Skorokhod Embeddings & LPs

Def^n 1. Let $(X_t)_{t\geq 0}$ be a stochastic process on $\mathbb{R}$, and suppose μ is a measure on $\mathbb{R}$. We say a stop. time T is a sol^n to the <u>Skorokhod Embedding Problem</u> (SEP) (for μ) if

$$X_T \sim \mu.$$

§1. SEP for BM.

Set B_t a BM, $B_0 = 0$, $F(z) = \mu((-\infty, z])$. Then

$$T := \inf\{t \geq 1 : B_t = F^{-1}(\Phi(B_1))\}$$

is an embedding since BM recurrent &

$$\mathbb{P}(F^{-1}(\Phi(B_1)) \leq x) = \mathbb{P}(B_1 \leq \Phi^{-1}(F(x)))$$

$$= \Phi(\Phi^{-1}(F(x))) = F(x) = \mu((-\infty, x])$$

However (unless $\mu \sim N(0,1)$) : $\mathbb{E}T = \infty$
(since this is the case for eg. $\sigma = \inf\{s : B_s = 1\}$)

Since $B_t^2 - t$ is a m'g, would like $\mathbb{E}B_T^2 = \mathbb{E}T$.

i.e. $\mathbb{E}T = \int x^2 \, d\mu$ for 'small' stopping times.

Def^n 2: ~~(Monroe)~~ An SE is <u>minimal</u> if

$\forall$ stop. times S,

$$S \leq T \text{ a.s. } \& \ B_S \sim \mu$$

$$\Rightarrow S = T \text{ a.s.}$$

<u>Thm</u> (Monroe) If μ centred, T is minimal iff $(B_{t\wedge T})_{t\geq 0}$ is UI.

Aim: find minimal solutions to SEP

For convenience, see μ as symmetric, integrable with +ve density & no atoms. Write $\bar{\mu}(x) = \mu([x,\infty))$

<u>Thm</u>: [Vallois]

Let
$$\psi(x) = \int_0^x \frac{s}{\bar{\mu}(s)} \mu(ds)$$

Then $\tau = \inf\{t \geq 0 : |B_t| \geq \psi^{-1}(L_t)\}$ solves the SEP & is minimal

<u>Proof</u>: Let $\mathcal{U}$ be the space of excursions, and for $\varepsilon \in \mathcal{U}$ write
$$\bar{\varepsilon} = \sup_{s \leq \zeta(\varepsilon)} \varepsilon(s), \quad \underline{\varepsilon} = \inf \varepsilon(s), \quad |\varepsilon| = \bar{\varepsilon} \vee (-\underline{\varepsilon})$$

where $\zeta(\varepsilon) = \inf\{u>0 : \varepsilon(u) = 0\}$

Then the excursion measure, n on $\mathcal{U}$ of Bm has:
$$n(|\varepsilon| \geq x) = 1/x$$

{Pf: $\mathbb{E} L_{H_{x}} = x$, also fst exc. above x}

Note that n on $\mathcal{U}$ induces a measure $\tilde{n} = \text{Leb} \times n$ on $\mathbb{R}_+ \times \mathcal{U}$
— No. of excurs. over time.

Let $A_t = \{(s,u) : s \leq t, |\varepsilon| \geq \psi^{-1}(s)\}$

then we stop at the smallest t for which our excursion PP places a point in A_t. (On local time scale)

Hence:
$$\mathbb{P}(L_\tau > l) = e^{-\tilde{n}(A_l)}$$

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But $\tilde{n}(A_\ell) = \int_0^\ell n(|\varepsilon| \geq \psi^{-1}(r))\, dr$

$$= \int_0^\ell \frac{1}{\psi^{-1}(r)}\, dr$$

Note ψ increasing, and so ψ^{-1} is also $\Rightarrow$
$\{L_T > \ell\} = \{|B_T| > \psi^{-1}(\ell)\}$

and ~~$\mathbb{P}(L_T > \ell) \leq$~~ $\int_0^\ell \frac{1}{\psi^{-1}(r)}\, dr = \int_0^{\psi^{-1}(\ell)} \frac{1}{s}\, \psi'(s)\, ds$

$$= \int_0^{\psi^{-1}(\ell)} \frac{\mu(ds)}{\bar{\mu}(s)} = \Big[-\log(\bar{\mu}(s))\Big]_0^{\psi^{-1}(\ell)}$$

$$= -\log\left(2\bar{\mu}(\psi^{-1}(\ell))\right)$$

$$\Rightarrow \quad \mathbb{P}(L_T > \ell) = 2\bar{\mu}(\psi^{-1}(\ell))$$

So $\mathbb{P}(B_T > x) = \frac{1}{2}\mathbb{P}(L_T > \psi(x)) = \bar{\mu}(x)$

as required.

Now check $B_{t\wedge T}$ is UI (and hence minimal)

Note: if $H_N = \inf\{t \geq 0 : |B_t| = N\}$, $(|B|_{t\wedge H_N} - L_{t\wedge H_N})$ is UI.

Consider a stop. t. S:

$$\mathbb{E}|B_{S\wedge T}|\, \mathbb{1}\{S\wedge T < H_N\} \xrightarrow{MC} \mathbb{E}|B_{S\wedge T}|$$

$$\wedge|$$

$$\mathbb{E}L_{S\wedge T\wedge H_N} \xrightarrow{MC} \mathbb{E}L_{S\wedge T}.$$

and $\mathbb{E}L_{S\wedge T} \leq \mathbb{E}L_T = \int_0^\infty \mathbb{P}(L_T > \ell)\, d\ell = \int_0^\infty 2\bar{\mu}(\psi^{-1}(\ell))\, d\ell$

$$\overset{\ell = \psi(u)}{=} \int_0^\infty 2u\, \mu(du) = 2\mathbb{E}(B_T)_+$$

Hence $\mathbb{E}\,|B_{s\wedge T}| \le 2\mathbb{E}\,(B_T)_+ \quad \forall\, s \le T.$

$\Rightarrow (B_{t\wedge T})$ is UI.

§2. ~~Levy~~ Spectrally Negative Lévy processes.

Now suppose X_t is a SNLP with a Gaussian components which does not drift to ~~infinity~~ $-\infty$

We wish to derive an excursion theory (from 0) for X_t. Note: we can ~~have~~ now have excursions which begin positive, but end negative (although not vice versa).

<u>Lemma</u> [Pistorius + Obłój ~~8~~ '07]: If W is the scale fn of X,

$$n(\{\bar{\varepsilon} \ge \eta\}) = \frac{1}{W(\eta)}.$$

$$n(\{\bar{\varepsilon} \ge \eta \text{ or } \underline{\varepsilon} \le -\delta\}) = \frac{W(\eta+\delta)}{W(\eta)\,W(\delta)}$$

$$n(\{\bar{\varepsilon} = 0,\ \underline{\varepsilon} \le -\delta\}) = \frac{\sigma^2}{2}\,\frac{W'(\delta)}{W(\delta)}.$$

"Idea":

Roughly, expect

$$\mathbb{P}_0(\tau_\eta^+ < \tau_\xi^-) \sim f(\xi) \times n(\{\bar{\varepsilon} \ge \eta\})$$

as $\xi \uparrow 0$,

where $f(\xi) \downarrow 0$ as $\xi \uparrow 0$ is indep of η.

But $\mathbb{P}_0(\tau_\eta^+ < \tau_\xi^-) = \frac{W(\xi)}{W(\xi+\eta)} \rightsquigarrow \frac{W(\xi)}{W(\eta)}$

constant will depend on ~~the~~ scaling of local time —.

Then :
$$\mathbb{P}_0(H_\eta < H_{-\delta}) = \frac{n(\{\bar{\varepsilon} \geq \eta\})}{n(\{\bar{\varepsilon} \geq \eta \text{ or } \underline{\varepsilon} \leq -\delta\})}$$

$$\Rightarrow \quad n(\{\bar{\varepsilon} \geq \eta \text{ or } \underline{\varepsilon} \leq -\delta\}) = \frac{n(\{\bar{\varepsilon} \geq \eta\})}{\mathbb{P}_0(\cdot \ - \ )}$$

$$= \frac{1}{W(\eta)} \Big/ \frac{W(\delta)}{W(\eta+\delta)}$$

$$= \frac{W(\eta+\delta)}{W(\eta)\, W(\delta)}$$

& $n(\{\bar{\varepsilon} = 0, \ \underline{\varepsilon} \leq -\delta\}) = \lim\limits_{\eta \downarrow 0} n(\{\bar{\varepsilon} < \eta \text{ or } \underline{\varepsilon} \leq -\delta\})$

$$= \lim \left[n(\{\bar{\varepsilon} \geq \eta \text{ or } \underline{\varepsilon} \leq -\delta\}) - n(\{\bar{\varepsilon} \geq \eta\}) \right]$$

$$= \lim_{\eta \downarrow 0} \left[\frac{1}{W(\eta)} \frac{(W(\eta+\delta) - W(\delta))}{W(\delta)} \right]$$

$$= \frac{W'(\delta+)}{W'(0+)\, W(\delta)} = \frac{\sigma^2}{2} \frac{W'(\delta)}{W(\delta)}$$

Now suppose μ is integrable, ~~non-atomic~~ with positive density ~~$\mu(dx) =$~~ $\rho(x)$

Theorem [Obłój - ~~B~~ Pistorius ~~Z~~ '09]

Define, for $x < 0 < y$,

$$D_\mu(y) = \int_0^y W(s)\ \mu(ds), \quad G_\mu(x) = \int_x^0 \frac{2}{\sigma^2}\frac{W(-s)}{W'(-s)}\ \mu(ds)$$

$$g(s) = G_\mu^{-1}(D_\mu(s))\ (\le 0), \quad f(s) = D_\mu^{-1}(G_\mu(s))\ (\ge 0)$$

$$\psi_+(y) = \int_0^y \frac{W(s)\ \mu(ds)}{(1+\bar{\mu}(s) - \bar{\mu}(g(s))}$$

$$\psi_-(z) = \int_{-z}^0 \frac{2}{\sigma^2}\ \frac{W(-s)\ \mu(ds)}{W'(-s)\ (1+\bar{\mu}(f(s)) - \bar{\mu}(s))}.$$

If $D_\mu(\infty) = G_\mu(-\infty)$ — i.e. $\int_0^\infty W(s)\,\mu(ds) = \int_{-\infty}^0 \frac{2}{\sigma^2}\frac{W(-s)}{W'(-s)}\mu(ds)$

Then $T = \inf\{t \ge 0 : X_t = \psi_+^{-1}(L_t)$
or $X_t = -\psi_-^{-1}(L_t)$
on an initially -ve excursion$\}$

Then T solves the SEP.

Pf: ~~(1)~~ ~~$g(\phi_+(l))$~~ Write $\phi_+(l) = \psi_+^{-1}(l)$ ~~$\phi_+(l) = \psi_+^{-1}(l)$~~

Then $g(\phi_+(l)) = -\phi_-(l)$.

Pf: $\Leftrightarrow \psi_-(-g(u)) = \psi_+(u)$

$$\psi_-(-g(u)) = \frac{2}{\sigma^2}\int_{g(u)}^0 \frac{W(-y)\ \rho(y)}{W'(-y)(1+\bar{\mu}(f(y)) - \bar{\mu}(y))}\,dy$$

$y = g(s)$
$s = f(y)$

$$= \int_0^u \frac{W(s)\ \ \mu(ds)}{(1+\bar{\mu}(s) - \bar{\mu}(g(s))}$$

$$[\phi_+(u)]' = \frac{1}{\gamma_+'(\phi_+(u))} = \frac{(1+\bar{p}(\phi_+(u)) - \bar{p}(g(\phi_+(u)))}{W(\phi_+(u))\ \rho(\phi_+(u))}$$

$$[\phi_-(u)]' = \frac{\sigma^2}{2}\ \frac{W'(\phi_-(u))\,(1+\bar{p}(f(-\phi_-(u))) - \bar{p}(-\phi_-(u)))}{W(\phi_-(u))\ \rho(-\phi_-(u))}$$

$$g'(s) = -\frac{\sigma^2}{2}\ \frac{W(s)\,W'(-g(s))}{W(-g(s))}\ \frac{\rho(s)}{\rho(g(s))}$$

—

& Note : $\quad g(\phi_+(\lambda)) = -\phi_-(\ell)$

$$\Rightarrow \quad f^{-1}(\phi_+(\lambda)) = -\phi_-(\lambda)$$

$$\Rightarrow \quad \phi_+(\lambda) = f(-\phi_-(\lambda))$$

—

Hence :

$$[1+\bar{p}(\phi_+(u)) - \bar{p}(g(\phi_+(u)))]'$$

$$= [1+\bar{p}(\phi_+(u)) - \bar{p}(g(\phi_+(u)))]$$

$$\times \Big\{ -\cancel{\rho(\phi_+(u))}\cdot\frac{1}{W(\phi_+(u))\,\cancel{\rho(\phi_+(u))}} \cancel{-\tfrac{\sigma^2}{2}\, W(\phi_+)\,W'(-g(\phi_+))}$$

$$-\frac{\sigma^2}{2}\ \frac{\cancel{W(\phi_+(u))}\,W'(-g(\phi_+(u)))}{W(-g(\phi_+(u)))}\ \frac{\cancel{\rho(\phi_+(u))}}{\cancel{\rho(g(\phi_+(u)))}}\cdot\frac{1}{\cancel{W(\phi_+(u))}\,\cancel{\rho(\phi_+(u))}}$$

$$\times\ \cancel{\rho(g(\phi_+(u)))} \Big\}$$

$$= [1+\bar{p}(\phi_+(u)) - \bar{p}(g(\phi_+(u)))]$$

$$\times \left\{ -\frac{1}{W(\phi_+(u))} - \frac{\sigma^2}{2}\ \frac{W'(-g(\phi_+(u)))}{W(-g(\phi_+(u)))} \right\}$$

$$\Rightarrow \frac{[1+\bar{\mu}(\phi_+(u)) - \bar{\mu}(g(\phi_+(u)))]'}{[1+\bar{\mu}(\phi_+(u)) - \bar{\mu}(g(\phi_+(u)))]} = -\left[n(\bar{\varepsilon} \geq \phi_+(u)) + n(\bar{\varepsilon}=0, \underline{\varepsilon} \leq -\phi_-(u))\right]$$

$$\Rightarrow \quad \mathbb{P}(L_T > k) = \exp\left(\int_0^k \frac{[\dots]'}{[\dots]} \, dl \right)$$

$$= 1 + \bar{\mu}(\phi_+(k)) - \bar{\mu}(-\phi_-(k)).$$

In particular, for $x > 0$

$$\mathbb{P}(X_T > x) = \int_0^\infty \mathbb{P}(L_T > l) \, n(\bar{\varepsilon} > \phi_+(l)) \, \mathbb{1}\{\phi_+(l) > x\} \, dl \quad (*)$$

subst. $u = \phi_+(l)$, $du = \frac{(1+\bar{\mu}(\cdots))}{W(u)\,\rho(u)} \, dl$

$$= \int_0^\infty \frac{1}{W(u)} W(u)\rho(u) \, \mathbb{1}\{u > x\} \, du$$

$$= \mu(x, \infty)$$

similarly for $\mathbb{P}(X_T < y)$, $y < 0$.

To deduce minimality, Bertoin - Le Jan $\Rightarrow$ $\mathbb{E} L_S \geq \int_0^\infty W(u)\rho(u)\,du$ for all S s.t. $B_S \sim \mu$.

So if $S \leq T$, $L_S \leq L_T$

But $\mathbb{E} L_T = \int_0^\infty \mathbb{P}(L_T > l) \, dl \overset{(\text{As in } (*))}{=} \int_0^\infty W(u)\,\rho(u)\,du$

So must have $L_S = L_T$ a.s. i.e. stop on same excursion

Spec neg $\Rightarrow$ $X_S \leq X_T$ if S occurs on +ve exc.
But $X_S \sim X_T \Rightarrow X_S = X_T$.

~~If stop~~ On -ve excursions, either ~~always~~ can stop before going below ϕ_- $\Rightarrow$ # by MP or never stop before ϕ_- $\Rightarrow X_S = X_T$.