on* $[-\pi, \pi]$.

Proof. Since $\{q_k\}$ is not bounded in $L^1_{2\pi}$ and (4.44) holds uniformly for $k \in \mathbb{N}$, it follows from Lemma 4.21 that, for a subsequence,

$$\lambda_k(w_k(t_k)) \to 0 \text{ for some } t_k \in [-\pi, \pi]. \tag{4.47}$$

From (**H**), $\{w_k(t_k)\}$ is bounded. From (4.45a), the sequence $\{g_k \circ w_k\}$ is uniformly bounded and Lipschitz continuous on $[-\pi, \pi]$, where

$$g_k(x) = \int_{w_k(t_k)}^x \sqrt{|\lambda_k(y)|}dy.$$

Therefore, by the Ascoli-Arzela Theorem, $\{g_k \circ w_k\}$ has a subsequence which is uniformly convergent, to v say, on $[-\pi, \pi]$. Now suppose (seeking a contradiction) that $\{w_k\}$ is not a bounded sequence of continuous functions. Say $|w_k(s_k)| \to \infty$ as $k \to \infty$. Since $\{g_k(w_k(s_k))\}$ is bounded, for k sufficiently large there exists y_k between $\frac{1}{2}w_k(s_k)$ and $w_k(s_k)$ such that $|\lambda_k(y_k)| \to 0$ as $k \to \infty$. By (**H**), this implies that $\{y_k\}$, and hence $\{w_k(s_k)\}$, is bounded. This contradiction shows that $\{w_k\}$ is a bounded sequence of continuous functions.

By (**H**), the sequence $\{\lambda_k\}$ converges uniformly on compact intervals, to λ say, where $\lambda \neq 0$ almost everywhere. Extracting a subsequence if necessary, suppose that, for $\{t_k\}$ in (4.47), $w_k(t_k) \to a^*$ and let

$$g(x) = \int_{a^*}^{x} \sqrt{|\lambda(y)|}dy = \lim_{k\to\infty} g_k(x).$$

Now define $w : [-\pi,\pi] \to \mathbb{R}$ by

$$g(w(t)) = v(t).$$

Then

$$g_k(w_k) - g(w_k) + \big(g(w_k) - g(w)\big) = g_k(w_k) - v \to 0$$

uniformly on $[-\pi,\pi]$ as $k \to \infty$. Since the first term on the left tends uniformly to zero we conclude that

$$g(w_k) - g(w) \to 0$$

uniformly on $[-\pi,\pi]$ as $k \to \infty$. Since g is a strictly monotone continuous function, $\{w_k\}$ (a subsequence of the original sequence) converges uniformly on $[-\pi,\pi]$. □

Henceforth suppose that $\{w_k\}$ converges uniformly to w on $[-\pi,\pi]$. In the proof of Lemma 4.29 it will be important to know that w is continuous.

LEMMA 4.26. *Suppose that $\mathcal{M}_k \le N$, $N \in \mathbb{N}$, for all $k \in \mathbb{N}$. Then there are at most $m \le N$ distinct points $t_j \in (-\pi,\pi]$, $1 \le j \le m$, with the property that, for all $\varepsilon > 0$ sufficiently small,*

$$\limsup_{k\to\infty} \int_{t_j-\varepsilon}^{t_j+\varepsilon} -q_k(t)dt = \infty. \tag{4.48}$$

Proof. Suppose that there are $N+1$ such points. Choose $\varepsilon > 0$ such that (4.48) holds and the intervals $[t_j - 2\varepsilon, t_j + 2\varepsilon]$, $1 \le j \le N+1$, are disjoint. Let ψ_j be a non-negative smooth function with compact support in $(t_j - \varepsilon, t_j + \varepsilon)$, with $\psi_j(t_j) > 0$ and with $\|\psi_j\|_{L^2_{2\pi}} = 1$. Then span $\{\psi_j : 1 \le j \le N+1\}$ is an $(N+1)$-dimensional space. Now let $\alpha_j^k \in \mathbb{R}$, $1 \le j \le N+1$, $k \in \mathbb{N}$, be such that $\sum_{j=1}^{N+1}(\alpha_j^k)^2 = 1$ and let

$$\phi_k = \sum_{j=1}^{N+1} \alpha_j^k \psi_j.$$

Then $\|\phi_k\|_{L^2_{2\pi}} = 1$.

Now the $(N+1)$st eigenvalue of the linear operator L_k defined by $L_k u = \mathcal{C}u' + q_k u$ is non-negative. Therefore, for all $k \in \mathbb{N}$, there is a unit vector $(\alpha_1^k, \cdots, \alpha_{N+1}^k) \in \mathbb{R}^{N+1}$ such that the corresponding function ϕ_k has the property that

$$\int_{-\pi}^{\pi} \{\phi_k \mathcal{C}\phi_k' + q_k \phi_k^2\}dt \ge -1.$$

Therefore
$$\frac{1}{2}\sum_{j=1}^{N+1}(\alpha_j^k)^2\int_{-\pi}^{\pi} -q_k\psi_j^2 dt \le 1 + \int_{-\pi}^{\pi}\phi_k \mathcal{C}\phi_k' dt.$$

But this is impossible since the right side is bounded independent of k and the left side is unbounded above as $k \to \infty$, because of the choice of the points t_j and the properties of the functions ψ_j. □

LEMMA 4.27. *Under the hypotheses of Theorem 4.24, let E be a compact interval in the complement of $\{t_j : 1 \le j \le m\}$ in $(-\pi,\pi]$, where the points t_j are defined in Lemma 4.26. Then there exists $\delta > 0$ such that*
$$\lambda_k(w_k(t)) \ge \delta, \quad t \in E, \quad k \in \mathbb{N}.$$

Proof. Let E' be a compact interval which contains none of the points t_j and which contains E in its interior. According to Lemma 4.26, there exists a constant $M(E')$ such that, in the notation of (4.35),
$$\int_{E'}\vartheta_k' dt - \frac{1}{2}\int_{E'}\lambda_k'(w_k)(1+\mathcal{C}w_k')\{{w_k'}^2 + (1+\mathcal{C}w_k')^2\}dt \le M(E'),$$
since E' is compact and disjoint from the points t_j. Therefore, by Theorem 3.4, since w_k has no stagnation points,
$$\frac{1}{2}\int_{E'}\frac{|\lambda_k'(w_k)|}{\lambda_k(w_k)}(1+\mathcal{C}w_k')dt \le M(E') + \pi.$$
It now follows from (4.44), (4.45a) and (**H**) that, for all $k \in \mathbb{N}$,
$$\frac{1}{2K}\int_{E'}(1+\mathcal{C}w_k')^{\frac{\varrho+2}{\varrho}} dt \le M(E') + \pi. \tag{4.49}$$
Therefore, by Lemma A.1 in the Appendix, $\{w_k'\}$ and $\{1+\mathcal{C}w_k'\}$ are bounded in $L^{\frac{\varrho+2}{\varrho}}(E)$. Recall that $\lambda_k \ge 0$ and suppose, for contradiction, that (for a subsequence) as $k \to \infty$
$$\beta_k := \lambda_k(w_k(\tau_k)) \searrow 0, \quad \tau_k \in E.$$
Then for any $t \in E$,
$$\begin{aligned}\lambda_k(w_k(t))^{1/\varrho} &\le \beta_k^{1/\varrho} + \frac{M_\varrho(w_k)}{\varrho}\int_{\tau_k}^{t}|w_k'(s)|ds\\ &\le \beta_k^{1/\varrho} + \text{const}\,|t-\tau_k|^{\frac{2}{2+\varrho}}.\end{aligned}$$
Therefore, for $t \in E$,
$$\begin{aligned}1 &= \Big\{\lambda_k(w_k(t))\big(|w'(t)|^2 + |1+\mathcal{C}w_k'(t)|\big)^2\Big\}^{\frac{\varrho+2}{2\varrho}}\\ &\le \text{const}\,\big(\beta_k^{\frac{\varrho+2}{2\varrho}} + |t-\tau_k|\big)\big(|w_k'(t)|^{\frac{\varrho+2}{\varrho}} + |1+\mathcal{C}w'(t)|^{\frac{\varrho+2}{\varrho}}\big),\end{aligned}$$

and so

$$\int_E \left(|w_k'(t)|^{\frac{\varrho+2}{\varrho}} + |1+\mathcal{C}w'(t)|^{\frac{\varrho+2}{\varrho}}\right)dt \geq \text{const} \int_E \frac{dt}{\beta_k^{\frac{\varrho+2}{2\varrho}} + |t-\tau_k|} \to \infty$$

as $k \to \infty$ since $\beta_k \searrow 0$. This is a contradiction which proves the result. $\square$

LEMMA 4.28. *Let E be the interval in Lemma 4.27. Then the sequence $\{w_k\}$, which is uniformly convergent to w on $[-\pi,\pi]$, has a subsequence which converges in $C^{2,\beta}(E)$ for all $\beta \in (0,\alpha)$, where α is as in* (**H**). *Moreover, $\{q_k\}$ is bounded in $L^\infty(E)$ and $w \in C^{2,\beta}(E)$.*

Proof. Equations (4.45b) satisfied by w_k imply that

$$\begin{aligned} 2\lambda_k(w_k)\mathcal{C}w_k' &= -\lambda_k(w_k) + \lambda_k(w_k)\mathcal{C}w_k' - \mathcal{C}\big(\lambda_k(w_k)w_k'\big) + 1 \\ &= -\lambda_k(w_k) + 1 + \mathfrak{L}_k(w_k) \end{aligned}$$

where $\mathfrak{L}_k$ is defined in (2.7) with $\lambda = \lambda_k$. Now the sequence $\{w_k\}$ is bounded with $\lambda_k(w_k)$ uniformly bounded away from zero on E.

From (**H**), there exists $N \in \mathbb{N}$ such that $|\lambda - \lambda_k| \leq \delta/2$ on $\mathcal{R}_0$ if $k \geq N$, with δ from Lemma 4.27. Therefore $\lambda \geq \delta/2$ on $\cup_{k\geq N} w_k(E)$. Since λ is continuous, $\lambda \geq \delta/3$ in some neighbourhood R of $\cup_{k\geq N} w_k(E)$, and it follows from (**H**) that λ_k converges to λ in $C^{1,\alpha}(R)$, $\alpha \in (0,1)$. The smoothing properties of the commutators $\mathfrak{L}_k$ on E (similar to those of $\mathcal{Q}$ in [6, 7, 37]; see also [26]), the compactness of the embedding of $C^{1,\alpha}(E)$ in $C^{1,\beta}(E)$ when $\beta \in (0,\alpha)$, and a bootstrap argument, yields the required compactness of $\{w_k\}$ in $C^{2,\beta}(E)$. That $w \in C^{2,\beta}(E)$ is now immediate. To see that $\{q_k\}$ is bounded in $L^\infty(E)$ we use formula (4.31):

$$\begin{aligned} q_k = \operatorname{Im}\Big\{\frac{W_k^{*\prime}}{W_k^*} + \frac{\lambda_k'(w_k)}{2\lambda_k(w_k)}W_k^*\Big\} &= \operatorname{Im}\Big\{\frac{\overline{W_k^*}W_k^{*\prime}}{|W_k^*|^2}\Big\} + \frac{\lambda_k'(w_k)(1+\mathcal{C}w_k')}{2\lambda_k(w_k)} \\ = \lambda_k(w_k)\big(w_k'\mathcal{C}w_k'' - w_k''(1+\mathcal{C}w_k')\big) &+ \frac{\lambda_k'(w_k)(1+\mathcal{C}w_k')}{2\lambda_k(w_k)}. \end{aligned}$$

The result follows from these observations and Remark A.2 in the Appendix. $\square$

LEMMA 4.29. *Suppose that* (**H**) *holds, $\{\mathcal{M}_k\}$ is bounded, $\|q_k\|_{L^1_{2\pi}} \to \infty$ and q_∞ is integrable. Then $\lambda(w) \neq 0$ on $[-\pi,\pi]$.*

Proof. Let E be any compact interval of $(-\pi,\pi] \setminus \{t_j : 1 \leq j \leq m\}$. Then

$$\int_E \vartheta_k' - \frac{1}{2}\lambda_k'(w_k)(1+\mathcal{C}w_k')\big({w_k'}^2 + (1+\mathcal{C}w_k')^2\big)dt = \int_E -q_k dt,$$

and therefore, by Theorem 3.4 and Lemma 4.28, the classical derivatives of w and $\mathcal{C}w$ exist on E and satisfy

$$\frac{1}{2}\int_E -\lambda'(w)(1+(\mathcal{C}w)')\big(w'^2+(1+(\mathcal{C}w)')^2\big)dt \le -\int_E q_\infty dt + \pi < \infty,$$

and $1+(\mathcal{C}w)' \ge 0$ on E. By the argument for (4.49),

$$\frac{1}{2K}\int_{-\pi}^{\pi}(1+(\mathcal{C}w)')^{\frac{2+\varrho}{\varrho}}dt \le \int_{-\pi}^{\pi} -q_\infty\, dt + m\pi,$$

where $\mathcal{C}w$ is continuous except possibly at t_j, $1 \le j \le m$. If it is not continuous at t_j, then $\mathcal{C}w$ has a simple jump discontinuity there because the classical derivative $(\mathcal{C}w)'$ is integrable on a neighbourhood of t_j. If there is a simple jump discontinuity at t_j then it is easily seen, from the Cauchy-principle-value formula for $\mathcal{C}$, that $\mathcal{C}(\mathcal{C}w)$ is unbounded in every neighbourhood of t_j. But $\mathcal{C}(\mathcal{C}w) = -w + [w]$, which is continuous by Lemma 4.25. Hence $\mathcal{C}w$ is continuous and 2π-periodic, and differentiable except at a finite number of points of $[-\pi,\pi]$. Thus, by Remark 4.6, $\mathcal{C}w$ and therefore w, is in $W^{1,(2+\varrho)/\varrho}$. Hence $\lambda(w)$ is nowhere zero, by Theorem 3.2 (a). $\square$

Proof of Theorem 4.24 concluded. If q_∞ is integrable, $\lambda(w)$ is nowhere zero by Lemma 4.29. This is a contradiction since (4.47) implies that $\lambda(w)$ has a zero on $[-\pi,\pi]$. This completes the proof of Theorem 4.24. $\square$

4.11.1. Proof of Theorem 3.12

LEMMA 4.30. *If* (**H**) *holds and* $\{\mathcal{M}_k\}$ *is bounded, then* $\{\|q_k\|_{L^1_{2\pi}}\}$ *is bounded.*

Proof. We argue by contradiction. Suppose without loss of generality that $\|q_k\|_{L^1_{2\pi}} \to \infty$ and let $L_k u = \mathcal{C}u' + q_k u$. Since, by assumption, at most N of the eigenvalues of L_k are negative we may, extracting a further subsequence if necessary, suppose that the first $l \le N$ eigenvalues of L_k tend to $-\infty$ as $k \to \infty$ and that all the remaining eigenvalues of L_k lie in the interval $[-K,\infty)$ for some $K \ge 0$. Let the eigenvalues of L_k, arranged in increasing order, be denoted by $\{\mu_i^k\}_{i\in\mathbb{N}}$ and let the corresponding $L^2_{2\pi}$-orthonormal sequence of eigenfunctions be $\{e_i^k\}_{i\in\mathbb{N}}$.

Let t_j, $1 \le j \le m \le N$ be the points identified in Lemma 4.26 ($m \ne 0$ since $q_k \le 0$ and $\int_{-\pi}^{\pi} q_k dt \to -\infty$ as $k \to \infty$). For $\varepsilon > 0$ sufficiently small let E_ε denote the ε-neighbourhood of the set $\{t_j : 1 \le j \le m\}$. Now let ϕ be any

$W^{1,2}_{2\pi}$-function which is zero on E_ε for some $\varepsilon > 0$. Then

$$\begin{aligned}\langle L_k\phi,\phi\rangle &= \sum_{i=1}^{\infty}\mu_i^k\langle\phi,e_i^k\rangle^2 = \sum_{i=1}^{l}(\mu_i^k)^{-1}\langle L_k\phi,e_i^k\rangle^2 + \sum_{i=l+1}^{\infty}\mu_i^k\langle\phi,e_i^k\rangle^2\\ &\geq \sum_{i=1}^{l}(\mu_i^k)^{-1}\langle L_k\phi,e_i^k\rangle^2 - K\|\phi\|^2_{L^2_{2\pi}}\\ &= \sum_{i=1}^{l}(\mu_i^k)^{-1}\Big(\langle\mathcal{C}\phi,e_i^k\rangle + \langle q_k\phi,e_i^k\rangle\Big)^2 - K\|\phi\|^2_{L^2_{2\pi}},\end{aligned}$$

where angle brackets denote the $L^2_{2\pi}$-inner product. Since $l \leq N < \infty$ and $\|e_i^k\|_{L^2_{2\pi}} = 1$, and since, by Lemma 4.28, $q_k(t)$ is uniformly bounded, independently of k, for t in the complement of E_ε, and since ϕ is zero on E_ε, it follows that

$$\liminf_{k\to\infty}\langle L_k\phi,\phi\rangle \geq -K\|\phi\|^2_{L^2_{2\pi}}, \quad K \geq 0, \tag{4.50}$$

where the constant K is independent of ϕ and of $\varepsilon > 0$. Now for each $\varepsilon > 0$ let

$$g_\varepsilon(t) = \begin{cases} 0 \text{ if } |t| \geq 2\varepsilon \\ 1 \text{ if } |t| \leq \varepsilon \\ \text{linear otherwise}\end{cases} \qquad \text{on } [-\pi,\pi].$$

Let $h_\varepsilon = 1 - g_\varepsilon$ on $[-\pi,\pi]$. Then, by the Cauchy-Schwarz inequality and the M. Riesz lemma on $L^2_{2\pi}$,

$$0 < \langle h_\varepsilon, \mathcal{C}h'_\varepsilon\rangle = \langle g_\varepsilon,\mathcal{C}g'_\varepsilon\rangle \leq \|g\|_{L^2_{2\pi}}\|g'\|_{L^2_{2\pi}} \leq \sqrt{\frac{8\varepsilon}{3}}\sqrt{\frac{2}{\varepsilon}} = \frac{4}{\sqrt{3}}. \tag{4.51}$$

For $\varepsilon > 0$ sufficiently small let

$$\phi_\varepsilon(t) = \prod_{j=1}^{N} h_\varepsilon(t - t_j).$$

Then $0 \leq \phi_\varepsilon \leq 1$ is a $W^{1,2}_{2\pi}$-function which is 0 in E_ε. Since $\phi_\varepsilon \to 1$ almost everywhere, $\|\phi_\varepsilon\|^2_{L^2_{2\pi}} \to 2\pi$ as $\varepsilon \to 0$, by the dominated convergence theorem.

Now, for an absolutely continuous 2π-periodic ϕ (see, for example, [37, Section 3]),

$$\langle\phi,\mathcal{C}\phi'\rangle = \frac{1}{8\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big\{\frac{\phi(s)-\phi(t)}{\sin\frac{1}{2}(s-t)}\Big\}^2 dsdt,$$

and therefore, for f, $g \in W^{1,2}_{2\pi}$ with $|f|$, $|g| \leq 1$,

$$\begin{aligned}\langle fg,\mathcal{C}(fg)'\rangle &\leq \frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big\{\frac{f(s)-f(t)}{\sin\frac{1}{2}(s-t)}\Big\}^2 dsdt + \Big\{\frac{g(s)-g(t)}{\sin\frac{1}{2}(s-t)}\Big\}^2 dsdt\\ &= 2\langle f,\mathcal{C}f'\rangle + 2\langle g,\mathcal{C}g'\rangle.\end{aligned}$$

Since $0 \le \phi_\varepsilon \le 1$ is a product of m translates of h_ε and since (4.51) holds, it follows by induction on m that, independent of ε,

$$\langle \phi_\varepsilon, \mathcal{C}\phi_\varepsilon' \rangle \le c_m \text{ where } c_m = (4/\sqrt{3})(2^m + 2^{m-1} - 2).$$

Hence, by (4.50),

$$\liminf_{k\to\infty} \int_{-\pi}^{\pi} q_k \phi_\varepsilon^2 dt \ge -K\|\phi_\varepsilon\|^2_{L^2_{2\pi}} - c_m \to -2\pi K - c_m$$

as $\varepsilon \to 0$. But, because of Theorem 4.24 and Fatou's lemma,

$$\liminf_{k\to\infty} \int_{-\pi}^{\pi} q_k \phi_\varepsilon^2 dt$$

can be made arbitrarily large and negative by choosing $\varepsilon > 0$ sufficiently small. This is a contradiction which proves Lemma 4.30. □

Theorem 3.12 now follows from Lemmas 4.22 and 4.30. □

A. Appendix – Background Material

The conjugate function $\mathcal{C}u$ is defined almost everywhere for any 2π-periodic locally integrable functions u by the Cauchy Principal Value integral

$$\mathcal{C}u(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(y)\cot(\tfrac{1}{2}(x-y))dy.$$

Alternatively,

$$\mathcal{C}\sin kx = -\cos kx, \quad \mathcal{C}\cos kx = \sin kx, \quad k \in \mathbb{N}, \quad \mathcal{C}1 = 0, \tag{A.1}$$

defines $\mathcal{C}$ for square-integrable functions.

The following notation is from [33, 37]. Let $L^p_{2\pi}$, $p > 0$, denote the space of 2π-periodic locally p^{th}-power summable real-valued functions. For $p \ge 1$, let $W^{1,p}_{2\pi}$ be the space of functions $w \in L^p_{2\pi}$ with weak first derivatives $w' \in L^p_{2\pi}$ and let $\mathcal{H}^{1,1}_{\mathbb{R}}$ be the real Hardy space of functions $w \in W^{1,1}_{2\pi}$ with w' in the usual Hardy space $\mathcal{H}^1_{\mathbb{R}} := \{u \in L^1_{2\pi} : \mathcal{C}u \in L^1_{2\pi}\}$. The space $\mathcal{H}^{1,1}_{\mathbb{R}}$ is a Banach algebra and $\lambda(u) \in \mathcal{H}^{1,1}_{\mathbb{R}}$ when $u \in \mathcal{H}^{1,1}_{\mathbb{R}}$, if λ is Lipschitz continuous [18, 38]. Let $\mathcal{H}^\infty_{\mathbb{R}}$ denote the real Hardy spaces of 2π-periodic functions u such that $u, \mathcal{C}u \in L^\infty_{2\pi}$ and let $\mathcal{H}^{1,\infty}_{\mathbb{R}}$ be the space of absolutely continuous functions with $w' \in \mathcal{H}^\infty_{\mathbb{R}}$. The usual space of k-times continuously differentiable functions on an interval I is denoted by $C^k(I)$.

The following background material is from [13, 15, 20, 32, 40]. Let $D \subset \mathbb{C}$ denote the open unit disc. For a holomorphic function $f : D \to \mathbb{C}$, let $f_r(t) = f(re^{it})$ for $t \in \mathbb{R}$ and $r \in (0,1)$.

We need a little more notation. The Nevanlinna class N consists of complex analytic functions $f : D \to \mathbb{C}$ such that

$$\sup_{r\in(0,1)} \int_0^{2\pi} \log^+ |f(re^{it})|dt < \infty.$$

If $f \in N$, $\lim_{r\nearrow 1} f(re^{it})$, denoted by $f^*(t)$, $t \in \mathbb{R}$, exists almost everywhere and $\log|f^*| \in L^1_{2\pi}$ if $f \not\equiv 0$. A function $f \in N$ belongs to the Nevanlinna–Smirnov class N^+ if

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ |f(re^{it})|dt = \int_0^{2\pi} \log^+ |f^*(t)|dt. \tag{A.2}$$

It is well known that, for any $p \in (0, \infty]$,

$$||f||_p := \lim_{r\to 1} \|f_r\|_{L^p_{2\pi}} = \sup_{r\in(0,1)} \|f_r\|_{L^p_{2\pi}}$$

is well defined. The Hardy class $\mathcal{H}^p_{\mathbb{C}}$ is the set of f with $||f||_p < \infty$. Note that $\mathcal{H}^p_{\mathbb{C}} \subset N^+$ and, for $f \in \mathcal{H}^p_{\mathbb{C}}$, $f^* \in L^p_{2\pi}$, $\|f^*\|_{L^p_{2\pi}} = ||f||_p$ and $\log|f^*| \in L^1_{2\pi}$ if $f \not\equiv 0$. By a theorem of Smirnov, $F \in N^+$ and $F^* \in L^p_{2\pi}$ together imply that $F \in \mathcal{H}^p_{\mathbb{C}}$ (see, e.g., [13, 2.5]). Moreover $u \in \mathcal{H}^1_{\mathbb{R}}$ if and only if $u + i\mathcal{C}u = U^*$ for some $U \in \mathcal{H}^1_{\mathbb{C}}$.

An outer function H is a holomorphic function on D which can be written in the form $H = \alpha\, \mathcal{O}(g)$ where $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and

$$\mathcal{O}(g)(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log|g|(t)dt\right\}, \quad z \in D, \tag{A.3}$$

for some 2π-periodic function $g : \mathbb{R} \to \mathbb{C}$ with $\log|g| \in L^1_{2\pi}$. Outer functions have the following properties [32, Theorems 17.16 & 17.7]:

$\mathcal{O}(g) \in N^+$ has no zeros in D and $\mathcal{O}(g)(0) \in \mathbb{R}$;

$|(\mathcal{O}(g))^*| = |g|$ for any g with $\log|g| \in L^1_{2\pi}$, and, for $p \in (0, \infty]$, $|g| \in L^p_{2\pi}$ if and only if $\mathcal{O}(g) \in \mathcal{H}^p_{\mathbb{C}}$;

for $f \in \mathcal{H}^p_{\mathbb{C}}$, $p \in (0, \infty]$ and $z \in D$, $|\mathcal{O}(f^*)(z)| \geq |f(z)|$;

for $f \in \mathcal{H}^p_{\mathbb{C}}$, $p > 0$, f is an outer function if and only if

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f^*|\, ds = \log|f(0)|;$$

if W and $1/W$ are in $\mathcal{H}^p_{\mathbb{C}}$, $p > 0$, then W is an outer function.

We used the following technical observation in the proof of Lemma 4.27.

LEMMA A.1. *Suppose that u is continuous on $[-\pi, \pi]$ with $u(\pi) = u(-\pi)$, and absolutely continuous on E' with $u' \in L^p(E')$, $1 < p < \infty$, where E' is an open interval. Let E be a compact sub-interval of E'. Then $\mathcal{C}u$ is absolutely continuous on E and*

$$\|\big(\mathcal{C}u\big)'\|_{L^p(E)} \leq K(E, E')(\|u\|_{L^\infty_{2\pi}} + \|u'\|_{L^p(E')}),$$

where $K(E, E')$ is a constant.

Proof. Let $E \subset E'' \subset E'$, where E'' is a compact interval with end points midway between those of E and E'. Let ϕ be a smooth function on $[-\pi, \pi]$ with $\phi \equiv 1$ on E'', $\phi \equiv 0$ on $[-\pi, \pi] \setminus E'$ and write $u = \phi u + (1 - \phi)u$. Then

$$\mathcal{C}u = \mathcal{C}(\phi u) + \mathcal{C}((1 - \phi)u).$$

It is clear that $(\mathcal{C}(\phi u))' = \mathcal{C}(\phi u') + \mathcal{C}(\phi' u) \in L^p_{2\pi}$, $1 < p < \infty$ with

$$\|(\mathcal{C}(\phi u))'\|_{L^p_{2\pi}} \leq K(\phi, E, E')(\|u\|_{L^p_{2\pi}} + \|u'\|_{L^p(E')}).$$

Moreover, for $x \in E$,

$$\mathcal{C}((1 - \phi)u)(x) = \frac{1}{2\pi} \int_{[-\pi,\pi]\setminus E''} \frac{(1 - \phi(y))u(y)}{\tan \frac{1}{2}(x - y)} dy,$$

which is clearly continuously differentiable on E with

$$|\big(\mathcal{C}((1 - \phi)u)\big)'(x)| \leq K(\phi, E, E')\|u\|_{L^1_{2\pi}}.$$

Combining these observations yields

$$\|\big(\mathcal{C}u\big)'\|_{L^p(E)} \leq K(\phi, E, E')(\|u\|_{L^\infty_{2\pi}} + \|u'\|_{L^p(E')}).$$

Now taking the infimum over ϕ completes the proof. □

REMARK A.2. *Almost the same argument leads to the following conclusion (similar to [37, Lemma A1 (i)]). Suppose that u is integrable on $[-\pi, \pi]$ and its restriction to an open interval E' is in $C^\alpha(E')$. Let E be a compact sub-interval of E'. Then $\mathcal{C}u\big|_E \in C^\alpha(E)$ and $\|\mathcal{C}u\|_{C^\alpha(E)}$ is bounded by a constant that depends only on the $L^1_{2\pi}$-norm of u and $\|u\|_{C^\alpha(E')}$.*

REMARK A.3. *If $f : \mathcal{R}(u) \to \mathbb{R}$ is Lipschitz continuous where u is absolutely continuous on $[-\pi, \pi]$, then $f \circ u$ is absolutely continuous. Moreover, $(f \circ u)'(t)$ exists for almost all $t \in [-\pi, \pi]$ and coincides with the weak derivative of $f \circ u$. Here $'$ denotes the classical derivative at points where it exists. If $f'(u(t))$ and $u'(t)$ exist, then the chain rule* (A.4) *holds at t. If $u'(t) = 0$, then $(f \circ u)'(t) = 0$ and the chain rule* (A.4) *holds at t whether or not $f'(u(t))$ exists, no matter what finite value is assigned to $f'(u(t))$.*

In the remaining case, $f'(u(t))$ does not exist but $u'(t)$ exists and is non-zero. Then it is easily to see that $f \circ u$ is not differentiable at t, and so t lies in a set of zero measure. It follows that, no matter what finite value is assigned to $f'(v)$ at points v where f is not differentiable,

$$(f \circ u)'(t) = f'(u(t))u'(t) \quad \text{for almost all } t \in [-\pi, \pi]. \tag{A.4}$$

Therefore this formula gives the weak derivative of $f \circ u$ on $[-\pi, \pi]$.

Acknowledgement. The results of Section 4.11 were obtained in response to questions raised by Boris Buffoni and Éric Séré. It is a pleasure to thank them for their interest. We are also grateful to Eugen Vărvărucă for his comments on a draft which helped us improve the presentation.

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