A few remarks about the Hilbert transform

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Abstract

A generalized integral similar to integrability B is used to study the Hilbert transform on $X = \oplus_{1 \leq p < \infty} L_p(\mathbb{R})$, with a view to obtaining (i) a mollifier which commutes with the Hilbert transform on X and coincides with the Friedrichs mollifier on $L_1^{loc}(\mathbb{R})$; (ii) estimates for nonlinear equations; (iii) an integral representation for the Hilbert transform of a regular Schwartz distribution; (iv) a generalized multiplier representation of the Hilbert transform on $L_1(\mathbb{R})$; (v) an elementary proof of the injectivity of $\mathcal{H}$ on X.

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1 Introduction

For any $p \in (1, \infty)$ the Hilbert transform $\mathcal{H}$, defined pointwise for $u \in L_p(\mathbb{R})$ by

$$\mathcal{H}u(x) = \lim_{\epsilon \searrow 0} \left(\frac{1}{\pi}\right) \int_{|y| \geq \epsilon} \frac{u(x-y)}{y} dy, \quad x \in \mathbb{R}, \tag{1.1}$$

is a bounded linear operator on $L_p(\mathbb{R})$ (see [18, Ch. II], [19, Ch. IV] or [20, Ch. II]). Since $(\mathcal{H}u)^\wedge = mu^\wedge$, when $u^\wedge$ denotes the Fourier transform of u in $L_2(\mathbb{R})$ and $m(k) = i \text{ sign}(k)$ [18, page 48] it follows from Plancherel's Theorem that $\mathcal{H}^2 = -I$ on $L_2(\mathbb{R})$ and thence, by density and continuity arguments, that $\mathcal{H}^2 = -I$ on $L_p(\mathbb{R}), 1 < p < \infty$.

When $u \in L_1(\mathbb{R})$ a function $\mathcal{H}u$ is defined almost everywhere on $\mathbb{R}$ by (1.1) ([19, Lemma III.1.2], [20, Section 5.14]), but $\mathcal{H}$ does not map $L_1(\mathbb{R})$ into itself. Indeed Kober [11] showed that if u and $\mathcal{H}u$ are in $L_1(\mathbb{R})$ then $\int_{-\infty}^{\infty} u(x)dx = 0$. However $\mathcal{H}$ is an operator of weak-type (1,1) on $L_1(\mathbb{R})$ [18, Ch. II] in the sense that there exists a constant A such that

$$\text{meas } \{x : |\mathcal{H}u(x)| \geq \alpha\} \leq A\|u\|_{L_1(\mathbb{R})}/\alpha \text{ for all } \alpha > 0 \,. \tag{1.2}$$

In a recent study of nonlinear equations involving the Hilbert transform [22] it was useful to know that if $u \in \oplus_{1 \leq p < \infty} L_p(\mathbb{R})$ (this notation is defined in the opening paragraph of Section 2) and $\mathcal{H}u = v \in \oplus_{1 < p \leq \infty} L_p(\mathbb{R})$ then both u and v are in $\oplus_{1 < p < \infty} L_p(\mathbb{R})$. The difficulty, which arises because $\mathcal{H}$ does not map $L_1(\mathbb{R})$ into itself, might be addressed by the complex analytic method of Hille and Tamarkin [9] who proved that when $u \in L_1(\mathbb{R})$ and $\mathcal{H}u \in L_1(\mathbb{R})$ then $\mathcal{H}(\mathcal{H}u) = -u$; see also [12, Theorem 35] for an extension of this result to generalised Hilbert transforms.

An equivalent complex-variable method is to change variables, mapping $x \in \mathbb{R}$ to $\phi \in (-\pi, \pi)$ by $x = \tan(\phi/2)$, and thereby replace the equation $\mathcal{H}u = v$, $u \in L_1(\mathbb{R})$, $v \in L_p(\mathbb{R})$ with $\mathcal{C}f = g + \text{const.}$, where $f, g \in L_1(S^1)$ and $\mathcal{C}$ denotes the conjugate operator on the unit circle S^1. That $\mathcal{H}v = -u$ then follows from the $(C, 1)$ convergence of Fourier and conjugate Fourier series, and Kolmogorov's theorem [24, Ch. III, (3.9) and (3.23), and Ch. VII, (4.4)] or, in a modern framework, from the equivalence of real- and complex-variable Hardy spaces. (This change of variables is derived from the Möbius transformation of the upper half z-plane to the unit disc in the ζ-plane given by $\zeta = (i - z)/(i + z)$.)

One purpose here is to give an elementary, real-variable proof of the above-mentioned result on $\mathcal{H}u = v$ and thereby to obtain simple proofs of its corollaries for nonlinear equations (Theorem 7, Corollary 8 and Theorem

9). A second purpose is to illustrate the utility in real-variable theory of an integral, similar to Integrability B [24, Ch. VIII] by using it to define a convolution operator which commutes with a general class of operators including the Hilbert transform on $L_p(\mathbb{R}), 1 \leq p < \infty$, and which coincides with the classical convolution for L_1-functions. According to Henstock [7], Integrability B is an idea due to Denjoy [4] and named by him after his pupil Boks (see [3]), which does not fit readily into general theories of integration [6]. A related idea, Integrability A, is attributed to Kolmogorov [13] by Bary in her version of conjugate-periodic-function theory [2, Ch.VIII, Section 18]; however see also [10] and [23].

The proofs of our results on the generalized convolution and the Hilbert transform are based on Zygmund's real-variable proofs of Kolmogorov's theorems on the Fourier series of conjugate functions [24, Ch. VII, (4.3) and (4.4)]. Since similar results clearly hold for a generalized (integral B) convolution and the conjugate operator on S^1, the use of Friedrichs mollifiers, as in the proof of Theorem 7, then gives easy real-variable proofs of innocent looking but subtle Hardy-space results such as

$$\{f \in L_1(S^1) : \tilde{f} \in L_2(S^1)\} = L_2(S^1) \ .$$

Here $\tilde{f}$ denotes the conjugate of $f \in L_1(S^1)$.

Pandey [15] has defined the Hilbert transform of a Schwartz distribution and shown that when the distribution corresponds to $u \in L_p(\mathbb{R})$, $1 < p < \infty$, then the Hilbert transform corresponds to $\mathcal{H}u \in L_p(\mathbb{R})$, in the usual way. Section 6 gives a generalised-integral representation of the Hilbert transform of distributions and ultradistributions corresponding to $u \in L_1(\mathbb{R})$ (Theorem 11).

Finally it is shown in Section 7 that the Fourier transform of $\mathcal{H}u$, calculated when $u \in L_1(\mathbb{R})$ using the generalised integral, is $mu^\wedge$, where m is defined above. This result, which gives a sense in which $\mathcal{H}$ is a multiplier on $L_1(\mathbb{R})$, is the precise analogue of Kolmogorov's theorem on the Fourier series of conjugate periodic functions [24, Ch. VII, (4.3)]. A corollary is that $\mathcal{H}u = v,\ u, v \in L_1(\mathbb{R})$ if, and only if, $mu^\wedge = v^\wedge$.

The generalized integral is introduced in Section 3; the convolution operator, the required result about $\mathcal{H}u = v$ and related matters follow using Friedrichs mollifiers in Sections 4 and 5. A simple corollary is that $\mathcal{H}$ is injective on $\oplus_{1\leq p<\infty} L_p(\mathbb{R})$.

Section 2 begins with some preliminary results and notation, and ends with a result about the Hardy operator and the Hilbert transform for a class of odd functions which is useful in studies of certain nonlinear equations.

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2 Preliminaries

The Lebesgue integral over $[a, b]$ of a Lebesgue measurable function f will be denoted by $\int_a^b f(x)dx$. As usual, $L_p(\mathbb{R})$ will denote the Banach space of (equivalence classes of) real-valued Lebesgue measurable functions whose pth power is integrable over $\mathbb{R}$, $1 \leq p < \infty$, or which are essentially bounded for $p = \infty$. The set of functions on $\mathbb{R}$ which can be written as a finite sum of functions in $L_p(\mathbb{R})$, $p \in P$, will be denoted by $\oplus_{p \in P} L_p(\mathbb{R})$. Let $X = \oplus_{1 \leq p < \infty} L_p(\mathbb{R})$ (so that each element of X is a finite sum of functions from $\cup_{1 \leq p < \infty} L_p(\mathbb{R})$). Similarly, let $X^0 = \oplus_{1 < p < \infty} L_p(\mathbb{R})$. Then $\mathcal{H} : X^0 \to X^0$, $\mathcal{H}^2 = -I$ on X^0 and $X = L_1(\mathbb{R}) \oplus X^0$. The linear space of functions which are pth-power-integrable or essentially bounded on every compact subset of $\mathbb{R}$ will be denoted by $L_p^{loc}(\mathbb{R})$, $1 \leq p \leq \infty$, and $C_b^1(\mathbb{R})$ will denote the space of functions which, along with their first derivatives, are bounded and continuous on $\mathbb{R}$. Let $C(\mathbb{R})$ denotes the space of continuous functions on $\mathbb{R}$. Finally, let $\mathcal{D}$ denote the space of all smooth functions with compact support on $\mathbb{R}$ endowed with the usual locally convex topology and let $\mathcal{D}'$ denote its dual, the space of (Schwartz) distributions.

For $\phi \in L_1(\mathbb{R})$, $u \in L_p(\mathbb{R})$, $1 \leq p \leq \infty$, let

$$\phi * u(x) = \int_{-\infty}^{+\infty} \phi(y)u(x-y)dy \ . \tag{2.1}$$

It is well-known [17, Ch. 8, Ex.4(a)], that $\phi * u(x)$ is well-defined for almost all $x \in \mathbb{R}$ and

$$\|\phi * u\|_{L_p(\mathbb{R})} \leq \|\phi\|_{L_1(\mathbb{R})} \|u\|_{L_p(\mathbb{R})} \text{ for } u \in L_p(\mathbb{R}),\ 1 \leq p \leq \infty \ . \tag{2.2}$$

For $\phi \in L_p(\mathbb{R})$, $u \in L_q(\mathbb{R})$, $p^{-1} + q^{-1} = 1$, $1 \leq p, q \leq \infty$,

$$\|\phi * u\|_{L_\infty(\mathbb{R})} \leq \|\phi\|_{L_p(\mathbb{R})} \|u\|_{L_q(\mathbb{R})} \ , \tag{2.3}$$

by Hölder's inequality. For any set $\Omega \subset \mathbb{R}$ let χ_Ω be its characteristic function.

Lemma 1. *Suppose $v \in C_b^1(\mathbb{R})$ and $u \in L_p(\mathbb{R})$. For $p = 1$*

$$\|\mathcal{H}(vu) - v\mathcal{H}(u)\|_{L_\infty(\mathbb{R})} \leq \frac{1}{\pi}\|u\|_{L_1(\mathbb{R})}\|v'\|_{L_\infty(\mathbb{R})} \ , \tag{2.4}$$

and for $1 < p < \infty$ there exists K_p such that

$$\|\mathcal{H}(vu) - v\mathcal{H}(u)\|_{L_\infty(\mathbb{R})} \leq K_p\|u\|_{L_p(\mathbb{R})}(\|v'\|_{L_\infty(\mathbb{R})} + \|v\|_{L_\infty(\mathbb{R})}) \ .$$

Proof. By definition, for almost all $x \in \mathbb{R}$,

$$\begin{aligned}\mathcal{H}(vu)(x) \quad &- \quad v(x)\mathcal{H}u(x) \\ &= \frac{1}{\pi}\lim_{\epsilon\searrow 0}\int_{|y|>\epsilon}\left\{\frac{v(x-y)-v(x)}{y}\right\}u(x-y)dy \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\Phi_x(y)u(x-y)dy = \left(\frac{1}{\pi}\right)\Phi_x * u(x),\end{aligned}$$

by the Dominated Convergence Theorem, where

$$\Phi_x(y) = \begin{cases} -v'(x), & \text{if} \quad y = 0, \\ \dfrac{v(x-y)-v(x)}{y} & \text{if} \quad y \neq 0 \ . \end{cases}$$

Since $\|\Phi_x\|_{L_\infty(\mathbb{R})} \leq \|v'\|_{L_\infty(\mathbb{R})}$, by the Mean Value theorem, for $u \in L_1(\mathbb{R})$, the first part follows by (2.3). Now observe that for $x \in \mathbb{R}$, and $1/p+1/q = 1$, $q > 1$,

$$\begin{aligned}\int_{-\infty}^{\infty}|\Phi_x(y)|^q dy \quad &\leq \quad 2\|v'\|^q_{L_\infty[-1,1]} + 2^q\|v\|^q_{L_\infty(\mathbb{R})}\int_{|y|\geq 1}\left(\frac{1}{|y|}\right)^q dy \\ &\leq \quad 2\|v'\|^q_{L_\infty(\mathbb{R})} + \frac{2^{q+1}}{(q-1)}\|v\|^q_{L_\infty(\mathbb{R})} \ .\end{aligned}$$

Hence for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$

$$\|\Phi_x\|_{L_q(\mathbb{R})} \leq K_p(\|v'\|_{L_\infty(\mathbb{R})} + \|v\|_{L_\infty(\mathbb{R})}) \ .$$

The result now follows by (2.3). □

Let $f^\wedge$ denote the Fourier transform of $f \in L_1(\mathbb{R}) \cup L_2(\mathbb{R})$. Then for $\phi \in L_1(\mathbb{R})$

$$(\phi * f)^\wedge(k) = \phi^\wedge(k)f^\wedge(k) \text{ almost everywhere} \tag{2.5}$$

and for $f \in L_2(\mathbb{R})$

$$(\mathcal{H}f)^\wedge(k) = m(k)f^\wedge(k) \text{ almost everywhere} \quad , \tag{2.6}$$

where [18, page 48]

$$m(k) = i\ \mathrm{sign}(k), \quad k \in \mathbb{R}\ .$$

It follows from (2.5) and (2.6) that $\mathcal{H}(\phi * f) = \phi * \mathcal{H}(f)$ for all $f \in L_2(\mathbb{R})$. Hence by the density of $L_2(\mathbb{R})$ in $L_p(\mathbb{R})$ and the continuity of $\mathcal{H}$ on $L_p(\mathbb{R})$, $1 < p < \infty$, it follows from (2.2) that for $\phi \in L_1(\mathbb{R})$

$$\mathcal{H}(f * \phi) = \mathcal{H}(\phi * f) = \phi * \mathcal{H}(f) \text{ for all } f \in L_p(\mathbb{R}), \quad 1 < p < \infty\ . \tag{2.7}$$

It also follows from (2.6) and Plancherel's theorem that for $f, g \in L_2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} g(x)\mathcal{H}f(x)dx = \int_{-\infty}^{\infty} g^\wedge(k)\overline{m(k)f^\wedge(k)}dk = -\int_{-\infty}^{\infty} \mathcal{H}g(x)f(x)dx\ .$$

It now follows, by similar density and continuity arguments, that

$$\int_{-\infty}^{\infty} g(x)\mathcal{H}f(x)dx = -\int_{-\infty}^{\infty} f(x)\mathcal{H}g(x)dx, \quad f \in L_p(\mathbb{R}), \quad g \in L_q(\mathbb{R})\ , \tag{2.8}$$

when $1/p + 1/q = 1$, $1 < p < \infty$. Similarly, since $\mathcal{H}^2 = -I$ on $L_2(\mathbb{R})$ from (2.6),

$$\mathcal{H}\mathcal{H}u = -u \text{ for all } u \in L_p(\mathbb{R}), \quad 1 < p < \infty\ . \tag{2.9}$$

Also, note the obvious fact that if $\phi \in \mathcal{D}$ then $\mathcal{H}\phi \in L_\infty(\mathbb{R})$. (Indeed Logan [14] has a sharp estimate

$$\|\mathcal{H}\phi\|_{L_\infty(\mathbb{R})} \le \frac{4\log 2}{\pi}\left\{\|\phi'\|_{L_\infty(\mathbb{R})}|\phi|\right\}^{\frac{1}{2}} \tag{2.10}$$

where $|\phi| = \sup_{a,b}\{|\int_a^b \phi(x)dx|\}$.)

The Hardy operator and Hilbert transform of some odd functions

It is clear from the definition that the Hilbert transform of an odd function is even and *vice versa*. It also follows easily that when $v \in X^0$ is odd and $v \ge 0$ on $(0,\infty)$,

$$\int_0^x \mathcal{H}v(t)dt \le 0, \quad x \in (0,\infty)\ . \tag{2.11}$$

To see this it suffices, by density and continuity arguments, to observe that (2.11) holds for smooth, odd functions v with compact support which are non-negative on $(0,\infty)$. Now

$$\begin{aligned}\mathcal{H}v(x) &= \frac{1}{\pi}\lim_{\epsilon\searrow 0}\int_{|y|>\epsilon}\frac{v(x-y)}{y}\ dy\\ &= \frac{1}{\pi}\lim_{\epsilon\searrow 0}\int_{\epsilon}^{\infty}\frac{v(x-y)-v(x+y)}{y}\ dy\ ,\end{aligned}$$

whence, for $x > 0$,

$$\begin{aligned}\int_0^x \mathcal{H}v(t)dt &= \frac{1}{\pi}\lim_{\epsilon\searrow 0}\int_\epsilon^\infty \frac{1}{y}\left\{\int_0^x (v(t-y)-v(t+y)dt\right\}dy\\ &= -\frac{1}{\pi}\lim_{\epsilon\searrow 0}\int_\epsilon^\infty \frac{1}{y}\left\{\int_{x-y}^{x+y} v(t)dt\right\}dy\ ,\end{aligned}$$

since v is odd,

$$= -\frac{1}{\pi}\lim_{\epsilon\searrow 0}\int_\epsilon^\infty \frac{1}{y}\left\{\int_{|x-y|}^{x+y} v(t)dt\right\}dy \le 0\ .$$

We conclude these preliminary observations with one about the Hilbert transform of an odd function v for which $v(x)/x$ is non-negative and non-increasing on $(0,\infty)$. Theorem 2 is implicit in a water-wave result [1, Theorem 2.6 and footnote] where a maximum-principle argument was used in the proof. The present proof of the Hilbert-transform result (which is similar to but simpler than an unpublished proof of the water-wave result due to L.E. Fraenkel) depends upon the following inequality, which is easily confirmed by differentiating both sides at $t \in (0,1)$ and $t \in (1,\infty)$ and using the behaviour at $t = 0$ and as $t \to \infty$:

$$\log\left(\frac{1+t}{|1-t|}\right) \ge \frac{2t}{1+t^2}\ ,\quad t \in [0,\infty),\ t \ne 1\ . \tag{2.12}$$

Let T denote the Hardy operator which is defined by

$$Tu(x) = \frac{1}{x}\int_0^x u(t)dt\ ,\quad u \in L_p(\mathbb{R})\ ,\quad 1 \le p \le \infty\ .$$

By Hardy's inequality ([17, Ch.3, Ex. 14], T is a bounded linear operator on $L_p(\mathbb{R}),\ 1 < p \le \infty$.

Theorem 2. *Suppose that $v \in L_p(\mathbb{R}),\ 1 < p < \infty$ is odd and $v(x)/x$ is non-negative and non-increasing for $x \in (0,\infty)$. Let $w = \mathcal{H}v$. Then Tw is non-positive and non-decreasing on $(0,\infty)$.*

Proof. We begin by proving the result for smooth functions v with compact support in $\mathbb{R}$ which satisfy the hypothesis.

That $\int_0^x w(t)dt \le 0$ on $(0,\infty)$ has already been established. Since v is odd, for $x > 0$

$$\begin{aligned} w(x) &= \frac{1}{\pi}\lim_{\epsilon\searrow 0}\int_{|x-y|>\epsilon}\frac{v(y)}{x-y}dy \\ &= \frac{1}{\pi}\lim_{\epsilon\searrow 0}\int_{\{y\ge 0:|x-y|>\epsilon\}} v(y)\left\{\frac{1}{x-y}-\frac{1}{x+y}\right\}dy \\ &= -\frac{1}{\pi}\lim_{\epsilon\searrow 0}\int_{\{y\ge 0:|x-y|>\epsilon\}} v(y)\frac{\partial}{\partial x}\left(\log\frac{x+y}{|x-y|}\right)\, dy\ . \end{aligned}$$

Since v is assumed to be odd and smooth with compact support,

$$\begin{aligned} \int_0^x w(t)dt &= -\frac{1}{\pi}\int_0^\infty v(y)\log\left(\frac{x+y}{|x-y|}\right)\, dy \\ &= \frac{1}{\pi}\int_0^\infty \frac{d}{dy}\left(\frac{v(y)}{y}\right)\left\{\int_0^y s\log\left(\frac{x+s}{|x-s|}\right)ds\right\}\, dy\ . \end{aligned}$$

The required result is then a consequence of the observation that

$$K(x,y) = \frac{\partial}{\partial x}\left\{\frac{1}{x}\int_0^y s\log\left(\frac{x+s}{|x-s|}\right)ds\right\} \le 0 \text{ for } x,y>0,\ x\ne y\ . \quad (2.13)$$

To see this, note that

$$\begin{aligned} \int_0^y s\log\left(\frac{x+s}{|x-s|}\right)ds &= \int_0^y s\log(x+s)ds - \int_0^y s\log|x-s|\ ds \\ &= \frac{1}{2}(y^2-x^2)\log\left(\frac{x+y}{|x-y|}\right) + xy \text{ for } y\ne x\ . \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial}{\partial x}\left\{\frac{1}{x}\int_0^y s\log\left(\frac{x+s}{|x-s|}\right)ds\right\} &= \frac{\partial}{\partial x}\left\{\frac{(y^2-x^2)}{2x}\log\left(\frac{x+y}{|x-y|}\right)\right\} \\ &= \frac{y}{x}-\frac{1}{2}\left(1+\left(\frac{y}{x}\right)^2\right)\log\left(\frac{1+y/x}{|1-y/x|}\right)\le 0 \end{aligned}$$

by (2.12).

To complete the proof it suffices to show that any $v \in L_p(\mathbb{R})$ which satisfies the hypotheses can be approximated to any degree of accuracy in $L_p(\mathbb{R})$ by a smooth function of compact support which satisfies the hypotheses. The result then follows by a simple limiting argument. Further, there is no loss of generality in supposing that the function v to be approximated is, at the outset, of compact support, in $L_p(\mathbb{R})$, linear on $[-\delta, \delta]$ for some $\delta > 0$ and satisfies the hypotheses of the theorem. So for such a function v let

$$V(x) = \frac{v(x)}{x} \ , \quad x \in \mathbb{R}\backslash\{0\} \ .$$

Then V is constant on $[-\delta, \delta]\backslash\{0\}$, even, non-increasing on $(-\infty, \infty)$ and V has compact support, in $[-N, N]$ say. Now Friedrichs mollifiers [18] can be used to find, for any $\epsilon > 0$, a smooth even function V_ϵ with compact support in $[-N-1, N+1]$ which is non-increasing on $[0, \infty)$ and $\|V_\epsilon - V\|_{L_\infty(\mathbb{R})} < \epsilon$. Now let

$$v_\epsilon(x) = xV_\epsilon(x)$$

to obtain the required approximation. This completes the proof. □

3 A generalized integral

Here we adapt to the present context the notion of Integrability B defined by Zygmund for periodic functions [24, Vol. 1, pages 263 and 381]. For any function $f : \mathbb{R} \to \mathbb{R}$ with compact support, $n \in \mathbb{N}$ and $t \in [0, 1]$, let

$$I_n(f)(t) = \frac{1}{2^n} \sum_{k=-\infty}^{\infty} f\left(t + \frac{k}{2^n}\right) \ . \tag{3.1}$$

Definition. For $I \in \mathbb{R}$ write

$$I = \sharp \int_{\mathbb{R}} f(x)dx$$

if, and only if,

$$I_n(f)(t) \to I \text{ in measure on } [0, 1] \text{ as } n \to \infty \ .$$

□

(Recall $I_n(f) \to I$ pointwise almost everywhere implies that $I_n(f) \to I$ in measure; conversely if $I_n(f) \to I$ in measure then a subsequence $I_{n_k}(f) \to I$ almost everywhere.)

Note that if g is a continuous function with compact support in $\mathbb{R}$ then for any $t \in [0, 1]$, $I_n(f)(t)$ is a Riemann partial sum of f corresponding to a regular partition of its support with intervals of length $1/2^n$. Consequently, when g has compact support and is continuous, $\sharp \int_{\mathbb{R}} g(x)dx$ exists and

$$I_n(g)(t) \to \int_{-\infty}^{\infty} g(x)dx \text{ uniformly for } t \in [0, 1] \tag{3.2}$$

as $n \to \infty$ [16, Theorem 6.4].

The extension of the $\sharp$ integral to functions which are not compactly supported is postponed to Section 6 where it is needed to calculate the Hilbert transform of regular distributions; in Section 7 it is used to calculate the Fourier transform of $\mathcal{H}u$ for $u \in L_1(\mathbb{R})$. The following result, which also holds in that more general context, is based on Saks' proof of the analogous result for periodic functions [24].

Lemma 3. *If $f \in L_1(\mathbb{R})$ and f has compact support then*

$$\sharp \int_{\mathbb{R}} f(x)dx = \int_{-\infty}^{\infty} f(x)dx \ .$$

Proof. Suppose that support$(f) \subset [-N, N]$. Let $\alpha > 0$ and $\epsilon \in (0,1)$ be given and let $f = f_1 + f_2$, where f_1 is continuous with compact support in $[-N, N]$ and $\|f_2\|_{L_1(\mathbb{R})} < \epsilon\alpha/4(N+1)$. Since $[k/2^n, 1+k/2^n] = [0,1] + k/2^n$ intersects $[-N, N]$ for at most $2^n(2N+1)+1$ values of k,

$$\begin{aligned} \int_0^1 |I_n(f_2)(t)|dt &\leq \frac{1}{2^n}\sum\nolimits_{-\infty}^{\infty}\int_0^1 |f_2\left(t + \frac{k}{2^n}\right)|dt \\ &\leq (2N+2)\|f_2\|_{L_1(\mathbb{R})} \leq \epsilon\alpha/2 \ . \end{aligned}$$

Hence, for all $n \in \mathbb{N}$,

$$\text{meas}\{t \in [0,1] : |I_n(f_2)(t)| \geq \alpha/2\} \leq \epsilon \ . \tag{3.3}$$

Since $\epsilon \in (0,1)$, the choice of f_1 and f_2 means that for all $t \in [0,1]$

$$\begin{aligned} |I_n(f)(t) - \int_{-\infty}^{\infty} f(x)dx| &\leq |I_n(f_1)(t) - \int_{-\infty}^{\infty} f_1(x)dx| \\ &\quad + |I_n(f_2)(t)| + \|f_2\|_{L_1(\mathbb{R})} \\ &\leq |I_n(f_1)(t) - \int_{-\infty}^{\infty} f_1(x)dx| + |I_n(f_2)(t)| + \alpha/4 \ . \end{aligned}$$

Therefore

$$\begin{aligned} &\text{meas}\ \{t \in [0,1] : |I_n(f)(t) - \int_{-\infty}^{\infty} f(x)dx| \geq \alpha\} \\ \leq\ & \text{meas}\{t \in [0,1] : |I_n(f_1)(t) - \int_{-\infty}^{\infty} f_1(x)dx| \geq \alpha/4\} \\ & + \text{meas}\{t \in [0,1] : |I_n(f_2)(t)| \geq \alpha/2\} \\ \leq\ & \text{meas}\{t \in [0,1] : |I_n(f_1)(t) - \int_{-\infty}^{\infty} f_1(x)dx| \geq \alpha/4\} + \epsilon \ , \text{ by (3.3)} \ , \\ \leq\ & \epsilon \end{aligned}$$

for all n sufficiently large, by (3.2), since f_1 is continuous with compact support. Hence $I_n(f) \to \int_{-\infty}^{\infty} f(x)dx$ in measure, and this is the required result.

□

The next result is the key step. It holds more generally for any linear operator of weak-type (1,1) which commutes with translations for which a version of (2.4) holds.

Lemma 4. *Suppose that $\phi \in \mathcal{D}$. Then there exists a constant $K = K(\phi)$ such that for $\epsilon \in (0,1)$, $\alpha > 0$ and $u \in L_1(\mathbb{R})$ with $\|u\|_{L_1(\mathbb{R})} \leq K\alpha\epsilon$*

$$\mathit{meas}\{t \in [0,1] : |I_n(\phi\mathcal{H}u)(t)| \geq \alpha\} < \epsilon$$

for all $n \in \mathbb{N}$.

Proof. Suppose that support$(\phi) \subset [-N,N]$. Then the set

$$A_{n,t} = \{k \in \mathbb{Z} : t + \frac{k}{2^n} \in [-N,N]\}, \quad t \in [0,1], \quad n \in \mathbb{N}\ ,$$

has at most $(1 + N2^{n+1})$ elements. By Lemma 1

$$\|\phi\mathcal{H}(u) - \mathcal{H}(\phi u)\|_{L_\infty(\mathbb{R})} \leq \left(\frac{1}{\pi}\right) \|u\|_{L_1(\mathbb{R})} \|\phi'\|_{L_\infty(\mathbb{R})}\ .$$

Therefore, for all $n \in \mathbb{N}$ and $t \in [0,1]$,

$$\begin{aligned} |I_n(\phi\mathcal{H}u)(t)| &= \left(\frac{1}{2^n}\right) \left| \sum_{k \in A_{n,t}} \phi\left(t + \frac{k}{2^n}\right) \mathcal{H}u\left(t + \frac{k}{2^n}\right) \right| \\ &\leq \left(\frac{1}{2^n}\right) \left| \sum_{k \in A_{n,t}} \mathcal{H}(\phi u)\left(t + \frac{k}{2^n}\right) \right| + \frac{(2N+1)}{\pi} \|u\|_{L_1(\mathbb{R})} \|\phi'\|_{L_\infty(\mathbb{R})} \\ &\leq \left(\frac{1}{2^n}\right) \left| \sum_{k \in A_{n,t}} \mathcal{H}(\phi u)\left(t + \frac{k}{2^n}\right) \right| + \frac{\alpha}{2} \qquad (3.4) \end{aligned}$$

if $\|u\|_{L_1(\mathbb{R})} \leq \pi\alpha/2(2N+1)\|\phi'\|_{L_\infty(\mathbb{R})}$. Now let $k \in A_{n,t}$ and for $x \in \mathbb{R}$ let

$$v_k(x) = \phi\left(x + \frac{k}{2^n}\right) u\left(x + \frac{k}{2^n}\right)\ .$$

Since $\mathcal{H}$ is linear and commutes with translations

$$\frac{1}{2^n}\sum_{k\in A_{n,t}}\mathcal{H}(\phi u)\left(t+\frac{k}{2^n}\right)=\mathcal{H}\left(\frac{1}{2^n}\sum_{k\in A_{n,t}}v_k\right)(t)\ , \tag{3.5}$$

where

$$\begin{aligned}\int_{-\infty}^{\infty}\left|\frac{1}{2^n}\sum_{k\in A_{n,t}}v_k(s)\right|ds &\leq (2N+1)\int_{-\infty}^{\infty}|\phi(s)u(s)|ds\\ &\leq (2N+1)\|u\|_{L_1(\mathbb{R})}\|\phi\|_{L_\infty(\mathbb{R})}\ .\end{aligned}$$

Therefore, by (1.2), there is a constant A such that for all $\alpha>0$

$$\begin{aligned}\text{meas}\{t\in\mathbb{R}\ &:\ \left|\mathcal{H}\left(\frac{1}{2^n}\sum_{k\in A_{n,t}}v_k\right)(t)\right|\geq\alpha/2\}\\ &\leq\ 2(2N+1)A\|\phi\|_{L_\infty(\mathbb{R})}\|u\|_{L_1(\mathbb{R})}/\alpha\leq\epsilon\end{aligned} \tag{3.6}$$

if $\|u\|_{L_1(\mathbb{R})}\leq\alpha\epsilon/2(2N+1)A\|\phi\|_{L_\infty(\mathbb{R})}$. Let

$$K=\min\left\{\frac{1}{2(2N+1)A\|\phi\|_{L_\infty(\mathbb{R})}},\frac{\pi}{2(2N+1)\|\phi'\|_{L_\infty(\mathbb{R})}}\right\}\ .$$

Then for $\epsilon\in(0,1)$ it follows from (3.4), (3.5) and (3.6) that

$$\begin{aligned}\text{meas}\{t\in[0,1]\ &:\ |I_n(\phi\mathcal{H}u)(t)|\geq\alpha\}\\ &\leq\ \text{meas}\left\{t\in[0,1]:\left|\left(\frac{1}{2^n}\right)\sum_{k\in A_{n,t}}\mathcal{H}(\phi u)\left(t+\frac{k}{2^n}\right)\right|\geq\frac{\alpha}{2}\right\}\\ &<\ \epsilon\text{ if }\|u\|_{L_1(\mathbb{R})}\leq K\alpha\epsilon\ .\end{aligned}$$

This completes the proof. □

The main result of this section, which holds more generally for linear operators of weak-type (1,1) for which a version of (2.4) holds, which are skew-symmetric on $L_2(\mathbb{R})$, map $\mathcal{D}$ into $C(\mathbb{R})$ and commute with translations, is the following.

Theorem 5. *For all $\phi\in\mathcal{D}$ and $u\in L_1(\mathbb{R})$*

$$\sharp\int_{\mathbb{R}}\phi(x)\mathcal{H}u(x)dx=-\int_{-\infty}^{\infty}u(x)\mathcal{H}\phi(x)dx\ .$$

Proof. Suppose ϕ has compact support in $[-N,N]$. Let $u = v_k + w_k$ where $v_k \in \mathcal{D}$ and $\|w_k\|_{L_1(\mathbb{R})} \to 0$ as $k \to \infty$. For $x \in \mathbb{R}$, let

$$U(x) = \phi(x)\mathcal{H}u(x), \quad V_k(x) = \phi(x)\mathcal{H}v_k(x), \quad W_k(x) = \phi(x)\mathcal{H}w_k(x)\ .$$

Let $\alpha > 0$ and $\epsilon \in (0,1)$ and, using Lemma 4, choose $K \in \mathbb{N}$ such that

$$\text{meas}\{t \in [0,1] : |I_n(W_k)(t)| \geq \alpha/4\} < \epsilon \tag{3.7}$$

for all $n \in \mathbb{N}$ and all $k \geq K$. Since $v_k \in L_2(\mathbb{R})$ and $\phi \in L_2(\mathbb{R})$

$$\begin{aligned} \int_{-\infty}^{\infty} V_k(x)dx &= \int_{-\infty}^{\infty} \phi(x)\mathcal{H}v_k(x)dx \\ &= -\int_{-\infty}^{\infty} v_k(x)\mathcal{H}\phi(x)dx, \text{ by (2.8)}, \\ &\to -\int_{-\infty}^{\infty} u(x)\mathcal{H}\phi(x)dx\ , \end{aligned} \tag{3.8}$$

since $\mathcal{H}\phi \in L_\infty(\mathbb{R})$ (see (2.10)) and $v_k \to u$ in $L_1(\mathbb{R})$. Choose K sufficiently large that (3.7) holds and

$$|\int_{-\infty}^{\infty} V_K(x)dx + \int_{-\infty}^{\infty} u(x)\mathcal{H}\phi(x)dx| \leq \alpha/2\ . \tag{3.9}$$

Now for $n \in \mathbb{N}$ and $t \in [0,1]$

$$\begin{aligned} |I_n(U)(t) &+ \int_{-\infty}^{\infty} u(x)\mathcal{H}\phi(x)dx| \\ &\leq |I_n(V_K)(t) - \int_{-\infty}^{\infty} V_K(x)dx| + |I_n(W_K)(t)| \\ &+ |\int_{-\infty}^{\infty} V_K(x)dx + \int_{-\infty}^{\infty} u(x)\mathcal{H}\phi(x)dx| \\ &\leq |I_n(V_K)(t) - \int_{-\infty}^{\infty} V_K(x)dx| + |I_n(W_K)(t)| + \alpha/2\ , \end{aligned}$$

by (3.9). Therefore

$$\begin{aligned} &\text{meas}\{t \in [0,1] : |I_n(U)(t) + \int_{-\infty}^{\infty} u(x)\mathcal{H}\phi(x)dx| \geq \alpha\} \\ &\quad \leq \text{meas}\{t \in [0,1] : |I_n(V_K)(t) - \int_{\infty}^{\infty} V_K(x)dx| \geq \alpha/4\} + \epsilon\ , \end{aligned}$$

by (3.7). Since $v_K \in L_2(\mathbb{R})$ is smooth, it follows that V_K is continuous with compact support in $[-N, N]$. Therefore by (3.2) for all n sufficiently large

$$\text{meas}\{t \in [0,1] : |I_n(U)(t) + \int_{-\infty}^{\infty} u(x)\mathcal{H}\phi(x)dx| \geq \alpha\} < \epsilon \ .$$

This proves that $\sharp \int_{\mathbb{R}} \phi(x)\mathcal{H}u(x)dx = -\int_{\infty}^{\infty} u(x)\mathcal{H}\phi(x)dx$, as required.

□

This observation leads to a definition of convolution which commutes with the Hilbert transform on $L_1(\mathbb{R})$ discussed in the next section.

4 The $\sharp$ convolution and its consequences

Suppose that $\psi \in \mathcal{D}$. Then Theorem 5 and the fact that $\mathcal{H}$ on $L_2(\mathbb{R})$ commutes with translations and anti-commutes with reflection gives that, for $u \in L_1(\mathbb{R})$ and all $x \in \mathbb{R}$,

$$\begin{aligned} \sharp \int_{\mathbb{R}} \psi(y)(\mathcal{H}u)(x-y)dy &= \int_{-\infty}^{\infty} u(x-y)\mathcal{H}\psi(y)dy \\ &= (u * \mathcal{H}\psi)(x) \\ &= \mathcal{H}(u * \psi)(x) = \mathcal{H}(\psi * u)(x) \ , \text{ by (2.7)} \ . \end{aligned}$$

Definition. For $\psi \in \mathcal{D}$ and $f : \mathbb{R} \to \mathbb{R}$ the $\sharp$ convolution $\psi \sharp f$ is defined by

$$\psi \sharp f(x) = \sharp \int_{\mathbb{R}} \psi(y) f(x-y)dy \ .$$ □

The preceding discussion shows that

$$\psi \sharp \mathcal{H}u(x) = \mathcal{H}(\psi * u)(x) \tag{4.1}$$

when $u \in L_1(\mathbb{R})$ and $\psi \in \mathcal{D}$. (Indeed (4.1) holds more generally for convolution operators of weak-type (1,1) for which Lemma 4 holds.)

Lemma 6. *If $u \in X$ and $\mathcal{H}u \in X$ then for $\phi \in \mathcal{D}$,*

$$\phi * \mathcal{H}u = \mathcal{H}(\phi * u) \ \textit{almost everywhere} \ .$$

Proof. Let $u \in X$ be written as $u = v + w$ where $v \in L_1(\mathbb{R})$ and $w \in X^0$. Then for almost all $x \in \mathbb{R}$,

$$\begin{aligned}\mathcal{H}(\phi * u)(x) &= \mathcal{H}(\phi * v)(x) + \mathcal{H}(\phi * w)(x) \\ &= \phi \sharp \mathcal{H}v(x) + \phi * \mathcal{H}w(x), \text{ by (4.1) and (2.7)} .\end{aligned}$$

But $\mathcal{H}u \in L_1^{loc}(\mathbb{R})$ since $\mathcal{H}u \in X$, and $\mathcal{H}w \in X^0$ since $\mathcal{H}$ maps X^0 into itself. Hence $\mathcal{H}v \in L_1^{loc}(\mathbb{R})$ and therefore $\phi * \mathcal{H}v = \phi \sharp \mathcal{H}v$ almost everywhere, by Lemma 3. This proves the required result.

□

The next result leads to the (known) Hardy-space results mentioned in the Introduction. For example, for $1 < p < \infty$,

$$\{f \in L_1(\mathbb{R}) : \mathcal{H}f \in L_p(\mathbb{R})\} = L_1(\mathbb{R}) \cap L_p(\mathbb{R}).$$

Theorem 7. *If $u, \mathcal{H}u \in X$ then*

$$\mathcal{H}(\mathcal{H}u) = -u \ .$$

Proof. Let $\{\phi_n\}$ be a sequence of regularising kernels [18, page 123] or [5, page 147]. In particular, for $v \in L_p(\mathbb{R}), 1 \leq p < \infty$, $\phi_n * v \to v$ in $L_p(\mathbb{R})$ as $n \to \infty$ and, by (2.2) and (2.3), $\phi_n * v \in L_\infty(\mathbb{R}) \cap L_p(\mathbb{R})$. Since by Lemma 6

$$\phi_n * \mathcal{H}(u) = \mathcal{H}(\phi_n * u)$$

it follows by (2.9) that

$$\mathcal{H}(\phi_n * \mathcal{H}u) = \mathcal{H}(\mathcal{H}(\phi_n * u)) = -\phi_n * u \ ,$$

since $\phi_n * u \in L_q(\mathbb{R})$ for all q sufficiently large.

Now $u \in X$ and therefore a subsequence $\{\phi_{n_k} * u\}$ converges almost everywhere to u as $k \to \infty$. Also $\mathcal{H}u \in X$, so $\mathcal{H}u = v + \sum_{i=1}^m w_i$, $v \in L_1(\mathbb{R})$, $w_i \in L_{p_i}(\mathbb{R})$ $1 < p_i < \infty$. Since $\{\phi_{n_k} * w_i\}$ converges to w_i in $L_{p_i}(\mathbb{R})$, a further subsequence exists such that $\{\mathcal{H}(\phi_{n_k} * w_i)\}$ converges almost everywhere to $\mathcal{H}(w_i), 1 \leq i \leq m$. Also $\phi_k * v \to v$ in $L_1(\mathbb{R})$ and hence, by (1.2), $\mathcal{H}(\phi_{n_k} * v) \to \mathcal{H}(v)$ in measure, and for a further subsequence the convergence is pointwise almost everywhere. This proves that $\mathcal{H}(\mathcal{H}u) = -u$.

□

Let $Y = \oplus_{1<p\leq\infty} L_p(\mathbb{R})$.

Corollary 8. *Suppose $u \in X, f \in Y$ and $\mathcal{H}u = f$. Then u and f are in X^0.*

Proof. Since $u \in X, u = v + w$ where $v \in L_1(\mathbb{R})$ and $w \in X^0$. Therefore

$$\mathcal{H}v = f - \mathcal{H}w = g - \sum_{i=1}^{m} w_i \ , \tag{4.2}$$

where $g \in L_\infty(\mathbb{R})$ and $w_i \in L_{q_i}(\mathbb{R}), 1 < q_i < q_{i+1} < \infty$.

Let $S = \{x \in \mathbb{R} : |\mathcal{H}v(x)| \geq 1\}$. Then $\text{meas}(S) < \infty$ by (1.2) and therefore

$$\int_{-\infty}^{\infty} |\chi_S \mathcal{H}v|^{q_1} dx < \infty, \text{ by (4.2) and Hölder's inequality .}$$

Also, since $q_1 > 1$

$$\begin{aligned}
\int_{-\infty}^{\infty} |(1-\chi_S)\mathcal{H}v|^{q_1} dx &\leq \sum_{k=0}^{\infty} \int_{\left\{x \in \mathbb{R} : \frac{1}{2^{k+1}} \leq |\mathcal{H}v(x)| \leq \frac{1}{2^k}\right\}} |\mathcal{H}v(x)|^{q_1} dx \\
&\leq \sum_{k=0}^{\infty} \frac{1}{2^{kq_1}} \text{meas} \left\{x \in \mathbb{R} : |\mathcal{H}v(x)| \geq \frac{1}{2^{k+1}}\right\} \\
&\leq 2A\|v\|_{L_1(\mathbb{R})} \sum_{k=0}^{\infty} \left(\frac{1}{2^{(q_1-1)}}\right)^k < \infty, \text{ by (1.2) .}
\end{aligned}$$

Hence $\mathcal{H}v \in L_{q_1}(\mathbb{R})$ and therefore $g \in X^0$, by (4.2). Hence $\mathcal{H}v \in X^0$ and so v, and hence u, is in X_0. Hence $f = \mathcal{H}u \in X^0$. This completes the proof.

□

We close this section by noting, from Kober's result [11] and Theorem 6, that $\mathcal{H}u = v, \quad u \in L_1(\mathbb{R}), \quad v \in L_1(\mathbb{R})$ implies that $\int_{-\infty}^{\infty} u(x)dx = 0 = \int_{-\infty}^{\infty} v(x)dx$.

5 A nonlinear diversion

Some important equations in continuum mechanics associated with nonlinear Neumann boundary-value problems in the upper half-plane can be written in the form

$$\mathcal{H}(u') = F(u) \tag{5.1}$$

for a (locally) absolutely continuous function $u : \mathbb{R} \to \mathbb{R}$ with $u' \in X$. Here F is a continuous function with the property

$$\left.\begin{array}{l}\text{there exists } \alpha > 0 \text{ such that both the sets} \\ \{t > 0 : |F(t)| \geq \alpha\} \text{ and } \{t < 0 : |F(t)| \geq \alpha\} \text{ have infinite measure .}\end{array}\right\} \quad \text{(P)}$$

The following result is both elementary and useful.

Theorem 9. *Suppose that u is a solution of (5.1) with $u' \in X$. If F satisfies (P) then $u \in L_\infty(\mathbb{R})$ and $u' \in X_0$.*

Proof. For $t \in \mathbb{R}$ let

$$g(t) = \int_0^t F_\alpha(s) ds$$

where for $s \in \mathbb{R}$,

$$F_\alpha(s) = \begin{cases} \alpha & \text{if} \quad |F(s)| \geq \alpha \\ 0 & \text{if} \quad |F(s)| < \alpha \ . \end{cases}$$

Then $|g(t)| \to \infty$ as $|t| \to \infty$, by property (P). Since $F(u) = \mathcal{H}(u')$ and $u' \in X$, $\text{meas}\{x : |F(u(x))| \geq \alpha\} < \infty$ and therefore $F_\alpha(u) \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ is a function which is non-zero only on a set of finite measure. It follows easily that $g \circ u$ has bounded variation on compact intervals and hence [8, 18.37] $g(u)$ is (locally) absolutely continuous. Now

$$|g(u(x))| = |\int_0^x g'(u(y))u'(y)dy| = |\int_0^x F_\alpha(u(y))u'(y)dy| \leq C$$

where C is a constant independent of x. This follows by Hölder's inequality because $u' \in \oplus_{1 \leq p < \infty} L_p(\mathbb{R})$ and $F_\alpha(u) \in \cap_{1 \leq p \leq \infty} L_p(\mathbb{R})$. Hence $u \in L_\infty(\mathbb{R})$, since $|g(t)| \to \infty$ as $|t| \to \infty$. That $u' \in X_0$ now follows from Corollary 8.

□

In the Peierls-Nabarro and Benjamin-Ono equations, $F(u) = \sin u$ and $u^2 - u$ respectively (see [22]). Both of these functions F satisfy (P). In Theorem 9 u itself need not be in any $L_p(\mathbb{R}), 1 \leq p < \infty$, as the Peierls-Nabarro example shows. There $u(x) = 2\tan^{-1}(x)$.

Let K denote the convex set of all odd functions u on $\mathbb{R}$ with $u \geq 0$ and $u(x)/x$ non-increasing on $(0, \infty)$. The following result indicates briefly how the inverse of the operator $u \mapsto \mathcal{H}(u')$ maps $Y \cap K$ into itself.

Theorem 10. *Suppose $f \in K \cap Y, u' \in X$ and $\mathcal{H}u' = f$. Then $u \in K \cap X^0$.*

Proof. This is immediate from Theorem 2 and Corollary 8. □

It is clear that if $F \in K$ and F is increasing on $(0, \infty)$ then the operator $u \mapsto F(u)$ maps K into itself. In such circumstances equation (5.1) may be regarded as a fixed-point problem on $K \cap X_0$.

6 The Hilbert transform of regular distributions

Pandey [15] defined the Hilbert transform of a distribution as follows. Consider $\mathcal{H}(\mathcal{D})$ as a topological space with the topology of $\mathcal{D}$ induced by $\mathcal{H}$ so that $\mathcal{H} : \mathcal{D} \to \mathcal{H}(\mathcal{D})$ is a homeomorphism. The dual space of $\mathcal{H}(\mathcal{D})$ (the space of continuous linear functionals on $\mathcal{H}(\mathcal{D})$) is what Pandey calls the space of ultradistributions, which he denotes by $\mathcal{H}'(\mathcal{D})$. After a discussion of $\mathcal{H}(\mathcal{D})$ and $\mathcal{H}'(\mathcal{D})$ he gives the following definition of the Hilbert transforms of a distribution.

Definition. [15] For $f \in \mathcal{D}'$, $\mathcal{H}f$ is the ultradistribution defined by

$$\langle \mathcal{H}f, \phi\rangle = -\langle f, \mathcal{H}\phi\rangle \text{ for all } \phi \in \mathcal{H}(\mathcal{D}) \ .$$

For an ultradistribution g, $\mathcal{H}g$ is the distribution defined by

$$\langle \mathcal{H}g, \psi\rangle = -\langle g, \mathcal{H}\psi\rangle \text{ for all } \psi \in \mathcal{D} \ .$$

□

Here $\langle\ ,\ \rangle$ denotes the duality bracket defined on $T' \times T$ where T is a topological space and T' its dual. Pandey notes that

$$\mathcal{H}(\mathcal{H}f) = -f \text{ and } \mathcal{H}(\mathcal{H}g) = -g, \text{ for } f \in \mathcal{D}', g \in \mathcal{H}'(\mathcal{D}) \ .$$

Also, when $u \in L_p(\mathbb{R})$, $1 < p < \infty$, is identified with $f_u \in \mathcal{D}'$ by the usual formula

$$\langle f_u, \phi\rangle = \int_{-\infty}^{\infty} u(x)\phi(x)dx, \quad \phi \in \mathcal{D}, \tag{6.1}$$

he observes that the ultradistribution $\mathcal{H}f_u$ is given by

$$\langle \mathcal{H}f_u, \psi\rangle = \int_{-\infty}^{\infty} \mathcal{H}u(x)\psi(x)dx \ ,$$

where $\mathcal{H}u$ is defined classically by (1.1). Here we give an integral representation for the ultradistribution $\mathcal{H}f_u$ when $u \in L_1(\mathbb{R})$ using the $\sharp$ integral. First we need to extend the $\sharp$ integral to functions defined on all of $\mathbb{R}$. For convenience, say that a sequence $\{\rho_N\} \subset \mathcal{D}$ is *admissible* if

(i) $0 \le \rho_N(x) \le 1, \quad x \in \mathbb{R}$;

(ii) $\{\|\rho_N'\|_{L_\infty(\mathbb{R})}\}, \{\|\rho_N''\|_{L_\infty(\mathbb{R})}\}$ and $\left\{\|\rho_N'\|_{L_1(\mathbb{R})}\right\}$ are bounded;

(iii) $\rho_N(x) = 1; \quad x \in [-N, N]$.

Definition. For $f : \mathbb{R} \to \mathbb{R}$ write

$$\sharp \int_{\mathbb{R}} f(x)dx = I$$

if, and only if, for every admissible sequence $\{\rho_N\} \subset \mathcal{D}$

$$I = \lim_{N\to\infty} \sharp \int_{\mathbb{R}} \rho_N(x) f(x)dx \ .$$

□

Clearly, because of (iii), this coincides with the previous definition of the $\sharp$ integral when f has compact support and, because of Lemma 3,

$$\int_{-\infty}^{\infty} f(x)dx = \sharp \int_{\mathbb{R}} f(x)dx \text{ for all } f \in L_1(\mathbb{R}) \ .$$

Now note that if $u \in L_1(\mathbb{R})$ then $f_u \in \mathcal{D}'$ may be defined using the formula (6.1); alternatively an ultradistribution $\tilde{f}_u$ may be defined by

$$\langle \tilde{f}_u, \psi \rangle = \int_{-\infty}^{\infty} u(x)\psi(x)dx, \ \psi \in \mathcal{H}'(\mathcal{D}) \ .$$

Recall that

$$\mathcal{H}(\mathcal{D}) \subset L_p(\mathbb{R}), \quad 1 < p \leq \infty \ . \tag{6.2}$$

Theorem 11. *For* $u \in L_1(\mathbb{R})$*,*

$$\langle \mathcal{H} f_u, \psi \rangle = -\sharp \int_{\mathbb{R}} \mathcal{H}u(x)\psi(x)dx, \quad \psi \in \mathcal{H}(\mathcal{D})$$

and

$$\langle \mathcal{H} \tilde{f}_u, \phi \rangle = -\sharp \int_{\mathbb{R}} \mathcal{H}u(x)\phi(x)dx, \quad \phi \in \mathcal{D} \ .$$

Proof. Let $\{\rho_N\}$ be an admissible sequence and let $\psi \in \mathcal{H}(\mathcal{D}) \subset L_p(\mathbb{R}), 1 < p < \infty$. Then $\rho_N \psi \to \psi$ in $L_p(\mathbb{R})$, by the Dominated Convergence Theorem, and therefore there is a subsequence with

$$\mathcal{H}(\rho_{N_k}\psi) \to \mathcal{H}\psi \text{ almost everywhere} \tag{6.3}$$

as $k \to \infty$, since $\mathcal{H} : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ is continuous. Moreover, from the admissibility of $\{\rho_N\}$ and Lemma 1, there exists a constant K independent of k such that

$$\|\mathcal{H}(\rho_{N_k}\psi)\|_{L_\infty(\mathbb{R})} \leq \|\mathcal{H}\psi\|_{L_\infty(\mathbb{R})} + K\|\psi\|_{L_p(\mathbb{R})} \text{ for all } k \in \mathbb{N} \ . \tag{6.4}$$

Therefore for $u \in L_1(\mathbb{R})$, Theorem 5 gives

$$\begin{aligned} \sharp \int_{\mathbb{R}} \rho_{N_k}(x)\psi(x)\mathcal{H}u(x)dx &= -\int_{-\infty}^{\infty} u(x)\mathcal{H}(\rho_{N_k}\psi)(x)dx \\ &\to -\int_{-\infty}^{\infty} u(x)\mathcal{H}(\psi)(x)dx \end{aligned} \tag{6.5}$$

by (6.3), (6.4) and the Dominated Convergence Theorem, since ψ is smooth. Since every admissible sequence $\{\rho_N\}$ has a subsequence for which (6.5) holds, it follows that for every admissible sequence

$$\sharp \int_{\mathbb{R}} \rho_N(x)\psi(x)\mathcal{H}u(x)dx \to -\int_{-\infty}^{\infty} u(x)\mathcal{H}(\psi)(x)dx \ .$$

Therefore, by definition, for $\psi \in \mathcal{H}(\mathcal{D})$ and $u \in L_1(\mathbb{R})$,

$$\sharp \int_{\mathbb{R}} \psi(x)\mathcal{H}u(x)dx = -\int_{-\infty}^{\infty} u(x)\mathcal{H}\psi(x)dx \ .$$

The proof of the formula for the distribution $\mathcal{H}\tilde{f}_u$ is identical.

□

7 A generalized Fourier transform

In this section the goal is to show that when $u \in L_1(\mathbb{R})$ the Fourier transform of $\mathcal{H}u$, calculated using the $\sharp$ integral, is a multiplication operator. This is immediate from Theorem 5 and the definitions once some technicalities have been dealt with.

Suppose $\rho \in \mathcal{D}$ and let

$$\begin{aligned} \nu(y) &= \frac{\rho(y) - \rho(0)}{y}, \quad y \neq 0 \ , \\ &= \rho'(0), \quad y = 0 \ . \end{aligned}$$

It follows from the Mean Value Theorem that $\|\nu\|_{L_\infty(\mathbb{R})} \leq \|\rho'\|_{L_\infty(\mathbb{R})}$. From first principles,

$$\nu'(0) = \frac{1}{2}\rho''(0) \text{ and } \nu'(y) = \frac{\rho(0) - \rho(y) + y\rho'(y)}{y^2}, \quad y \neq 0 \ . \tag{7.1}$$

Hence by the Mean Value Theorem

$$\|\nu'\|_{L_\infty(\mathbb{R})} \leq \|\rho''\|_{L_\infty(\mathbb{R})}$$

and it follows from (7.1) that for $M > 0$

$$\begin{aligned}
\|\nu'\|_{L_1(\mathbb{R})} &\leq 2M\|\rho''\|_{L_\infty(-M,M)} + \int_{|y|\geq M} \left|\frac{\rho'(y)}{y}\right| dy + \int_{|y|\geq M} \frac{2\|\rho\|_{L_\infty(\mathbb{R})}}{y^2}\, dy \\
&\leq 2M\|\rho''\|_{L_\infty(-M,M)} + M^{-1}\left(\|\rho'\|_{L_1(\mathbb{R})} + 4\|\rho\|_{L_\infty(\mathbb{R})}\right) . \qquad (7.2)
\end{aligned}$$

Let $\{\rho_N\}$ be an admissible sequence and for $k \in \mathbb{R}\backslash\{0\}$ let

$$\psi_N^k(y) = \rho_N(y)e^{iky}, \quad y \in \mathbb{R} .$$

Lemma 12. *There is a constant C such that for $k \neq 0$*

$$\|\mathcal{H}\psi_N^k\|_{L_\infty(\mathbb{R})} \leq 1 + \frac{C}{|k|} \text{ for all } N \in \mathbb{N}$$

and

$$\mathcal{H}\psi_N^k(x) \to -m(k)e^{ikx} \text{ as } N \to \infty$$

where $m(k) = i\ sign(k)$.

Proof. By definition

$$\begin{aligned}
\mathcal{H}\psi_N^k(x) &= \frac{1}{\pi}\lim_{\epsilon\searrow 0}\int_{|y|\geq\epsilon} \frac{\rho_N(x-y)e^{ik(x-y)}}{y}dy \\
&= \frac{e^{ikx}}{\pi}\lim_{\epsilon\searrow 0}\int_{\epsilon\leq|y|\leq\epsilon^{-1}} \frac{\rho_N(x-y)e^{-iky}}{y}dy \\
&= \frac{e^{ikx}}{\pi}\left\{\lim_{\epsilon\searrow 0}\int_{\epsilon\leq|y|\leq\epsilon^{-1}} \left(\frac{\rho_N(x-y)-\rho_N(x)}{y}\right)e^{-iky}dy \right.\\
&\quad \left. +\rho_N(x)\lim_{\epsilon\searrow 0}\int_{\epsilon\leq|y|\leq\epsilon^{-1}} \frac{e^{-iky}}{y}dy\right\} \\
&= \frac{e^{ikx}}{\pi}\left\{\frac{1}{ik}\int_{-\infty}^{+\infty} {\nu'}_{N,x}(y)(e^{-iky}-1)dy - i\pi\rho_N(x)\ \mathrm{sign}(k)\right\}
\end{aligned}$$

(see [21, Art. 290] for example) where

$$\nu_{N,x}(y) = \begin{cases} \dfrac{\rho_N(x-y)-\rho_N(x)}{y}, & y \neq 0 \\ -{\rho'}_N(x), & y = 0 . \end{cases}$$

Since $0 \leq \rho_N(x) \leq 1$ and $\rho_N(x) \to 1$ as $N \to \infty$, it will suffice to show that

$$\int_{-\infty}^{\infty} \left|{\nu'}_{N,x}(y)\right| dy \text{ is bounded independent of } x \text{ and } N \qquad (7.3)$$

and

$$\int_{-\infty}^{\infty} \left|\nu'_{N,x}(y)dy\right| \to 0 \text{ as } N \to \infty \tag{7.4}$$

for each $x \in \mathbb{R}$. Clearly (7.3) is ensured by taking $M = 1$ and $\rho(y) = \rho_N(x - y)$, for any x and N, in (7.2) and using the fact that $\{\rho_N\}$ is admissible.

Let $x \in \mathbb{R}$ and $M > 0$. Then for $N \geq |x| + M$

$$\nu_{N,x}(y) = 0 \text{ for all } y \in [-M, M]$$

and hence, by (7.2)

$$\int_{-\infty}^{\infty} \left|\nu'_{N,x}(y)\right| dy \leq \frac{1}{M} \left(\|\rho'_N\|_{L_1(\mathbb{R})} + 4\|\rho_N\|_{L_\infty(\mathbb{R})}\right)$$

for all $N \geq M + |x|$. Since $\{\rho_N\}$ is admissible, this ensures (7.4), and the proof is complete.

□

Theorem 13. *For $u \in L_1(\mathbb{R})$ and $k \neq 0$*

$$\sharp \int_{\mathbb{R}} e^{iky} \mathcal{H}u(y) dy = i \ sign(k) u^\wedge(k) \ .$$

Proof. Let $\{\rho_N\}$ be an admissible sequence. Then by Theorem 5, for $k \neq 0$,

$$\sharp \int_{\mathbb{R}} \rho_N(y) e^{iky} \mathcal{H}u(y) dy = -\int_{-\infty}^{\infty} u(y) \mathcal{H}\psi_N^k(y) dy$$
$$\to i \ \mathrm{sign}(k) \int_{-\infty}^{\infty} e^{iky} u(y) dy \text{ as } N \to \infty \ ,$$

by Lemma 12 and the Dominated Convergence Theorem,

$$= m(k) u^\wedge(k) \ .$$

Therefore

$$\sharp \int_{\mathbb{R}} e^{iky} \mathcal{H}u(y) dy = m(k) u^\wedge(k)$$

as required.

□

This result is the exact analogue for Hilbert transforms of Kolmogorov's result [24, Ch. VII, (4.3)] for Fourier series and enables one to reconcile the meaning of the statement '$f \in L_1(\mathbb{R})$ and $\mathcal{H}f \in L_1(\mathbb{R})$' based on the classical formula and the alternative interpretation based upon Fourier transforms [18, page 221].

Theorem 14. *Suppose that u and v are in $L_1(\mathbb{R})$. Then $\mathcal{H}u = v$ if, and only if, $mu^\wedge = v^\wedge$ almost everywhere.*

Proof. Suppose that $mu^\wedge = v^\wedge$. Let $\{\phi_n\}$ be a sequence of regularising kernels as in the proof of Theorem 7. Then $\phi_n * u \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}) \subset L_2(\mathbb{R})$ and $\phi_n * u \to u$ in $L_1(\mathbb{R})$ as $n \to \infty$. Hence, by (2.5) and (2.6) for each $n \in \mathbb{N}$,

$$(\mathcal{H}(\phi_n * u))^\wedge = m\phi_n^\wedge u^\wedge = \phi_n^\wedge v^\wedge = (\phi_n * v)^\wedge \ .$$

Therefore for $n \in \mathbb{N}$,

$$\mathcal{H}(\phi_n * u) = \phi_n * v \text{ in } L_2(\mathbb{R}) \ .$$

Now $\phi_n * u \to u$ in $L_1(\mathbb{R})$, a subsequence of $\{\phi_{n_k} * v\}$ converges to v almost everywhere. Therefore $\mathcal{H}(u) = v$ since $\mathcal{H}(\phi_{n_k} * u) \to \mathcal{H}(u)$ in measure.

Now suppose that $\mathcal{H}(u) = v$. Then the Fourier transform of $\mathcal{H}(u)$, calculated using the $\sharp$ integral, coincides with the Fourier transform of v in the usual sense, because, for any $f \in L_1(\mathbb{R})$,

$$\int_{-\infty}^{\infty} f(x)dx = \sharp \int_{\mathbb{R}} f(x)dx \ .$$

Hence $mu^\wedge = v^\wedge$, by the preceding theorem. This completes the proof. □

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