Evolutionary problems with energies with linear growth

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(joint work with Martin Kružík)

We study a rate-independent evolution of problems where the energy W is a function of the deformation gradient, $W = W(Du)$, and grows linearly at infinity,

$$c\,|s| - c_2 \le W(x,s) \le C(1+|s|) \text{ for } x \in \bar{\Omega}\ , \tag{1}$$

with constants $0 < c \le C$. Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary.

The aim of this note is to sketch a framework within which the existence of a rate-independent process with an energy of type (1) can be proved. Rate-independent processes are understood here in the energetic formulation, i.e., characterised by stability, energy inequality, and compatibility with initial conditions. This is made precise below.

Before moving on to the evolutionary process, we should motivate the functional analytic framework in the static context. The setting needs to be chosen such that oscillation and concentration effects are taken into account. This can be seen in the following toy model, where the task is to minimise the functional

$$\min I(u) := \int_0^1 \left[\frac{(u'(x))^2}{1+(u'(x))^4} + \theta^2\,|u'(x)| + (u(x)-x)^2 \right] \mathrm{d}x,$$

with $\theta \ge 0$ among $u \in W^{1,1}(0,1)$ with $u(0) = 0$. The second term is introduced to make the functional coercive; the third term favours solutions close to the identity. The decisive term is the first one, which becomes minimal for $u'(x) = 0$ or in the limit $u'(x) \to \pm\infty$. One would thus expect approximative solution (minimising sequences) to *oscillate* between gradient 0 and gradients which become arbitrarily large in modulus. A particular point here is that the minimising sequences thus do not oscillate between finite values for the deformation gradient (as for the toy model $\int_0^1 \left((u'(x))^2 - 1\right)^2 \mathrm{d}x$ between ± 1), but *concentrate* mass at $\pm\infty$. Young measures [5, 1, 3, 4] are an appropriate tool to deal with oscillations, while DiPerna-Majda measures [2] describe the limits of sequences with oscillations and concentrations.

We use DiPerna-Majda measures to describe the evolution of rate-independent processes with linear energies. Let $u\colon \Omega \to \mathbb{R}^m$ denote the deformation, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. We write $q := (u, \eta, \lambda)$ for a state; u denotes the deformation, η is the associated DiPerna-Majda measure, and λ is derived from η. (To be precise, for a suitable compactification $\beta\mathbb{R}^{m\times n}$ of $\mathbb{R}^{m\times n}$ and for $\eta \cong (\hat{\nu}, \sigma)$ via slicing, we set $\lambda(x) = \int_{\beta\mathbb{R}^{m\times n}} \frac{\Lambda}{1+|s|} \nu_x(ds) d\sigma(x)$ with Λ bounded).

The following definitions are natural in the context of DiPerna-Majda measures (we write $\tilde{g}(s) := \frac{g(s)}{1+|s|}$ and recall that $\beta\mathbb{R}^{m\times n}$ is a suitable compactification of

$\mathbb{R}^{m\times n}$). The applied body force f give rise to

$$F(q) := \int_\Omega f(x,t)\cdot u(x)\,\mathrm{d}x \quad\text{and}\quad \dot F(t,q) = \int_\Omega \frac{\partial f(x,t)}{\partial t}\cdot u(x)\,\mathrm{d}x;$$

the time-dependent elastic energy $E(t,q)$ is

$$E(t,q) = \int_{\bar\Omega\times\beta\mathbb{R}^{m\times n}} \tilde W(x,s)\eta(\mathrm{d}s\mathrm{d}x) - \int_\Omega f(x,t)\cdot u(x)\,\mathrm{d}x. \tag{2}$$

Γ is the energy augmented by a spatial regularisation,

$$\Gamma(t,q) := E(t,q) + \int_\Omega \varrho\,|\nabla\lambda(x)|^2\,\mathrm{d}x,$$

with $\rho > 0$.

The *dissipation distance* $\mathcal{D}$ describes the energetic loss between two states of the system characterised by η_1 and η_2. We choose $\mathcal{D}(q_1,q_2) = \int_\Omega \|\lambda_1 - \lambda_2\|\mathrm{d}x$. The *temporal dissipation* is then given by

$$\mathrm{Diss}(q,[t_1,t_2]) := \sup_{L\in\mathbb{N}}\left\{\sum_{l=1}^{L}\mathcal{D}\left(\eta(\tau_l),\eta(\tau_{l-1})\right)\ \Big|\ t_1=\tau_0<\cdots<\tau_L=t_2\right\}.$$

For given q_0 in the state space Q, the process $q\colon[0,T]\to Q$ is a *solution* if the following three conditions hold:

(1) *Stability:* For every $t\in[0,T]$, we have

$$\Gamma(t,q(t)) \le \Gamma(t,\tilde q) + D(q(t),\tilde q) \text{ for every } \tilde q\in Q.$$

(2) *Energy inequality:* For every $0\le t_1\le t_2\le T$, we have

$$\Gamma(t_1,q(t_1)) + \mathrm{Diss}(q,[t_1,t_2]) \le \Gamma(t_2,q(t_2)) - \int_{t_1}^{t_2}\dot F(t,q(t))\mathrm{d}t.$$

(3) *Initial condition:* $q(0) = q_0$.

In this setting, the existence of a process satisfying the above conditions can be proved and suitable regularity assumptions for sufficiently small forces.

References

[1] J. M. Ball, *A version of the fundamental theorem for Young measures*, in: PDEs and continuum models of phase transitions (Nice, 1988) (M. Rascle, D. Serre, M. Slemrod, eds.), Lecture Notes in Physics **344**, Springer (1989), 207–215.

[2] R. J. DiPerna, A. J. Majda, *Oscillations and concentrations in weak solutions of the incompressible fluid equations*, Comm. Math. Phys. **108** (1987), 667–689.

[3] L. Tartar, *Compensated compactness and applications to partial differential equations*, in: Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV (R. J. Knops, ed.) Pitman Res. Notes in Math. **39** (1979), 136–212.

[4] L. Tartar, *Mathematical tools for studying oscillations and concentrations: From Young measures to H-measures and their variants.* In: Multiscale problems in science and technology. Challenges to mathematical analysis and perspectives. (N. Antonič et al., eds.) Proceedings of the conference on multiscale problems in science and technology, held in Dubrovnik, Croatia, September 3–9, 2000. Springer, Berlin, 2002.

[5] L. C. Young, *Generalized curves and the existence of an attained absolute minimum in the calculus of variations*, Comptes Rendus de la Société des Sciences et des Lettres de Varsovie, Classe III **30** (1937), 212–234.