Robust hedging of double touch barrier options

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Abstract

We consider model-free pricing of digital options, which pay out if the underlying asset has crossed both upper and lower barriers. We make only weak assumptions about the underlying process (typically continuity), but assume that the initial prices of call options with the same maturity and all strikes are known. Under such circumstances, we are able to give upper and lower bounds on the arbitrage-free prices of the relevant options, and further, using techniques from the theory of Skorokhod embeddings, to show that these bounds are tight. Additionally, martingale inequalities are derived, which provide the trading strategies with which we are able to realise any potential arbitrages. We show that, depending of the risk aversion of the investor, the resulting hedging strategies can outperform significantly the standard delta/vega-hedging in presence of market frictions and/or model misspecification.

1 Introduction

In the standard approach to pricing and hedging, one postulates a model for the underlying, calibrates it to the market prices of liquidly traded vanilla options and then uses the model to derive prices and associated hedges for exotic over-the-counter products. Prices and hedges will be correct only if the model describes perfectly the real world, which is not very likely. The model-free approach uses market data to deduce bounds on the prices consistent with *no-arbitrage* and the associated super- and sub- replicating strategies, which are robust to model misspecification. In this work we adopt such an approach to derive model-free prices and hedges for digital double barrier options.

The methodology, which we now outline, is based on solving the *Skorokhod embedding problem* (SEP). We assume *no arbitrage* and suppose we know the market prices of calls and puts for all strikes at one maturity T. We are interested in pricing an exotic option with payoff given by a path-dependent functional $O(S)_T$. The example we consider here is a *digital double touch* barrier option struck at $(\underline{b}, \overline{b})$ which pays 1 if the stock price reaches both $\underline{b}$ and $\overline{b}$ before maturity T. Our aim is to construct a model-free super-replicating strategy of the form

$$O(S)_T \leq F(S_T) + N_T, \tag{1}$$

where $F(S_T)$ is the payoff of a finite portfolio of European puts and calls and N_T are gains from a self-financing trading strategy (typically forward transactions). Furthermore, we want (1) to be tight in the sense that we can construct a market model which matches the market prices of calls and puts and in which we have equality in (1). The initial price of the portfolio $F(S_T)$ is then the least upper bound on

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the price of the exotic $O(S)_T$ and the right hand side of (1) gives a simple super-replicating strategy at that cost. There is an analogous argument for the lower bound and an analogous sub-replicating strategy.

In fact, in order to construct (1), we first construct the market model which induces the upper bound on the price of $O(S)_T$ and hence will attain equality in (1). To do so we rely on the theory of Skorokhod embeddings (cf. Obłój [Obł04]). We assume *no arbitrage* and consider a market model in the risk-neutral measure so that the forward price process $(S_t : t \leq T)$ is a martingale[1]. It follows from Monroe's theorem [Mon78] that $S_t = B_{\rho_t}$, for a Brownian motion (B_t) with $B_0 = S_0$ and some increasing sequence of stopping times $\{\rho_t : t \leq T\}$ (possibly relative to an enlarged filtration). Knowing the market prices of calls and puts for all strikes at maturity T is equivalent to knowing the distribution μ of S_T (cf. [BL78]). Thus, we can see the stopping time $\rho = \rho_T$ as a solution to the SEP for μ. Conversely, let τ be a solution to the SEP for μ, i.e. $B_\tau \sim \mu$ and $(B_{t\wedge\tau} : t \geq 0)$ is a uniformly integrable martingale. Then the process $\tilde{S}_t := B_{\tau\wedge\frac{t}{T-t}}$ is a model for the stock-price process consistent with the observed prices of calls and puts at maturity T. In this way, we obtain a correspondence which allows us to identify market models with solutions to the SEP and vice versa. In consequence, to estimate the fair price of the exotic option $\mathbb{E}O(S)_T$, it suffices to bound $\mathbb{E}O(B)_\tau$ among all solutions τ to the SEP. More precisely, if $O(S)_T = O(B)_{\rho_T}$ a.s., then we have

$$\inf_{\tau: B_\tau\sim\mu} \mathbb{E}O(B)_\tau \leq \mathbb{E}O(S)_T \leq \sup_{\tau: B_\tau\sim\mu} \mathbb{E}O(B)_\tau, \tag{2}$$

where all stopping times τ are such that $(B_{t\wedge\tau})_{t\geq 0}$ is uniformly integrable. Once we compute the above bounds and the stopping times which achieve them, we usually have a good intuition how to construct the super- (and sub-) replicating strategies (1).

A more detailed description of the SEP-driven methodology outlined above can be found in Obłój [Obł09]. The idea of *no-arbitrage* bounds on the prices goes back to Merton [Mer73]. The methods for robust pricing and hedging of options sketched above go back to the works of Hobson [Hob98] (lookback option) and Brown, Hobson and Rogers [BHR01] (single barrier options). More recently, Dupire [Dup05] investigated volatility derivatives using the SEP and Cox, Hobson and Obłój [CHO08] designed pathwise inequalities to derive price range and robust super-replicating strategies for derivatives paying a convex function of the local time.

Unlike in previous works, e.g. Brown, Hobson and Rogers [BHR01], we don't find a unique inequality (1) for a given barrier option. Instead we find that depending on the market input (i.e. prices of calls and puts) and the pair of barriers different strategies may be optimal. We characterise all of them and give precise conditions to decide which one should be used. This new difficulty is coming from the dependence of the payoff on both the running maximum and minimum of the process. Solutions to the SEP which maximise or minimise $\mathbb{P}(\sup_{u\leq\tau} B_u \geq \overline{b}, \inf_{u\leq\tau} B_\tau \leq \underline{b})$ have not been developed previously and they are introduced in this paper. As one might suspect, they are considerably more involved that the ones by Perkins [Per86] or Azéma and Yor [AY79] exploited by Brown, Hobson and Rogers [BHR01].

From a practical point of view, the no-arbitrage price bounds which we obtain are too wide to be used for pricing. However, our super- or sub- hedging strategies can still be used. Specifically, suppose an agent sells a double touch barrier option $O(S)_T$ for a premium p. She can then set up our superhedge (1) for an initial premium $\overline{p} > p$. At maturity T she holds $H = -O(S)_T + F(S)_T + N_T + p - \overline{p}$ which on average is worth zero, $\mathbb{E}H = 0$, but is also bounded below: $H \geq p - \overline{p}$. In reality, in the presence of model uncertainty and market frictions, this can be an appealing alternative to the standard delta/vega hedging. Indeed, our numerical simulations in Section 3.3 show that in the presence of transaction costs a risk averse agent will generally prefer the hedging strategy we construct to a (daily monitored) delta/vega-hedge.

The paper is structured as follows. First we present the setup: our assumptions and terminology and explain the types of double barriers considered in this and other papers. Then in Section 2 we consider digital double touch barrier option mentioned above. We first present super- and sub- replicating strategies and then prove in Section 2.3 that they induce tight model-free bounds on the admissible prices of the double touch options. In Section 3 we reconsider our assumptions and investigate some applications. Specifically, in Section 3.1 we consider the case when calls and puts with only a finite number of strikes are observed and in Section 3.2 we discuss discontinuities in the price process (S_t). In Section 3.3 we

[1] Equivalently, under a simplifying assumption of zero interest rates S_t is simply the stock price process.

present a numerical investigation of the performance of our super- and sub- hedging strategies. Section 4 contains the proofs of main theorems. Additional figures are attached in Section 5.

1.1 Setup

In what follows $(S_t)_{t\geq 0}$ is the forward price process. Equivalently, we can think of the underlying with zero interest rates, or an asset with zero cost of carry. In particular, our results can be directly applied in Foreign Exchange markets for currency pairs from economies with similar interest rates. Moving to the spot market with non-zero interest rates is not immediate as our barriers become time-dependent.

We assume that $(S_t)_{t\geq 0}$ has continuous paths. We comment in Section 3.2 how this assumption can be removed or weakened to a requirement that given barriers are crossed continuously. We fix a maturity $T > 0$, and assume we observe the initial spot price S_0 and the market prices of European calls for all strikes $K > 0$ and maturity T:

$$\Big(C(K) : K \geq 0\Big), \tag{3}$$

which we call the *market input*. For simplicity we assume that $C(K)$ is twice differentiable and strictly convex on $(0,\infty)$. Further, we assume that we can enter a forward transaction at no cost. More precisely, let ρ be a stopping time relative to the natural filtration of $(S_t)_{t\leq T}$ such that $S_\rho = \overline{b}$. Then the portfolio corresponding to selling a forward at time ρ has final payoff $(\overline{b} - S_T)\mathbf{1}_{\rho\leq T}$ and we assume its initial price is zero. The initial price of a portfolio with a constant payoff K is K. We denote $\mathcal{X}$ the set of all calls, forward transactions and constants and $\mathrm{Lin}(\mathcal{X})$ is the space of their finite linear combinations, which is precisely the set of portfolios with given initial market prices. For convenience we introduce a pricing operator $\mathcal{P}$ which, to a portfolio with payoff X at maturity T, associates its initial (time zero) price, e.g. $\mathcal{P}K = K$, $\mathcal{P}(S_T - K)^+ = C(K)$ and $\mathcal{P}(\overline{b} - S_T)\mathbf{1}_{t\geq\rho} = 0$. We also assume $\mathcal{P}$ is linear, whenever defined. Initially, $\mathcal{P}$ is only given on $\mathrm{Lin}(\mathcal{X})$. One of the aims of the paper is to understand extensions of $\mathcal{P}$ which do not introduce arbitrage to $\mathrm{Lin}(\mathcal{X}\cup\{Y\})$, for double touch barrier derivatives Y. Note that linearity of $\mathcal{P}$ on $\mathrm{Lin}(\mathcal{X})$ implies call-put parity holds and in consequence we also know the market prices of all European put options with maturity T:

$$P(K) := \mathcal{P}(K - S_T)^+ = K - S_0 + C(K).$$

Finally, we assume the market admits *no arbitrage* (or quasi-static arbitrage) in the sense that any portfolio of initially traded assets with a non-negative payoff has a non-negative price:

$$\forall X \in \mathrm{Lin}(\mathcal{X}) : X \geq 0 \Longrightarrow \mathcal{P}X \geq 0. \tag{4}$$

As we do not have any probability measure yet, by $X \geq 0$ we mean that the payoff is non-negative for any continuous non-negative stock price path $(S_t)_{t\leq T}$.

By a *market model* we mean a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with a continuous $\mathbb{P}$-martingale (S_t) which matches the market input (3). Note that we consider the model under the risk-neutral measure and the pricing operator is just the expectation $\mathcal{P} = \mathbb{E}$. Saying that (S_t) matches the market input is equivalent to saying that it starts in the initial spot S_0 a.s. and that $\mathbb{E}(S_T - K)^+ = C(K)$, $K > 0$. This in turn is equivalent to knowing the distribution of S_T (cf. [BL78, BHR01]). We denote this distribution μ and often refer to it as the *law of S_T implied by the call prices.* Our regularity assumptions on $C(K)$ imply that

$$\mu(\mathrm{d}K) = C''(K),\ \ K > 0, \tag{5}$$

so that μ has a positive density on $(0,\infty)$. We could relax this assumption and take the support of μ to be any interval $[a, b]$. Introducing atoms would complicate our formulae (essentially without introducing new difficulties).

The running maximum and minimum of the price process are denoted respectively $\overline{S}_t = \sup_{u\leq t} S_u$ and $\underline{S}_t = \inf_{u\leq t} S_u$. We are interested in this paper in derivatives whose payoff depends both on $\overline{S}_T$ and $\underline{S}_T$. It is often convenient to express events involving the running maximum and minimum in terms of the first hitting times $H_x = \inf\{t : S_t = x\}$, $x \geq 0$. As an example, note that $\mathbf{1}_{\overline{S}_T\geq\overline{b},\,\underline{S}_T\leq\underline{b}} = \mathbf{1}_{H_{\overline{b}}\vee H_{\underline{b}}\leq T}$.

We use the notation $a << b$ to indicate that a is *much smaller* than b – this is only used to give intuition and is not formal. The minimum and maximum of two numbers are denoted $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ respectively, and the positive part is denoted $a^+ = a \vee 0$.

1.2 Types of digital double barrier options

This paper is part of a larger project of describing model-independent pricing and hedging of all digital double barrier options. As mentioned above, in this paper we consider double touch barrier options. Given barriers $\underline{b} < S_0 < \overline{b}$ there are 8 digital barrier options which pay 1 depending conditional on the underlying crossing/not-crossing the barriers[2]. Naturally they come in pairs, e.g. $\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \geq \underline{b}} = 1 - \mathbf{1}_{\overline{S}_T \leq \overline{b} \text{ or } \underline{S}_T \leq \underline{b}}$ a.s.[3]. In consequence a model-free super- and sub- hedge of a double touch/no-touch barrier option $\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \geq \underline{b}}$ implies respectively a model-free sub- and super- hedge of a barrier option with payoff $\mathbf{1}_{\overline{S}_T \leq \overline{b} \text{ or } \underline{S}_T \leq \underline{b}}$ (and vice versa). Furthermore, there are two double touch/no-touch options with payoffs $\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \geq \underline{b}}$ and $\mathbf{1}_{\overline{S}_T \leq \overline{b},\, \underline{S}_T \leq \underline{b}}$ but by symmetry it suffices to consider one of them. In consequence to understand model-free pricing and hedging of all digital double barrier options it is enough to consider three options

- the *double touch* option with payoff $\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}}$,
- the *double touch/no-touch* option with payoff $\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \geq \underline{b}}$,
- the *double no-touch* option with payoff $\mathbf{1}_{\overline{S}_T \leq \overline{b},\, \underline{S}_T \geq \underline{b}}$.

The present papers deals with the first type, while the second and third types are treated in Cox and Obłój [CO08b] and Cox and Obłój [CO08a].

2 Model-free pricing and hedging

We investigate now model-free pricing and hedging of double touch option which pays 1 if and only if the stock price goes above $\overline{b}$ and below $\underline{b}$ before maturity: $\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}}$. We present simple quasi-static super- and sub- replicating strategies which prove to be optimal (i.e. replicating) in some market model.

2.1 Superhedging

We present here four super-replicating strategies. All our strategies have the same simple structure: we buy an initial portfolio of calls and puts and when the stock price reaches $\underline{b}$ or $\overline{b}$ we buy or sell forward contracts. Naturally our goal is not only to write a super-replicating strategy but to write *the smallest super-replicating strategy* and to do so we have to choose judiciously the parameters. As we will see in Section 2.3, for a given pair of barriers $\underline{b}, \overline{b}$ exactly one of the super-replicating strategies will induce a tight bound on the derivative's price. We will provide an explicit criterion determining which strategy to use.

$\overline{H}^I$*: superhedge for* $\underline{b} << S_0 < \overline{b}$.

We buy α puts with strike $K \in (\underline{b}, \infty)$ and when the stock price reaches $\underline{b}$ we buy β forward contracts, see Figure 1. The values of α, β are chosen so that the final payoff on $(0, K)$, provided the stock price has reached $\underline{b}$, is constant and equal to 1. One easily computes that $\alpha = \beta = (K - \underline{b})^{-1}$. Formally, the

[2] Naturally, there are further 4 'degenerate' options which only involve one of the barriers – these were treated in [BHR01] as mentioned above.

[3] We assume the distribution of S_T has no atoms and this implies that distributions of $\overline{S}_T$ and $\underline{S}_T$ also have no atoms, cf. Cox and Obłój [CO08b].

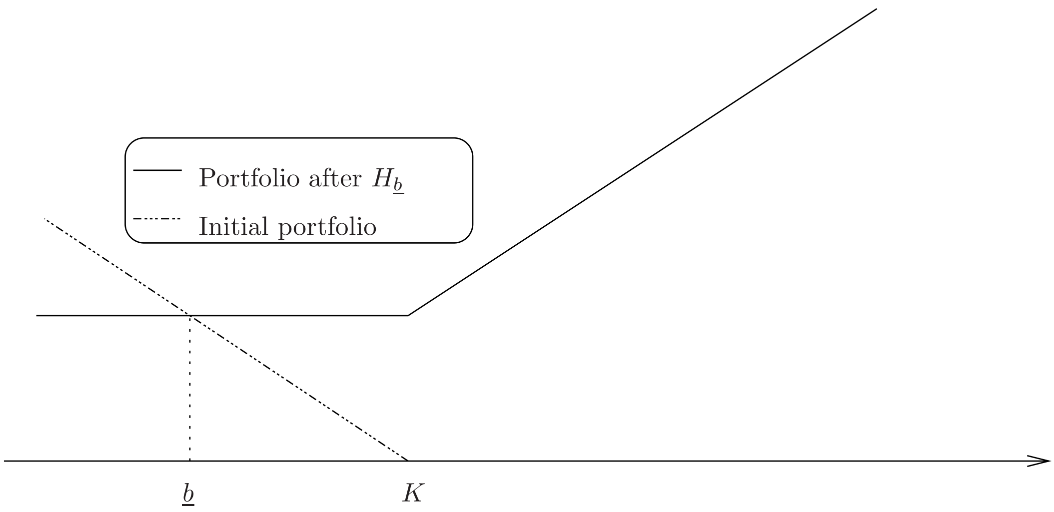

Figure 1: Superhedge H^I

super-replication follows from the following inequality

$$\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} \leq \frac{(K - S_T)^+}{K - \underline{b}} + \frac{S_T - \underline{b}}{K - \underline{b}} \mathbf{1}_{\underline{S}_T \leq \underline{b}} =: \overline{H}^I(K), \tag{6}$$

where the last term corresponds to a forward contract entered into, at no cost, when $S_t = \underline{b}$. Note that $\mathbf{1}_{\underline{S}_T \leq \underline{b}} = \mathbf{1}_{H_{\underline{b}} \leq T}$.

$\overline{H}^{II}$*: superhedge for* $\underline{b} < S_0 << \overline{b}$.

This is a mirror image of $\overline{H}^I$: we buy α calls with strike $K \in (0, \overline{b})$ and when the stock price reaches $\overline{b}$ we sell β forward contracts. The values of α, β are chosen so that the final payoff on (K, ∞), provided the stock price reached $\overline{b}$, is constant and equal to 1. One easily computes that $\alpha = \beta = (\overline{b} - K)^{-1}$. Formally, the super-replication follows from the following inequality

$$\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} \leq \frac{(S_T - K)^+}{\overline{b} - K} + \frac{\overline{b} - S_T}{\overline{b} - K} \mathbf{1}_{\overline{S}_T \geq \overline{b}} =: \overline{H}^{II}(K). \tag{7}$$

$\overline{H}^{III}$*: superhedge for* $\underline{b} << S_0 << \overline{b}$.

This superhedge involves a static portfolio of 4 calls and puts and at most 4 dynamic trades. The choice of parameters is judicious which makes the strategy the most complex to describe. Choose

$$0 < K_4 < \underline{b} < K_3 < K_2 < \overline{b} < K_1 \tag{8}$$

and buy α_i calls with strike K_i, $i = 1, 2$ and α_j puts with strike K_j, $j = 3, 4$. If the stock price reaches $\overline{b}$ without having hit $\underline{b}$ before, that is when $H_{\overline{b}} < H_{\underline{b}} \wedge T$, sell β_1 forward. If $H_{\underline{b}} < H_{\overline{b}} \wedge T$, at $H_{\underline{b}}$ buy β_2 forwards. When the stock price reaches $\underline{b}$ after having hit $\overline{b}$, that is when $H_{\overline{b}} < H_{\underline{b}} \leq T$, buy $\beta_3 = \alpha_3 + \beta_1$ forwards. Finally, if $H_{\underline{b}} < H_{\overline{b}} \leq T$, sell $\beta_4 = \alpha_2 + \beta_2$ forwards. The choice of β_3 and β_4 is such that the final payoff after hitting $\overline{b}$ and then $\underline{b}$ (resp. $\underline{b}$ and then $\overline{b}$) is constant and equal to 1 on $[K_4, K_3]$ (resp. $[K_2, K_1]$). We now proceed to impose conditions which determine other parameters. A pictorial representation of the superhedge is given in Figure 2.

Note that the initial payoff on $[K_3, K_2]$ is zero. After hitting $\overline{b}$ and before hitting $\underline{b}$ the payoff should be zero on $[K_1, \infty)$ and equal to 1 at $\underline{b}$. Likewise, after hitting $\underline{b}$ and before hitting $\overline{b}$, the payoff should be zero on $[0, K_4]$ and equal to 1 at $\overline{b}$. This yields 6 equations

$$\begin{cases} \alpha_1 + \alpha_2 - \beta_1 = 0 \\ \alpha_2(K_1 - K_2) - \beta_1(K_1 - \overline{b}) = 0 \\ \alpha_3(K_3 - \underline{b}) - \beta_1(\underline{b} - \overline{b}) = 1 \end{cases} \qquad \begin{cases} \alpha_3 + \alpha_4 - \beta_2 = 0 \\ \alpha_3(K_3 - K_4) + \beta_2(K_4 - \underline{b}) = 0 \\ \alpha_2(\overline{b} - K_2) + \beta_2(\overline{b} - \underline{b}) = 1 \end{cases}. \tag{9}$$

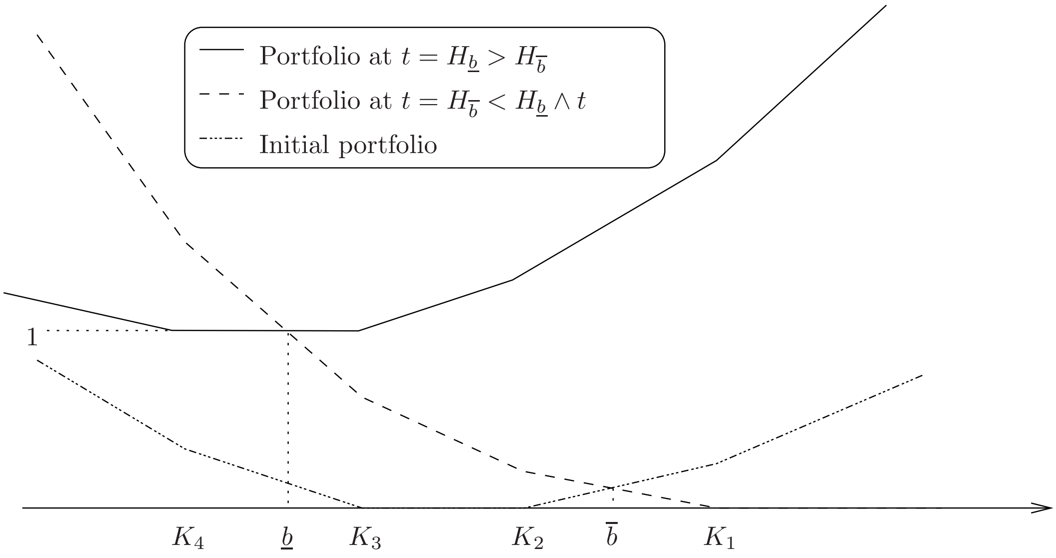

Figure 2: Superhedge H^{III}

The superhedging strategy corresponds to an a.s. inequality

$$\begin{aligned}\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} \leq\ & \alpha_1(S_T - K_1)^+ + \alpha_2(S_T - K_2)^+ + \alpha_3(K_3 - S_T)^+ + \alpha_4(K_4 - S_T)^+ \\ & - \beta_1(S_T - \overline{b})\mathbf{1}_{H_{\overline{b}} < H_{\underline{b}} \wedge T} + \beta_2(S_T - \underline{b})\mathbf{1}_{H_{\underline{b}} < H_{\overline{b}} \wedge T} \\ & + \beta_3(S_T - \underline{b})\mathbf{1}_{H_{\overline{b}} < H_{\underline{b}} \leq T} - \beta_4(S_T - \overline{b})\mathbf{1}_{H_{\underline{b}} < H_{\overline{b}} \leq T} \\ & =: \overline{H}^{III}(K_1, K_2, K_3, K_4),\end{aligned} \tag{10}$$

where the parameters, after solving (9), are given by

$$\begin{aligned}&\alpha_3 = \frac{(K_1 - K_2)(\underline{b} - K_4)(\overline{b} - \underline{b}) - (K_1 - \overline{b})(\overline{b} - K_2)(\underline{b} - K_4)}{(K_1 - K_2)(K_3 - K_4)(\overline{b} - \underline{b})^2 - (K_3 - \underline{b})(K_1 - \overline{b})(\overline{b} - K_2)(\underline{b} - K_4)} \\ &\begin{cases} \alpha_1 = \left(1 - \alpha_3 \frac{K_3 - K_4}{\underline{b} - K_4}(\overline{b} - \underline{b})\right)(K_1 - \overline{b})^{-1} \\ \alpha_2 = \left(1 - \alpha_3 \frac{K_3 - K_4}{\underline{b} - K_4}(\overline{b} - \underline{b})\right)(\overline{b} - K_2)^{-1} \\ \alpha_4 = \frac{K_3 - \underline{b}}{\underline{b} - K_4}\alpha_3 \end{cases} \qquad \begin{cases} \beta_1 = \alpha_1 + \alpha_2 \\ \beta_2 = \alpha_3 + \alpha_4 \end{cases}.\end{aligned} \tag{11}$$

Using (8) one can verify that α_3 and α_1 are non-negative and thus also α_2 and α_4, and all $\beta_1, \ldots, \beta_4$.

$\overline{H}^{IV}$ *: superhedge for* $\underline{b} < S_0 < \overline{b}$.
Choose $0 < K_2 < \underline{b} < S_0 < \overline{b} < K_1$. The initial portfolio is composed of α_1 calls with strike K_1, α_2 puts with strike K_2, α_3 forward contracts and α_4 in cash. If we hit $\overline{b}$ before hitting $\underline{b}$ we sell β_1 forwards, and if we hit $\underline{b}$ before hitting $\overline{b}$ we buy β_2 forwards. The payoff of the portfolio should be zero on $[K_1, \infty)$ (resp. $[0, K_2]$) and equal to 1 at $\underline{b}$ (resp. $\overline{b}$) in the first (resp. second) case. Finally, when we hit $\underline{b}$ after having hit $\overline{b}$ we buy β_3 forwards, and when we hit $\overline{b}$, having hit $\underline{b}$, we sell β_4 forwards. In both cases the final payoff should then be equal to 1 on $[K_2, K_1]$, see Figure 3. The superhedging strategy corresponds to the following a.s. inequality

$$\begin{aligned}\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} \leq\ & \alpha_1(S_T - K_1)^+ + \alpha_2(K_2 - S_T)^+ + \alpha_3(S_T - S_0) + \alpha_4 \\ & - \beta_1(S_T - \overline{b})\mathbf{1}_{H_{\overline{b}} < H_{\underline{b}} \wedge T} + \beta_2(S_T - \underline{b})\mathbf{1}_{H_{\underline{b}} < H_{\overline{b}} \wedge T} \\ & + \beta_3(S_T - \underline{b})\mathbf{1}_{H_{\overline{b}} < H_{\underline{b}} \leq T} - \beta_4(S_T - \overline{b})\mathbf{1}_{H_{\underline{b}} < H_{\overline{b}} \leq T} \\ & =: \overline{H}^{IV}(K_1, K_2),\end{aligned} \tag{12}$$

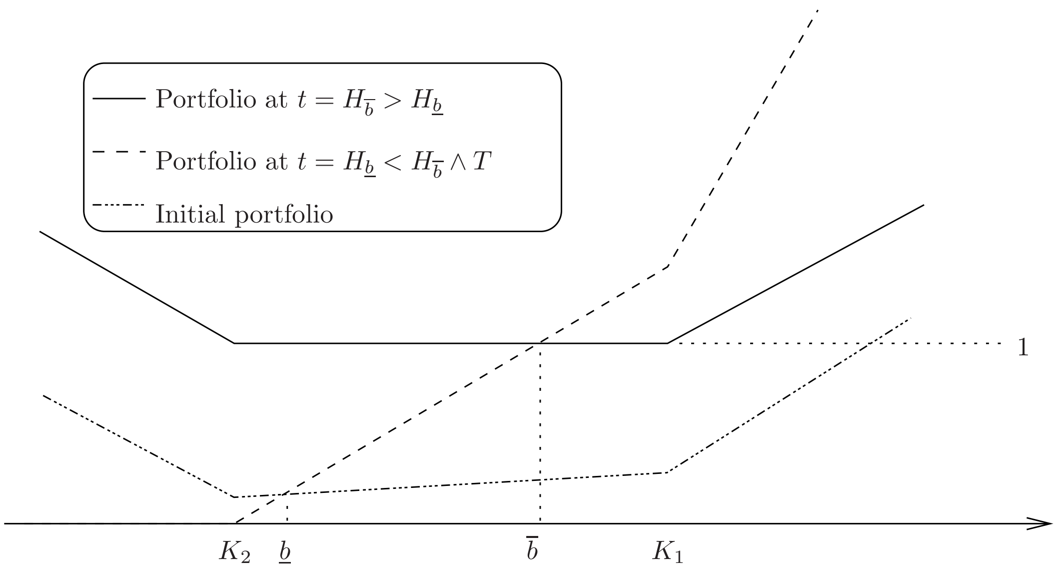

Figure 3: Superhedge H^{IV}

where, working out the conditions on α_i, β_i, the parameters are

$$\begin{cases} \alpha_1 = 1/(K_1 - \underline{b}) \\ \alpha_2 = 1/(\overline{b} - K_2) \\ \alpha_3 = \frac{(K_1-\underline{b})-(\overline{b}-K_2)}{(K_1-\underline{b})(\overline{b}-K_2)} \\ \alpha_4 = \frac{\underline{b}\overline{b}-K_1K_2}{(K_1-\underline{b})(\overline{b}-K_2)} + \alpha_3 S_0 \end{cases} \qquad \begin{cases} \beta_1 = \alpha_1 + \alpha_3 = 1/(\overline{b} - K_2) \\ \beta_2 = \alpha_2 - \alpha_3 = 1/(K_1 - \underline{b}) \\ \beta_3 = \alpha_1 = 1/(K_1 - \underline{b}) \\ \beta_4 = \alpha_2 = 1/(\overline{b} - K_2) \end{cases}. \tag{13}$$

2.2 Subhedging

We present now three constructions of subhedges which will turn out to be the best (i.e. the most expensive) model-free subhedges depending on the relative distance of barriers to the spot. We note however that there is also a fourth (trivial) subhedge, which has payoff zero and corresponds to an empty portfolio. In fact this will be the most expensive subhedge when $\underline{b} << S_0 << \overline{b}$ and we can construct a market model in which both barriers are never hit. Details will be given in Theorem 2.3.4.

$\underline{H}_I$*: subhedge for* $\underline{b} < S_0 < \overline{b}$.
Choose $0 < K_2 < \underline{b} < S_0 < \overline{b} < K_1$. The initial portfolio will contain a cash amount, a forward, calls with 5 different strikes and additionally will also include two digital options, which pay off £1 provided S_T is above a specified level. Figure 3 demonstrates graphically the hedging strategy, and we note the effect of the digital options is to provide a jump in the payoff at the points $\underline{b}$, $\overline{b}$.

As in the previous cases, the optimality of the construction will follow from an almost-sure inequality. The relevant inequality is now:

$$\begin{aligned} \mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} \geq\ & \alpha_0 + \alpha_1(S_T - S_0) - \alpha_2(S_T - K_2)^+ + \alpha_3(S_T - \underline{b})^+ - \alpha_3(S_T - K_3)^+ \\ & + \alpha_3(S_T - \overline{b})^+ - (\alpha_3 - \alpha_2)(S_T - K_1)^+ - \gamma_1 \mathbf{1}_{\{S_T > \underline{b}\}} + \gamma_2 \mathbf{1}_{\{S_T \geq \overline{b}\}} \\ & + (\alpha_2 - \alpha_1)(S_T - \underline{b})\mathbf{1}_{\{H_{\underline{b}} < H_{\overline{b}} \wedge T\}} - \alpha_2(S_T - \overline{b})\mathbf{1}_{\{H_{\underline{b}} < H_{\overline{b}} < T\}} \\ & - (\alpha_3 - \alpha_2 + \alpha_1)(S_T - \overline{b})\mathbf{1}_{\{H_{\overline{b}} < H_{\underline{b}} \wedge T\}} + (\alpha_3 - \alpha_2)(S_T - \underline{b})\mathbf{1}_{\{H_{\overline{b}} < H_{\underline{b}} < T\}} \end{aligned}. \tag{14}$$

Specifically, we can see that the hedging strategy consists of a portfolio which contains cash α_0, α_1 forwards, is short α_2 calls at strike K_2 etc. The novel terms here are the digital options; we note further that the digital options can be considered also as the limit of portfolios of calls (see for example [BC94]). In our context, we can use their limiting argument to write: $\mathcal{P}\mathbf{1}_{\{S_T \geq \overline{b}\}} = -C'(K)$.

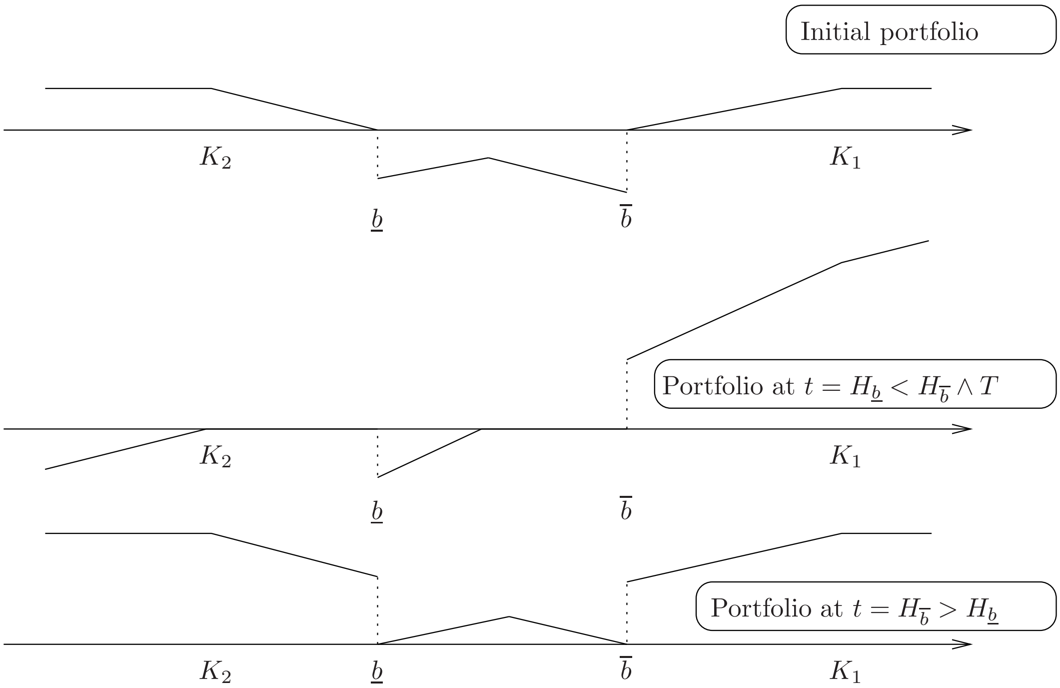

Figure 4: Subhedge $\underline{H}_I$.

The strategy to be followed is then: initially, run to either $\underline{b}$ or $\overline{b}$; supposing that $\underline{b}$ is hit first, we buy $(\alpha_2 - \alpha_1)$ forward, then if we later hit $\overline{b}$, we sell forward α_2 units of the underlying. A similar strategy is followed if $\overline{b}$ is hit first. As previously, the structure imposes some constraints on the parameters. The relevant constraints are:

$$\begin{aligned}
0 &= \alpha_0 + \alpha_1(\underline{b} - S_0) - \alpha_2(\underline{b} - K_2) && (15)\\
0 &= \alpha_0 + \alpha_1(\overline{b} - S_0) - \alpha_2(\overline{b} - K_2) + \alpha_3(K_3 - \underline{b}) - \gamma_1 + \gamma_2 && (16)\\
1 &= \alpha_0 + \alpha_1(K_2 - S_0) + (\alpha_2 - \alpha_1)(K_2 - \underline{b}) - \alpha_2(K_2 - \overline{b}) && (17)\\
1 &= \alpha_0 + \alpha_1(K_2 - S_0) - (\alpha_3 - \alpha_2 + \alpha_1)(K_2 - \overline{b}) + (\alpha_3 - \alpha_2)(K_2 - \underline{b}) && (18)\\
\gamma_1 &= (K_3 - \underline{b})\alpha_3 && (19)\\
\gamma_2 &= (\overline{b} - K_3)\alpha_3 && (20)\\
\frac{K_3 - \underline{b}}{K_1 - \underline{b}} &= \frac{\overline{b} - K_3}{\overline{b} - K_2} && (21)
\end{aligned}$$

The equations (15) and (16) arise from the constraint that initially the payoff is zero at $\underline{b}, \overline{b}$; constraints (17) and (18) come from the constraint that the final payoff is 1 at K_2 when both barrier are hit (in either order); (19) and (20) represent the fact that, in the intermediate step, at K_3 the gap at $\underline{b}$ (resp. $\overline{b}$) is the size of the respective digital option. The final constraint, (21) follows from noting that K_3 is the intersection point of the lines from $(\underline{b}, 0)$ to $(K_1, 1)$ and from $(K_2, 1)$ to $(\overline{b}, 0)$. Note that it follows that the initial payoff on $(0, K_1)$ and (K_2, ∞) are co-linear, or that the finial payoff in K_1 is 1 when both barriers are hit.

The given equations can be solved to deduce:

$$\left\{\begin{aligned}
\alpha_0 &= \frac{S_0(K_1+K_2-\overline{b}-\underline{b})+\overline{b}\underline{b}-K_1K_2}{(\overline{b}-K_2)(K_1-\underline{b})}\\
\alpha_1 &= \frac{K_1+K_2-\overline{b}-\underline{b}}{(\overline{b}-K_2)(K_1-\underline{b})}\\
\alpha_2 &= \frac{1}{\overline{b}-K_2}\\
\alpha_3 &= \frac{\overline{b}-K_2+K_1-\underline{b}}{(\overline{b}-K_2)(K_1-\underline{b})}
\end{aligned}\right.
\qquad
\left\{\begin{aligned}
K_3 &= \frac{\overline{b}K_1-\underline{b}K_2}{\overline{b}-K_2-\underline{b}+K_1}\\
\gamma_1 &= \frac{\overline{b}-\underline{b}}{\overline{b}-K_2}\\
\gamma_2 &= \frac{\overline{b}-\underline{b}}{K_1-\underline{b}}
\end{aligned}\right. \qquad (22)$$

We note from the above that $\alpha_2, \alpha_3, \gamma_1$ and γ_2 are all (strictly) positive; further, it can be checked that

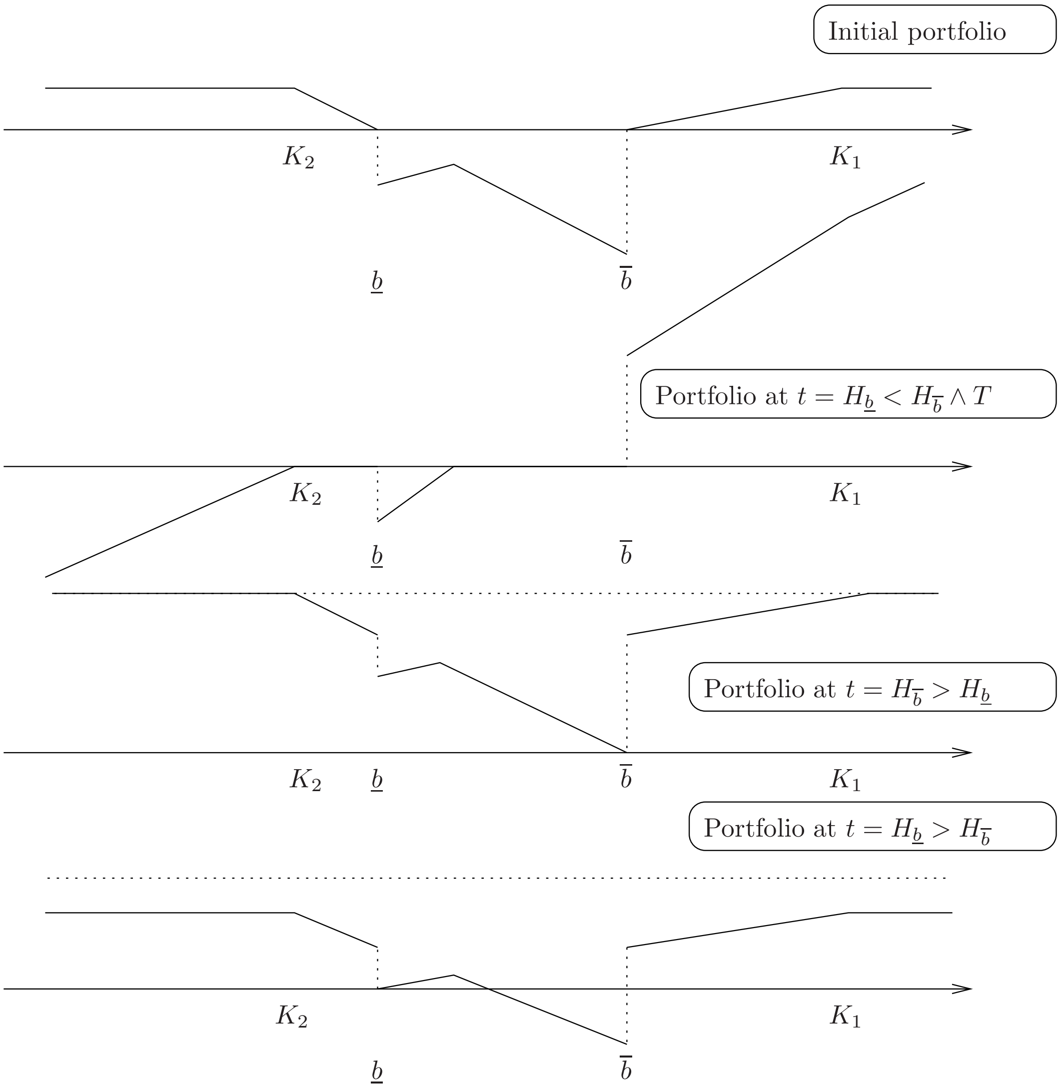

Figure 5: Subhedge $\underline{H}_{II}$

the quantities $(\alpha_3 - \alpha_2), (\alpha_2 - \alpha_1), (\alpha_3 - \alpha_2 + \alpha_1)$ are all positive. It follows that the construction holds for all choices of K_1, K_2 with $K_2 < \underline{b}$, and $K_1 > \overline{b}$.

For future reference, we define $\underline{H}_I(K_1, K_2)$ to be the random variable given by the right hand side of (14), where the coefficients are given by the solutions of (15)–(21).

$\underline{H}_{II}$*: subhedge for* $\underline{b} < S_0 << \overline{b}$.
While the above hedge can be considered to be the 'typical' subhedge for the option, there are also two further cases that need to be considered when the initial stock price, S_0, is much closer to one of the barriers than the other. The resulting subhedge will share many of the features of the previous construction, however the main difference concerns the behaviour in the tails; we now have the hedge taking the value one in the tails only under one of the possible ways of knocking in (specifically, in the case where $\underline{b} < S_0 << \overline{b}$, we get equality in the tails only when $\underline{b}$ is hit first.)

A graphical representation of the construction is given in Figure 5. In this case, rather than specifying only K_1 and K_2, we also need to specify $K_3 \in (\underline{b}, \overline{b})$ satisfying:

$$\frac{\overline{b} - K_3}{\overline{b} - K_2} \geq \frac{K_3 - \underline{b}}{K_1 - \underline{b}}$$

which implies the function is larger just above $\underline{b}$ than just below $\overline{b}$. This can be rearranged to get:

$$K_3 \leq \overline{b}\frac{K_1 - \underline{b}}{(K_1 - \underline{b}) + (\overline{b} - K_2)} + \underline{b}\frac{\overline{b} - K_2}{(K_1 - \underline{b}) + (\overline{b} - K_2)}$$

The actual inequality we use remains the same as in the previous case (14), as do some of the constraints:

$$\begin{aligned}
0 &= \alpha_0 + \alpha_1(\underline{b} - S_0) - \alpha_2(\underline{b} - K_2) && (23)\\
0 &= \alpha_0 + \alpha_1(\overline{b} - S_0) - \alpha_2(\overline{b} - K_2) + \alpha_3(K_3 - \underline{b}) - \gamma_1 + \gamma_2 && (24)\\
1 &= \alpha_0 + \alpha_1(K_2 - S_0) + (\alpha_2 - \alpha_1)(K_2 - \underline{b}) - \alpha_2(K_2 - \overline{b}) && (25)\\
1 &= \alpha_0 + \alpha_1(\underline{b} - S_0) + \alpha_2(K_2 - K_1 + \overline{b} - \underline{b}) + \alpha_3(K_3 + K_1 - \underline{b} - \overline{b}) - \gamma_1 + \gamma_2 && (26)\\
\gamma_1 &= (K_3 - \underline{b})\alpha_3 && (27)\\
\gamma_2 &= (\overline{b} - K_3)\alpha_3 && (28)
\end{aligned}$$

(23) and (24) refer still to having an initial payoff of 0 at $\overline{b}$ and $\underline{b}$, (27) and (28) also still relate the size of the digital options to the slopes. The change is in the constraints (25) and (26) which now ensure that the function at K_1 and K_2, after hitting first $\underline{b}$ and then $\overline{b}$, takes the value 1. We note that, in the previous example, where (21) held, these are in fact equivalent to (17) and (18); the fact that (21) no longer holds means that we need to be more specific about the constraints.

The solutions to the above are now:

$$\left\{\begin{aligned}
\alpha_0 &= \tfrac{(\overline{b}\underline{b}+S_0K_3)(K_1-K_2)-(\underline{b}K_1+S_0\overline{b})(K_3-K_2)-(\overline{b}K_2+S_0\underline{b})(K_1-K_3)}{(\overline{b}-\underline{b})(K_1-K_3)(\overline{b}-K_2)}\\
\alpha_1 &= \tfrac{K_3(K_1-K_2)-\underline{b}(K_1-K_3)-\overline{b}(K_3-K_2)}{(\overline{b}-\underline{b})(K_1-K_3)(\overline{b}-K_2)}\\
\alpha_2 &= \tfrac{1}{\overline{b}-K_2}\\
\alpha_3 &= \tfrac{K_1-K_2}{(K_1-K_3)(\overline{b}-K_2)}
\end{aligned}\right. \qquad
\left\{\begin{aligned}
\gamma_1 &= \tfrac{(K_3-\underline{b})(K_1-K_2)}{(\overline{b}-K_2)(K_1-K_3)}\\
\gamma_2 &= \tfrac{(\overline{b}-K_3)(K_1-K_2)}{(\overline{b}-K_2)(K_1-K_3)}
\end{aligned}\right. \qquad (29)$$

As before, we write $\underline{H}_{II}(K_1, K_2, K_3)$ for the random variable on the right hand side of (14) where the constants are chosen as the solutions to the above equations.

$\underline{H}_{III}$: *subhedge for* $\underline{b} << S_0 < \overline{b}$.
The third case here is the corresponding version of the above where we have a large value of K_3, specifically,

$$K_3 \geq \overline{b}\frac{K_1 - \underline{b}}{(K_1 - \underline{b}) + (\overline{b} - K_2)} + \underline{b}\frac{\overline{b} - K_2}{(K_1 - \underline{b}) + (\overline{b} - K_2)}$$

and we need to modify equations (25) and (26) appropriately:

$$\begin{aligned}
1 &= \alpha_0 + \alpha_1(\overline{b} - S_0) + (\alpha_3 - \alpha_2)(\overline{b} - \underline{b})\\
1 &= \alpha_0 + \alpha_1(\overline{b} - S_0) + \alpha_2(K_2 - K_1 + \underline{b} - \overline{b}) + \alpha_3(K_3 + K_1 - 2\underline{b}) - \gamma_1 + \gamma_2
\end{aligned}$$

and the solutions are now:

$$\left\{\begin{aligned}
\alpha_0 &= \tfrac{(\overline{b}\underline{b}+S_0K_3)(K_1-K_2)-(\underline{b}K_1+S_0\overline{b})(K_3-K_2)-(\overline{b}K_2+S_0\underline{b})(K_1-K_3)}{(\overline{b}-\underline{b})(K_3-K_2)(K_1-\underline{b})}\\
\alpha_1 &= \tfrac{K_3(K_1-K_2)-\underline{b}(K_1-K_3)-\overline{b}(K_3-K_2)}{(\overline{b}-\underline{b})(K_3-K_2)(K_1-\underline{b})}\\
\alpha_2 &= \tfrac{K_1-K_3}{(K_3-K_2)(K_1-\underline{b})}\\
\alpha_3 &= \tfrac{K_1-K_2}{(K_3-K_2)(K_1-\underline{b})}
\end{aligned}\right. \qquad
\left\{\begin{aligned}
\gamma_1 &= \tfrac{(K_3-\underline{b})(K_1-K_2)}{(K_3-K_2)(K_1-\underline{b})}\\
\gamma_2 &= \tfrac{(\overline{b}-K_3)(K_1-K_2)}{(K_3-K_2)(K_1-\underline{b})}
\end{aligned}\right. \qquad (30)$$

As before, we write $\underline{H}_{III}(K_1, K_2, K_3)$ for the random variable on the right hand side of (14) where the constants are chosen as the solutions to the above equations.

2.3 Pricing

Consider the double touch digital barrier option with the payoff $\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}}$. As an immediate consequence of the superhedging strategies described in Section 2.1 we get an upper bound on the price of this derivative:

Proposition 2.3.1. Given the market input (3), no-arbitrage (4) in the class of portfolios $\mathrm{Lin}(\mathcal{X} \cup \{\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}}\})$ implies the following inequality between the prices

$$\mathcal{P}\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} \leq \inf\left\{\mathcal{P}\overline{H}^{I}(K), \mathcal{P}\overline{H}^{II}(K'), \mathcal{P}\overline{H}^{III}(K_1,K_2,K_3,K_4), \mathcal{P}\overline{H}^{IV}(K_1,K_4)\right\}, \tag{31}$$

where the infimum is taken over $K > \underline{b}$, $K' < \overline{b}$ and $0 < K_4 < \underline{b} < K_3 < K_2 < \overline{b} < K_1$, and where $\overline{H}^{I}, \overline{H}^{II}, \overline{H}^{III}, \overline{H}^{IV}$ are given by (6),(7),(10) and (12) respectively.

The purpose of this section is to show that given the law of S_T and the pair of barriers $\underline{b}, \overline{b}$ we can determine explicitly which superhedges and with what strikes the infimum on the RHS of (31) is achieved. We present formal criteria but we also use labels, e.g. '$\underline{b} << S_0 < \overline{b}$', which provide an intuitive classification. An example is considered later in Figure 10. Furthermore, we will show that we can always construct a market model in which the infimum in (31) is the actual price of the double barrier option and therefore exhibit the model-free least upper bound for the price of the derivative. Subsequently, an analogue reasoning for subhedging and the lower bound is presented.

Let μ be the market implied law of S_T given by (5). The barycentre of μ associates to a non-empty Borel set $\Gamma \subset \mathbb{R}$ the mean of μ over Γ via

$$\mu_B(\Gamma) = \frac{\int_\Gamma u\mu(\mathrm{d}u)}{\int_\Gamma \mu(\mathrm{d}u)}. \tag{32}$$

For $w < \underline{b}$ and $z > \overline{b}$ let $\rho_-(w) > \underline{b}$ and $\rho_+(z) < \overline{b}$ be the unique points such that the intervals $[w, \rho_-(w)]$ and $[\rho_+(z), z]$ are centered respectively around $\underline{b}$ and $\overline{b}$, that is

$$\begin{cases} \rho_- : [0,\underline{b}] \to [\underline{b},\infty) & \text{defined via } \mu_B([w,\rho_-(w)]) = \underline{b}, \\ \rho_+ : [\overline{b},\infty) \to [0,\overline{b}] & \text{defined via } \mu_B([\rho_+(z),z]) = \overline{b}. \end{cases} \tag{33}$$

Note that $\rho_\pm$ are decreasing and well defined as $\mu_B([0,\infty)) = S_0 \in (\underline{b},\overline{b})$. We need to define two more functions:

$$\begin{cases} \gamma_+(w) \geq \overline{b} & \text{defined via } \mu_B\big([0,w] \cup [\rho_+(\gamma_+(w)),\gamma_+(w)]\big) = \underline{b},\ w \leq \underline{b}, \\ \gamma_-(z) \leq \underline{b} & \text{defined via } \mu_B\big([\gamma_-(z),\rho_-(\gamma_-(z))] \cup [z,\infty)\big) = \overline{b},\ z \geq \overline{b}, \end{cases} \tag{34}$$

so that $\gamma_+(\cdot)$ is increasing, $\gamma_-(\cdot)$ is decreasing, and:

$$\gamma_+(w) \downarrow \overline{b} \text{ as } w \downarrow 0, \gamma_-(z) \uparrow \underline{b} \text{ as } z \uparrow \infty.$$

Note that γ_+ is defined on $[0, w_0]$ where $w_0 = \underline{b} \wedge \sup\{w < \underline{b} : \gamma_+(w) < \infty\}$ and similarly γ_- is defined on $[z_0, \infty]$. We are now ready to state our main theorem.

Theorem 2.3.2. *Let μ be the law of S_T inferred from the prices of vanillas via* (5) *and consider the double barrier derivative paying $\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}}$ for a fixed pair of barriers $\underline{b} < S_0 < \overline{b}$. Then exactly one of the following is true*

$\boxed{I}$ *'$\underline{b} << S_0 < \overline{b}$':*

There exists $z_0 > \overline{b}$ such that[4]

$$\gamma_-(z) \downarrow 0 \text{ as } z \downarrow z_0, \text{ and } \rho_-(0) \leq \overline{b}. \tag{35}$$

Then there is a market model in which $\mathbb{E}\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} = \mathbb{E}\overline{H}^{I}(\rho_-(0)) = \frac{C(\rho_-(0))}{\rho_-(0) - \underline{b}}$.

[4]note that here and subsequently, we use $\uparrow \infty$ and $\downarrow 0$ as meaning only the case where the increasing/decreasing sequence is itself finite/strictly positive, so that in (35), we strictly mean

$$\gamma_-(z) \to 0 \text{ as } z \downarrow z_0, \text{ and } \gamma_-(z) > 0 \text{ for } z > z_0$$

$\boxed{II}$ *'$\underline{b} < S_0 << \overline{b}$':*
There exists $w_0 < \underline{b}$ such that

$$\gamma_+(w) \uparrow \infty \text{ as } w \uparrow w_0, \text{ and } \rho_+(\infty) \geq \underline{b}.$$

Then there is a market model in which $\mathbb{E}\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} = \mathbb{E}\overline{H}^{II}(\rho_+(\infty)) = \frac{C(\rho_+(\infty))}{\overline{b}-\rho_+(\infty)}$.

$\boxed{III}$ *'$\underline{b} << S_0 << \overline{b}$':*
There exists $0 \leq w_0 \leq \underline{b}$ such that $\gamma_-(\gamma_+(w_0)) = w_0$ and $\rho_-(w_0) \leq \rho_+(\gamma_+(w_0))$.
Then there is a market model in which

$$\begin{aligned}\mathbb{E}\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} &= \mathbb{E}\overline{H}^{III}\big(\gamma_+(w_0), \rho_+(\gamma_+(w_0)), \rho_-(w_0), w_0\big) \\ &= \alpha_1 C\big(\gamma_+(w_0)\big) + \alpha_2 C\big(\rho_+(\gamma_+(w_0))\big) + \alpha_3 P\big(\rho_-(w_0)\big) + \alpha_4 P\big(w_0\big),\end{aligned} \tag{36}$$

where α_i are given in (11).

$\boxed{IV}$ *'$\underline{b} < S_0 < \overline{b}$':*
We have $\overline{b} < \rho_-(0), \underline{b} > \rho_+(\infty)$ and $\rho_+(\rho_-(0)) < \rho_-(\rho_+(\infty))$.
Then there is a market model in which

$$\begin{aligned}\mathbb{E}\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} &= \mathbb{E}\overline{H}^{IV}\big(\rho_-(0), \rho_+(\infty)\big) \\ &= \alpha_1 C\big(\rho_-(0)\big) + \alpha_2 P\big(\rho_+(\infty)\big) + \alpha_4,\end{aligned} \tag{37}$$

where α_i are given in (13).

We present now the analogues of Proposition 2.3.1 and Theorem 2.3.2 for the subhedging case. Whereas above we find an upper bound on the price of the derivative, in this case we will construct a lower bound.

Proposition 2.3.3. Given the market input (3), no-arbitrage (4) in the class of portfolios $\mathrm{Lin}(\mathcal{X} \cup \{\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}}, \mathbf{1}_{\{S_T > \underline{b}\}}, \mathbf{1}_{\{S_T \geq \overline{b}\}}\})$ implies the following inequality between the prices

$$\mathcal{P}\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} \geq \sup\left\{\mathcal{P}\underline{H}_I(K_1,K_2), \mathcal{P}\underline{H}_{II}(K_1,K_2,K_3), \mathcal{P}\underline{H}_{III}(K_1,K_2,K_3), 0\right\}, \tag{38}$$

where the supremum is taken over $0 < K_2 < \underline{b} < K_3 < \overline{b} < K_1$ and $\underline{H}_I, \underline{H}_{II}, \underline{H}_{III}$ are given by (14) and the solutions to the relevant set of equations: (22), (29) and (30).

Again, an important aspect of (38) is that we can in fact show that the bound is tight — that is, given a set of call prices, there exists a process under which equality is attained. Recall that under no-arbitrage the prices of digital calls are essentially specified by our market input via $\mathcal{P}\mathbf{1}_{\{S_T \geq \overline{b}\}} = -C'(K)$.

In order to classify the different states, we make the following definitions. Let μ be the law of S_T implied by the call prices. Fix $\underline{b} < S_0 < \overline{b}$, and, given $v \in [\underline{b}, \overline{b}]$, define:

$$\begin{aligned}\psi(v) &= \inf\left\{z \in [0,\underline{b}] : \int_{(z,\underline{b})\cup(v,\overline{b})} u\,\mu(\mathrm{d}u) + \overline{b}\left(\frac{\overline{b}-S_0}{\overline{b}-\underline{b}} - \mu((z,\underline{b})\cup(v,\overline{b}))\right) = \underline{b}\frac{\overline{b}-S_0}{\overline{b}-\underline{b}}\right. \\ &\qquad \left.\text{and } \mu((z,\underline{b})\cup(v,\overline{b})) \leq \frac{\overline{b}-S_0}{\overline{b}-\underline{b}}\right\}\end{aligned} \tag{39}$$

$$\begin{aligned}\theta(v) &= \sup\left\{z \geq \overline{b} : \int_{(\underline{b},v)\cup(\overline{b},z)} u\,\mu(\mathrm{d}u) + \underline{b}\left\{\frac{S_0-\underline{b}}{\overline{b}-\underline{b}} - \mu((\underline{b},v)\cup(\overline{b},z))\right\} = \overline{b}\frac{S_0-\underline{b}}{\overline{b}-\underline{b}}\right. \\ &\qquad \left.\text{and } \mu((\underline{b},v)\cup(\overline{b},z)) \leq \frac{S_0-\underline{b}}{\overline{b}-\underline{b}}\right\}\end{aligned} \tag{40}$$

where we use the convention $\sup\{\emptyset\} = -\infty, \inf\{\emptyset\} = \infty$.

In particular, the definition of ψ ensures that, on the set where $\psi(v) \neq \infty$, we can run all the mass initially from S_0 to $\{\underline{b}, \overline{b}\}$ and then embed from $\underline{b}$ to $(\psi(v), \underline{b}) \cup (v, \overline{b})$ and a compensating atom at $\overline{b}$

with the remaining mass. Note further that the functions ψ and θ are both decreasing on the sets $\{v \in [\underline{b}, \overline{b}] : \psi(v) < \infty\}$ and $\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\}$, which are both closed intervals. Specifically, we will be interested in the region where both the functions allow for a suitable embedding; define

$$\begin{aligned} \overline{v} &= \min\left\{\sup\{v \in [\underline{b}, \overline{b}] : \psi(v) < \infty\}, \sup\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\}\right\}, \\ \underline{v} &= \max\left\{\inf\{v \in [\underline{b}, \overline{b}] : \psi(v) < \infty\}, \inf\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\}\right\}, \quad \text{and} \\ \kappa(v) &= \overline{b}\frac{\theta(v) - \underline{b}}{\theta(v) - \underline{b} + \overline{b} - \psi(v)} + \underline{b}\frac{\overline{b} - \psi(v)}{\theta(v) - \underline{b} + \overline{b} - \psi(v)}, \end{aligned} \tag{41}$$

where $\sup\{\emptyset\} = -\infty$ and $\inf\{\emptyset\} = \infty$.

Theorem 2.3.4. *Let μ be the law of S_T inferred from the prices of vanillas via* (5) *and consider the double barrier derivative paying $\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}}$ for a fixed pair of barriers $\underline{b} < S_0 < \overline{b}$, and recall* (39)-(41). *Then exactly one of the following is true*

$\boxed{I}$ *'$\underline{b} < S_0 < \overline{b}$':*
We have $\overline{v} \geq \underline{v}$ and there exists $v_0 \in [\underline{v}, \overline{v}]$ such that $\kappa(v_0) = v_0$. Then there exists a market model in which:

$$\begin{aligned} \mathbb{E}\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}} &= \mathbb{E}\underline{H}_I(\theta(v_0), \psi(v_0)) \\ &= \alpha_0 + \alpha_2(C(\theta(v_0)) - C(\psi(v_0))) + \gamma_2 D(\overline{b}) - \gamma_1 D(\underline{b}) \\ &\quad + \alpha_3\left[C(\underline{b}) + C(\overline{b}) - C(v_0) - C(\theta(v_0))\right] \end{aligned} \tag{42}$$

where $D(x)$ is the price of a digital option with payoff $\mathbf{1}_{\{S_T \geq x\}}$, and the values of $\alpha_0, \alpha_2, \alpha_3, \gamma_1, \gamma_2$ are given by (22).

$\boxed{II}$ *'$\underline{b} < S_0 << \overline{b}$':*
We have $\overline{v} \geq \underline{v}$ and $\overline{v} < \kappa(\overline{v})$. Then there exists a market model in which:

$$\begin{aligned} \mathbb{E}\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}} &= \mathbb{E}\underline{H}_{II}(\theta(\overline{v}), \psi(\overline{v}), \overline{v}) \\ &= \alpha_0 + \alpha_2(C(\theta(\overline{v})) - C(\psi(\overline{v}))) + \gamma_2 D(\overline{b}) - \gamma_1 D(\underline{b}) \\ &\quad + \alpha_3\left[C(\underline{b}) + C(\overline{b}) - C(\overline{v}) - C(\theta(\overline{v}))\right] \end{aligned} \tag{43}$$

where $D(x)$ is the price of a digital option with payoff $\mathbf{1}_{\{S_T \geq x\}}$, and the values of $\alpha_0, \alpha_2, \alpha_3, \gamma_1, \gamma_2$ are given by (29).

$\boxed{III}$ *'$\underline{b} << S_0 < \overline{b}$':*
We have $\overline{v} \geq \underline{v}$ and $\underline{v} > \kappa(\underline{v})$. Then there exists a market model in which:

$$\begin{aligned} \mathbb{E}\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}} &= \mathbb{E}\underline{H}_{III}(\theta(\underline{v}), \psi(\underline{v}), \underline{v}) \\ &= \alpha_0 + \alpha_2(C(\theta(\underline{v})) - C(\psi(\underline{v}))) + \gamma_2 D(\overline{b}) - \gamma_1 D(\underline{b}) \\ &\quad + \alpha_3\left[C(\underline{b}) + C(\overline{b}) - C(\underline{v}) - C(\theta(\underline{v}))\right] \end{aligned}$$

where $D(x)$ is the price of a digital option with payoff $\mathbf{1}_{\{S_T \geq x\}}$, and the values of $\alpha_0, \alpha_2, \alpha_3, \gamma_1, \gamma_2$ are given by (30).

$\boxed{IV}$ *'$\underline{b} << S_0 << \overline{b}$':*
We have $\overline{v} < \underline{v}$. Then there exists a market model in which $\mathbb{E}\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}} = 0$.

Furthermore, in cases $\boxed{I}$–$\boxed{III}$ we have $\underline{v} = \inf\{v \in [\underline{b}, \overline{b}] : \psi(v) < \infty\} \leq \sup\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\} = \overline{v}$.

3 Applications and Practical Considerations

3.1 Finitely many strikes

One important practical aspect where reality differs from the theoretical situation described above concerns the availability of calls with arbitrary strikes. Generally, calls will only trade at a finite set of strikes, $0 = x_0 \leq x_1 \leq \ldots \leq x_N$ (with $x_0 = 0$ corresponding to the asset itself). It is then natural to ask: how does this affect the hedging strategies introduced above? In full generality, this question results in a rather large number of 'special' cases that need to be considered separately (for example, the case where no strikes are traded above $\overline{b}$, or the case where there are no strikes traded with $\underline{b} < K < \overline{b}$). In addition, there are differing cases, dependent on whether the digital options at $\underline{b}$ and $\overline{b}$ are traded. Consequently, we will not attempt to give a complete answer to this question, but we will consider only the cases where there are 'comparatively many' traded strikes, and assume that digital calls are not available to trade. Furthermore, we will apply the theorems of previous sections to measures with atoms. It should be clear how to do this, but a formal treatment would be rather lengthy and tedious, with some extra care needed when the atoms are at the barriers. For that reason we only state the results of this section informally.

Mathematically, the presence of atoms in the measure μ means that the call prices are no longer twice differentiable. The function is still convex, but we now have possibly differing left and right derivatives for the function. The implication for the call prices is the following:

$$\mu([x,\infty)) = -C'_{-}(K), \text{ and } \mu((x,\infty)) = -C'_{+}(K).$$

In particular, atoms of μ will correspond to 'kinks' in the call prices.

The first remark to make in the finite-strike case is that, if we replace the supremum/infimum over strikes that appear in expressions such as (31) and (38) by the supremum/infimum over traded strikes, then the arguments that conclude that these are lower/upper bounds on the price are still valid. The argument only breaks down when we wish to show that these are the best possible bounds. To try to replace the latter, we now need to consider which models might be possible under the given call prices. Our approach will be based on the following type of argument:

(i) suppose that using only calls and puts with traded strikes we may construct $\tilde{H}^i$, for $i \in \{I, \ldots, IV\}$, such that $\tilde{H}^i \geq \overline{H}^i$ as a function of S_T;

(ii) suppose further that we can find an admissible call price function $C(K)$, $K > 0$, which agrees with the traded prices, and such that in the market model $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}^*)$ associated by Theorem 2.3.2 with the upper bound (31) we have $\tilde{H}^i(S_T) = \overline{H}^i(S_T)$, $\mathbb{P}^*$-almost surely;

then the smallest upper bound on the price of a digital double touch barrier option is the cost of the cheapest portfolio $\tilde{H}^i$. This is fairly easy to see: clearly the price is an upper bound on the price of the option, since $\tilde{H}^i$ superhedges, and under $\mathbb{P}^*$ this upper bound is attained. Indeed, by assumption on $\tilde{H}^i$, in the market model associated with $\mathbb{P}^*$ we have $\mathcal{P}\tilde{H}^i = \mathbb{E}^*\tilde{H}^i = \mathbb{E}^*\overline{H}^i = \mathcal{P}\overline{H}^i$. Consequently, by Theorem 2.3.2, the price of the traded portfolio $\tilde{H}^i$ and the price of the digital double touch barrier option are equal under the market model $\mathbb{P}^*$. Note that in (ii) above it is in fact enough to have $\tilde{H}^i(S_T) = \overline{H}^i(S_T)$ just for the $\overline{H}^i$ which attains equality in (31).

We now wish to understand the possible models that might correspond to a given set of call prices $\{C(x_i); 0 \leq i \leq N\}$. Simple arbitrage constraints (see e.g. [DH06, Theorem 3.1][5]) require that the call prices at other strikes (if traded) imply that the function $C(K)$ is convex and decreasing. This allows us to deduce that, for K such that $x_j < K < x_{j+1}$ for some j, we must have

$$C(K) \leq C(x_j)\frac{x_{j+1} - K}{x_{j+1} - x_j} + C(x_{j+1})\frac{K - x_j}{x_{j+1} - x_j} \tag{44}$$

$$C(K) \geq C(x_j) + \frac{C(x_j) - C(x_{j-1})}{x_j - x_{j-1}}(K - x_j) \tag{45}$$

$$C(K) \geq C(x_{j+1}) - \frac{C(x_{j+2}) - C(x_{j+1})}{x_{j+2} - x_{j+1}}(x_{j+1} - K) \tag{46}$$

These inequalities therefore provide upper and lower bounds on the call price at strike K, and it can be seen that the upper bound and lower bound are tight by choosing suitable models: in the upper bound,

[5] We suppose also that our call prices do not exhibit what is termed here as weak arbitrage

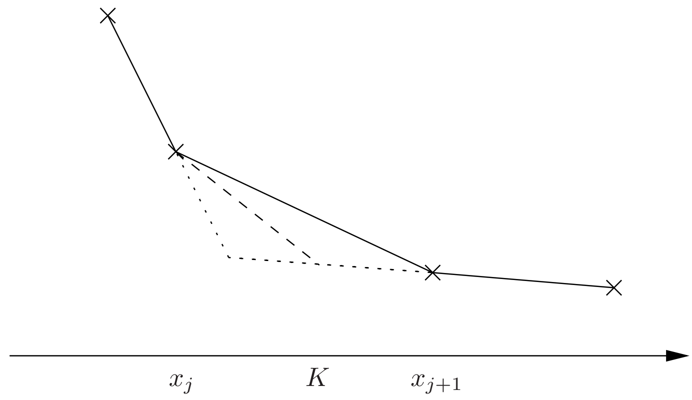

Figure 6: Possible call price surfaces as a function of the strike. The crosses indicate the prices of traded calls, the solid line corresponds to the upper bound, which corresponds to placing all possible mass at traded strikes. The lower bound is indicated by the dotted line, and the dashed line indicates the surface we will choose when we wish to minimise the call price at K. In this case, we note that there will be mass at K and x_j, but not at x_{j+1}. There is a second case where K is below the kink in the dotted line, when the resulting surface would place mass at K and x_{j+1}, but not at x_j.

the corresponding model places all mass of the law of S_T at the strikes x_i; in the lower bounds, the larger of the two possible terms can be attained with a law that places mass at K, and at other x_i's except, in (45), at x_j, and in (46), at x_{j+1}. Moreover, provided that there are sufficiently many traded calls between two strikes K, K', we can (for example) choose a law that attains the maximum possible call price at K, and the minimum possible price at K'. We will assume that this property holds for all 'relevant' points in the sequel.

Consider firstly the case where we wish to superhedge the double touch option. We will consider the model $\mathbb{P}^*$ which corresponds (through Theorem 2.3.2) to the call prices obtained by linearly interpolating the prices at $x_0, \ldots, x_n$ — in particular, $C(K)$ is the maximal possible value it may take under the assumption of no arbitrage. The key idea is to consider the portfolio depicted in Figure 7, where the smaller payoff is the optimal payoff under the assumption that all strikes may be traded, and the upper payoff is one that may be constructed using only strikes that are actually traded. Then although the upper payoff is strictly larger between x_j and x_{j+1} say, where $x_j < K_1 < x_{j+1}$, the points at which this occurs are not points at which the asset will finish, since under $\mathbb{P}^*$, the law of S_T is supported only by the points x_i. Consequently, both the upper and lower payoff have almost surely the same payoff under $\mathbb{P}^*$, and therefore the same expectation and price — in particular, the upper portfolio is a superhedging portfolio, which has the same price as the lower portfolio, which is the smallest upper bound for the double touch under $\mathbb{P}^*$, and this payoff is in turn equal to the payoff of the double touch under $\mathbb{P}^*$. Since we have a superhedge for all models, and a model under which the superhedge is a hedge, we must have the least upper bound. We note additionally that the same choice of $C(K)$ and the same $\mathbb{P}^*$ will work in a similar manner for the other hedges $\overline{H}^I, \overline{H}^{II}$ and $\overline{H}^{IV}$.

Consider now the lower bound. To keep things simple. we begin by altering slightly the problem: rather than the payoff $\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}}$, we consider a subhedge of the option with payoff $\mathbf{1}_{\overline{S}_T > \overline{b}, \underline{S}_T < \underline{b}}$. Then (14) still holds[6] with the new term $\mathbf{1}_{\overline{S}_T > \overline{b}, \underline{S}_T < \underline{b}}$ on the left-hand side provided we modify the digital options on the right-hand side to $\mathbf{1}_{\{S_T \geq \underline{b}\}}$ and $\mathbf{1}_{\{S_T > \overline{b}\}}$. So we may still consider the optimal portfolio (in all three cases) as being short a collection of calls at strikes K_1, K_2 and K_3, long calls at $\underline{b}$ and $\overline{b}$, and holding digital options at each of these points. Intuitively, we should look for a model which will maximise the cost of the calls at K_1, K_2 and K_3, and minimise the cost at $\overline{b}$ and $\underline{b}$, as well as maximising the cost of a digital call at $\underline{b}$ and minimising the cost of the digital call at $\overline{b}$. The former conditions correspond to choosing the call prices which give the upper bound (44) — so we choose the call price which linearly interpolates $C(x_i)$ except when $x_i \leq \underline{b} \leq x_{i+1}$, and $x_i \leq \overline{b} \leq x_{i+1}$. In the latter cases, we wish to minimise the call price, so we choose the prices corresponding to the appropriate lower bound (45) or (46), which

[6] Technically, for (14) to still hold, we actually need to modify the hitting times so that we consider the entrance times of e.g. $(0, \underline{b})$ rather than the hitting time of $\underline{b}$. In practice, this will not be crucial.

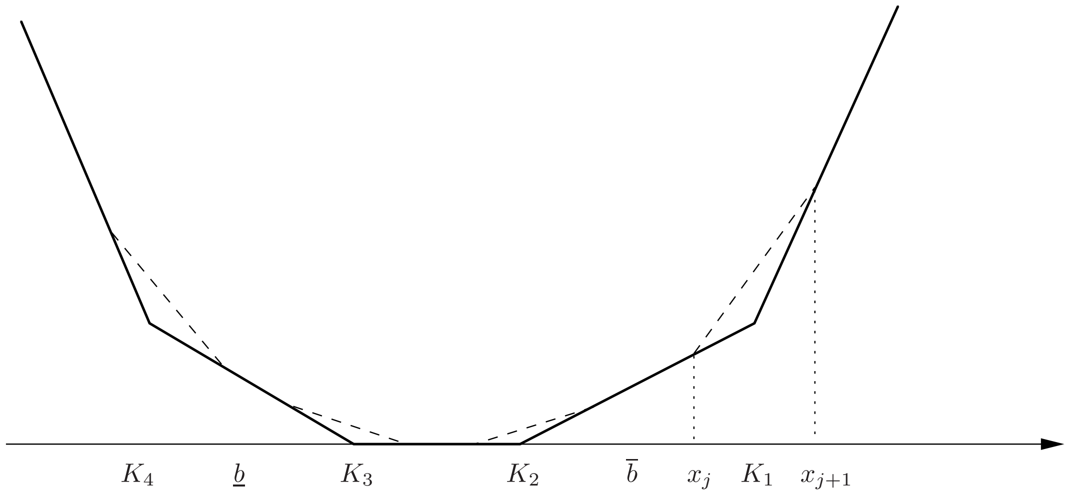

Figure 7: An optimal superhedge $\overline{H}^{III}$ in the case where only finitely many strikes are traded. The lower (solid) payoff denotes the optimal construction under the chosen extension of call prices to all strikes, and the upper (dashed) payoff denotes the payoff actually constructed. Note that, for example, x_j is the largest traded strike below K_1, and x_{j+1} is the smallest traded strike greater than K_1.

have a kink at $\underline{b}$ and at (exactly) one of its two adjacent traded strikes and likewise for $\overline{b}$.

We note that the prices of the digital calls (which are either the left gradient or the right gradient of the call prices) will also now be optimised when they trade in exactly the forms specified above (that is, the digital call at $\underline{b}$ only pays out if the asset is greater than or equal to $\underline{b}$, while the call at $\overline{b}$ will pay out if the asset is strictly larger than $\overline{b}$ at maturity).

The above procedure specifies uniquely a complete set of call prices $C(K)$, which match the market input and which are our candidate for the smallest lower bound among the possible models. Then we note that a construction similar to that given in Figure 8 will work — the main difference to the superhedge case is that, at the discontinuity, there are two possible cases that need to be considered, and the optimal subhedge will depend on behaviour of $C(K)$ at the strikes adjacent to $\underline{b}$ and $\overline{b}$. More precisely, the portfolio given by the dotted line in Figure 8 corresponds to the case when $C(K)$ has no kinks at the traded strikes to the immediate right of both barriers (i.e. S_T has no atoms at these strikes). The other three possibilities are straightforward modifications. The argument then proceeds as above: in the model given by Theorem 2.3.4 subhedge $\underline{H}_i$ achieves equality in (38) and the portfolio constructed in Figure 8 is almost-surely equal[7] to $\underline{H}_i$, so that they must have the same price. The resulting subhedge, constructed using only calls and puts with traded strikes, is therefore a hedge under the chosen model, and therefore the optimal lower bound. Note carefully though the behaviour at the barriers: the optimal model has atoms at the barriers which will be embedded by mass which has already hit the other barrier. With the modified payoff, $\mathbf{1}_{\overline{S}_T>\overline{b},\underline{S}_T<\underline{b}}$, this will still give equality in the subhedge. One could now consider the option $\mathbf{1}_{\overline{S}_T\geq\overline{b},\underline{S}_T\leq\underline{b}}$ by approximating by an option of the form $\mathbf{1}_{\overline{S}_T>\overline{b}-\varepsilon,\underline{S}_T<\underline{b}+\varepsilon}$, which would (in the optimal embedding) place atoms 'just inside the barriers' — it is this behaviour of the 'optimal' construction that lead us to consider the modified payoff in this case.

3.2 Jumps in the underlying

Throughout the paper, we have assumed that $(S_t)_{t\geq0}$ has continuous paths. In fact we can relax this assumption considerably. First of all it is relatively simple to see that if we only assume that barriers $\underline{b},\overline{b}$ are crossed in a continuous manner then all of our results remain true. Secondly, if we make no continuity assumptions[8] then all our superhedges still work – jumping over the barrier only makes the appropriate forward transaction more profitable. In contrast, our subhedges do not work and lower bounds on prices

[7] Assuming that the optimal K_3 chosen is separated from $\underline{b}$ and $\overline{b}$ by a traded call price
[8] We always suppose the processes have càdlàg paths.

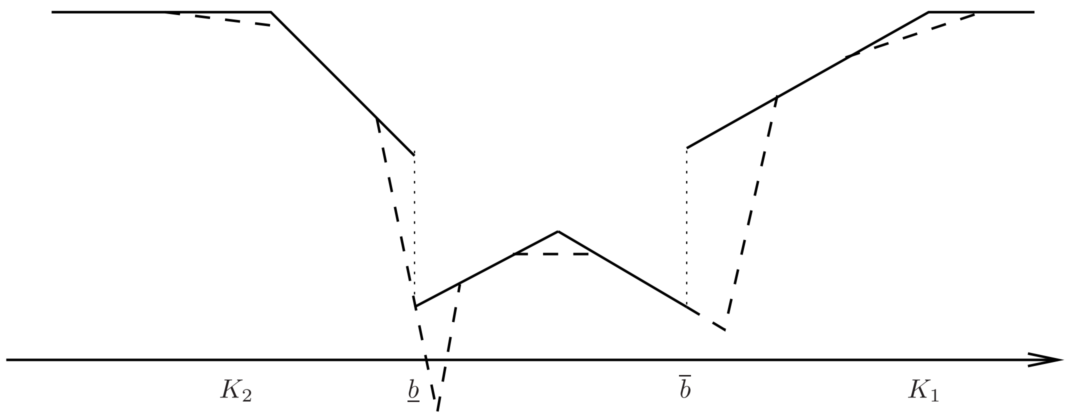

Figure 8: An optimal subhedge in the case where only finitely many strikes are traded. The upper (solid) payoff denotes the optimal construction under the chosen extension of call prices to all strikes, and the lower (dashed) payoff denotes the payoff actually constructed. Observe that there are two possible constructions at the discontinuity — the choice of which type of construction is optimal will depend on whether the optimal lower bound on the call price at $\underline{b}$ or $\overline{b}$ corresponds to the bound (45) or (46)

are trivially zero and are attained in the model where $S_t = S_0$ for all $t < T$ and then it jumps to the final position S_T.

3.3 Hedging comparisons

In practice, one would expect that the prices we derive as upper and lower bounds using the techniques of this paper will be rather wide, and well outside typical bid/ask spreads. Consequently, the use of these techniques as a method for pricing is unlikely to be successful. However the technique also provides superhedging and subhedging strategies that may be helpful. Consider a trader who has sold a double barrier option (at a price determined by some model perhaps), and who wishes to hedge the resulting risk. In a Black-Scholes world, the trader could remove the risk from his position by delta-hedging the short position. However, there are a number of practical considerations that would interfere with such an approach:

Discrete Hedging: A notable source of errors in the hedge will be the fact that the hedging portfolio cannot be continuously adjusted; rather the delta of the position might be adjusted on a periodic basis, resulting in an inexact hedge of the position. While including a gamma hedge could improve this, the hedge will never be perfect. In addition, there is an organisational cost (and risk) to setting up such a hedging operation that might be important.

Transaction Costs: A second consideration is that each trade will incur a certain level of transaction costs. These might be tiny for delta-hedging but not so anymore for vega-hedging. To minimise the total transaction costs, the trader would like to be able to trade as infrequently as possible. Of course, this means that there will be a necessary trade-off with the discretisation errors incurred above.

Model Risk The final concern for the trader would be: am I hedging with the correct model? Using an incorrect model will of course result in systematic hedging errors due to e.g. incorrectly estimated volatility, but could also lead to large losses should the model fail to incorporate structural effects such as jumps. A delta-hedge would typically be improved with a vega-hedge which in turn raises the issue of transaction costs as mentioned above.

It would appear that the constructions developed in Section 2 may be able to address some of these issues: there is no need for regular recalculation of the Greeks of the position, although the breaching of the barrier still requires monitoring; since there are only a small number of transactions, it seems likely that the transaction costs may be reduced; our hedge has been derived using model-free techniques, so that we will still be hedged even if the market does not behave according to our initial model, and behaviour such as jumps (at least for the upper bound of the double touch) will not affect this.

A further consideration that is likely to be of importance to a hedger is the likely distribution of the returns. Under the hedging strategy suggested above, so that the trader has sold the option using the 'correct' price (plus a small profit), and set up the superhedging strategy suggested at a higher price, on average the trader will come out even, as he will if he delta hedges. His comparison between the approaches would then come down to the respective risk involved in the different hedges. For a delta/vega hedge, this is typically symmetric about zero, however the superhedge will be very asymmetric as it is bounded below. One might expect that a large number of paths will hedge 'correctly,' resulting in a loss close to the difference between the price the option was sold at, and the price that the hedge was bought at, and the remaining paths will do better, when the superhedge strictly dominates. If the trader is particularly worried about the possible tail of his trading losses as a measure of risk, this strict cut-off could be very advantageous. The delta/vega hedge, on the other hand, has the appeal of having a lower variance of hedging errors.

Of course a variety of such strategies (typically known as static, semi-static or robust) have been suggested in the literature, under a variety of more or less restrictive assumptions on the price process, and mostly for single barrier options, and variants such as knock-out calls. We have already mentioned the paper [BHR01] which makes very limited restrictions on the underlying price process. More restrictive is the work of [BC94], and subsequent papers [CC97, CEG98]. Here the authors assume that the volatility satisfies a symmetry assumption, and as a consequence, one can for example hedge a knock-out call with the barrier above the strike by holding the vanilla call, and being short a call at a certain strike above the barrier. By the assumption on the volatility, whenever the underlying hits the barrier, both calls have the same value, and the position may be closed out for zero value. A related technique is due to [DEK95], and followed up by [AAE02] and [Fin03]. The idea here is to use other traded options to make the value of the hedging portfolio equal to zero along the barrier when liquidated. In the simplest form, a portfolio of calls above the barrier at different strikes and/or maturities are purchased so that the portfolio value at selected times before maturity is zero. Extensions allow this idea to be used for stochastic volatility, and even to cover jumps, at the expense of needing possibly a very large portfolio of options. More recently, work of [NP06] unifies both these approaches, and allows a fairly general set of asset dynamics, as does [GM07], where the authors find an optimal portfolio by setting up an optimisation problem. Note however that all these strategies assume a known model for the underlying, and also that the hedging assets will be liquid enough for the portfolio to be liquidated at the price specified under the model. In addition, since some of the hedging portfolios can involve a large number of options, it is not clear that the static hedges here will be efficient at resolving the issues of transaction costs (and operational simplicity) or model risk. A numerical investigation of the performance of a range of these options in practice has been conducted in [EFNS06], and similar investigations for a different class of static hedges appear in [DST01, Tom97].

There are also more classical, theoretical approaches to the problem of hedging where the problem is considered in an incomplete market (without which, of course, a perfect hedge would be possible). In this situation (for example [FS90]) one wants to solve an optimal control problem where the aim is to minimise the 'risk' of the hedging error, where 'risk' is interpreted suitably (perhaps with regards to a utility function or a risk measure). One can further modify the approach to restrict the class of trading strategies available (e.g. [cJP04]). More recently, in a combination of the static and dynamic approaches, [İJS08] have considered the problem of risk minimisation over an initial static portfolio and a dynamic trading strategy in the underlying.

The most notable difference between the studies described above, and the ideas of the previous sections is that we make very little modelling assumptions on the underlying, whereas most of the approaches listed require a single model to be specified, with respect to which the results will then be optimal. In particular, these techniques are unable to say anything about hedging losses should the assumed model not actually be correct.

Of course, the criteria under which we have constructed our hedges — that they are the smallest model free superhedging strategy, or the greatest model-free subhedging strategy — do not necessarily mean that the behaviour of the hedges in 'normal' circumstances will be particularly suitable: we would expect that the hedge would perform best in extreme market conditions, however in order for it to be suitable as a hedge against model risk, one would also want the performance of the hedge to generally be reasonable. To see how this strategy compares, we will now consider some Monte Carlo based comparisons with the standard delta/vega-hedging techniques. We only look at the double touch options treated in Section 2. The comparisons will take the following form:

(i) We choose the Heston model for the 'true' underlying asset, and compute the time-0 call prices under this model at a range of strike prices, and the time-0 price of a double barrier option;

(ii) We compute the optimal super- and sub- hedges for the digital double touch barrier option based on the observed call prices, and suppose that the hedger purchases these portfolios using the cash received from the buyer (and borrowing/investing the difference between the portfolios);

(iii) For comparison purposes, we also hedge the option using a suitable delta/vega hedge with daily updating (for comparison purposes, the hitting of the barrier in both cases is also monitored on a daily basis).

In the numerical examples, we assume that the underlying process is the Heston stochastic volatility model ([Hes93])

$$\begin{cases} dS_t &= \sqrt{v_t} S_t dW_t^1, & S_0 = S_0,\ v_0 = \sigma_0 \\ dv_t &= \kappa(\theta - v_t)dt + \xi \sqrt{v_t} dW_t^2, & d\langle W^1, W^2 \rangle_t = \rho dt, \end{cases} \tag{47}$$

with parameters

$$S_0 = 100,\ \sigma_0 = 0.5,\ \kappa = 0.6,\ \theta = 1,\ \xi = 1.3 \text{ and } \rho = 0.15\ . \tag{48}$$

Transactions in S_t carry a 0.5% transaction cost and buying or selling call/put options carries a 1% transaction cost. The delta/vega hedge is constructed using the Black-Scholes delta of the option, but using the at-the-money implied volatility assuming that the call prices are correct (i.e. they follow the Heston model). While not perfect as a hedge, empirical evidence [DFW98] or [EFS06] suggests that the hedge is reasonable even without the vega component, although it is also the case that more sophisticated methods should result in an improvement of this benchmark.

We consider a short and a long position in a digital double touch barrier option with payoff $\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}}$ for $\overline{b} = 117$ and $\underline{b} = 83$ and compare hedging performance of our quasi-static super- and sub- hedges and the standard delta/vega hedges running 40000 Monte Carlo simulations[9].

The cumulative distributions of hedging errors are given in the upper graphs in Figure 9. Quasi-static super- and sub- hedges introduced in this paper also incur large losses – this is due to barrier crossing being monitored daily. However, a closer inspection reveals that this comparatively large loses are less frequent then when using delta/vega hedging strategy. Indeed, in Table 1 we show that an agent with an exponential utility $U(x) = 1 - \exp(-x)$ would prefer the error distribution of our hedges to that of the delta/vega hedge. We could also ask what happens if we allow our hedges to monitor the barrier crossings exactly. The corresponding cumulative distributions of hedging errors are given in the lower graphs in Figure 9 – we clearly see how the losses are bounded below. It is interesting however to note that in terms of utility of hedging errors (cf. Table 1) this doesn't really change the performance of our hedges - it eliminates some extremely rare larger losses but at the cost of frequent small profits (cf. Section 3.2).

Finally, we investigate how the performance of our hedges compares if we vary the barrier levels. Appropriate cumulative distributions (for 20000 MC runs) are reported in Figures 11-15 in Section 5 and exponential utilities are reported in Table 1. Our superhedge consistently outperforms delta/vega hedging. Our subhedge on the other hand performs worse as the barriers get closer together. In fact, for barriers at 105 & 95 or 103 & 97, an exponential utility agent having a long position in a double touch barrier option would prefer to use the standard delta/vega hedging to our sub-replicating strategy. This would appear to be due to the fact that typically the option knocks-in quickly and delta/vega-hedging then carries smaller transactions costs.

[9] The resulting hedging errors were mean-adjusted as in Tompkins [Tom97] for consistency. The adjustments are of order 0.001 and have no qualitative influence on our results.

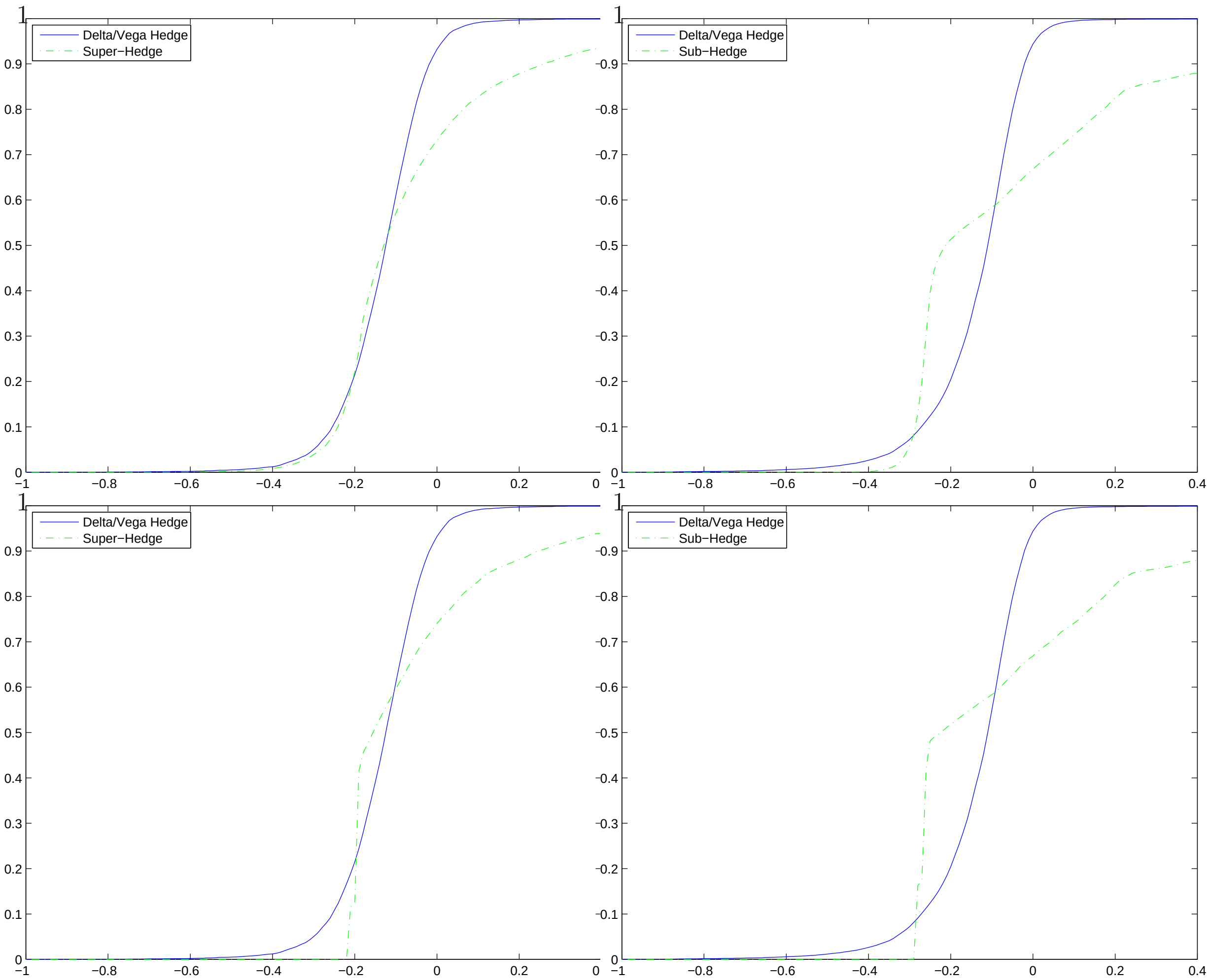

Figure 9: Cumulative distributions of hedging errors under different scenarios of a short position (*left*) and a long position (*right*) in a double touch option with barriers at 117 and 83 under Heston model (47)–(48). In the lower graphs exact monitoring of barrier crossings was allowed.

Barriers	Position	Delta/Vega	Model-free	Model-free (barriers monitored exactly)
130–70	short	−0.076	**−0.026**	−0.025
130–70	long	−0.078	**−0.027**	−0.027
120–80	short	−0.125	**−0.047**	−0.045
120–80	long	−0.127	**−0.069**	−0.068
117–83	short	−0.137	**−0.065**	−0.062
117–83	long	−0.139	**−0.078**	−0.077
115–85	short	−0.144	**−0.085**	−0.082
115–85	long	−0.147	**−0.086**	−0.085
110–90	short	−0.139	**−0.063**	−0.061
110–90	long	−0.144	**−0.115**	−0.113
105–95	short	−0.096	**−0.048**	−0.047
105–95	long	**−0.102**	−0.171	−0.167
103–97	short	−0.070	**−0.039**	−0.038
103–97	long	**−0.075**	−0.199	−0.194
120–95	short	−0.127	**−0.059**	−0.057
120–95	long	−0.130	**−0.099**	−0.098
105–80	short	−0.155	**−0.076**	−0.075
105–80	long	−0.161	**−0.124**	−0.121

Table 1: Comparison of exponential utilities of hedging errors of positions in a double touch options under Heston model (47)–(48) resulting from delta/vega hedging and our model-free super- or sub-hedging strategies. In each case, the preferred hedge is highlighted. The last column reports the change in utility when our strategies are allowed to monitor exactly the moments of barrier crossings.

Naturally when the barriers vary the types of super- and sub- hedges which we use change. Figure 10 shows which types are optimal depending on the values of the barriers. This provides an illustrations of the intuitive labels we gave to each case in Section 2. In this implementation we assumed one thousands strikes between 0 and 500 are available and digital calls are not traded. This results in non-smoothness of the borders between the cases.

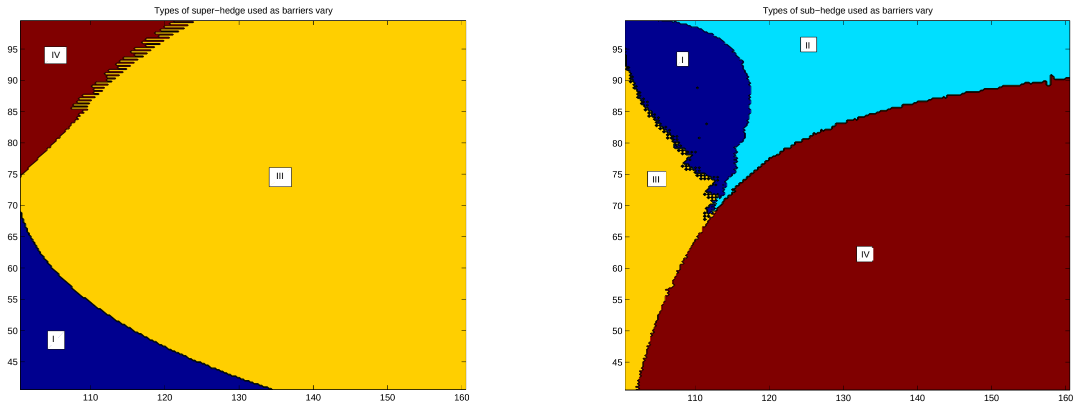

Figure 10: Optimal types of superhedges $\overline{H}$ (*left*) and subhedges $\underline{H}$ (*right*) as barriers vary under the Heston model (47)–(48). $\underline{H}^{IV} \equiv 0$ is the trivial subhedge. Superhedge $\overline{H}^{II}$ is not visible as it only appears for upper barrier levels above 325.

The results presented in this section clearly show that the hedges we advocate are in many circumstances an improvement on the classical hedges. Naturally, there is a large literature (e.g. [CK96]) on more sophisticated techniques that might offer a considerable improvement over the classical hedge we have implemented and it is possible that such improvements would reverse the relative performance of the hedges. However, we hope that the numerical evidence does at least convince the reader that the quasi-static hedges are competitive with dynamic hedging, and that their differing nature means that they might well prove a more suitable approach in situations where market conditions are dramatically different to the idealised Black-Scholes world — e.g. large transaction costs, illiquidity or large jumps. There also seems scope for a more sophisticated approach based on the quasi-static hedges, but allowing for some model-based trading: for example, a hybrid of a quasi-static portfolio and some dynamic trading could be used to reduce some of the over-hedge in the simple quasi-static hedge. Finally, in our simulations we did not really incorporate model misspecification risk. We would expect that if a trader believes in a model which is significantly different from the real world model then our model-free hedging strategies would outperform hedging 'using the wrong model.'

4 Proofs

Let $(B_t)_{t\geq 0}$ be a standard real valued Brownian motion starting from B_0. We recall that for any probability measure ν on $\mathbb{R}_+$ with $\nu_B(\mathbb{R}_+) = B_0$ we can find a stopping time τ such that $B_\tau \sim \nu$ and $(B_{t\wedge\tau})$ is a uniformly integrable martingale. Such stopping is simply a solution to the Skorokhod embedding problem and number of different explicit solutions are known, see Obłój [Obł04] for an overview of the domain. Note also that when $\nu([a, b]) = 1$ then $(B_{t\wedge\tau})$ is a uniformly integrable martingale if and only if $B_t \in [a, b]$, $t \leq \tau$, a.s. In the sequel when speaking about embedding a measure we implicitly mean embedding it in a UI manner in (B_t).
Recall that if $B_0 = S_0$ and τ is an embedding of μ then $S_t := B_{\tau\wedge\frac{t}{T-t}}$, $t \leq T$, is a market model which matches the market input (3). In what follows we will be constructing embeddings τ of μ such that the associated market model attains equality in our super- or sub- hedging inequalities. Stopping times τ will often be compositions of other stopping times embedding (rescaled) restrictions of μ or some other intermediary measures. Unless specified otherwise, the choice of particular intermediary stopping times has no importance and we do not specify it – one's favorite solution to the Skorokhod embedding problem can be used.

Proof of Theorem 2.3.2. We start with some preliminary lemmas and then prove Theorem 2.3.2. In the body of the proof, cases $\boxed{\text{I}}$ to $\boxed{\text{IV}}$ refer to the cases stated in Theorem 2.3.2. We note that, by considering $z_0 = \gamma_+(w_0)$, $\boxed{\text{III}}$ is equivalent to:

$$\text{There exists } z_0 \geq \overline{b} \text{ such that } \gamma_+(\gamma_-(z_0)) = z_0 \text{ and } \rho_+(z_0) \geq \rho_-(\gamma_-(z_0)).$$

We recall, without proof, straightforward properties of the barycentre function (32) which will be useful in the sequel.

Lemma 4.0.1. *The barycentre function defined in* (32) *satisfies*

- $\mu_B(\Gamma) \geq a$, $\Gamma_1 \subset (0, a) \Longrightarrow \mu_B(\Gamma \setminus \Gamma_1) \geq a$,
- $a \leq \mu_B(\Gamma) \leq \mu_B(\Gamma_1) \leq b$ *and* $\mu(\Gamma \cap \Gamma_1) = 0 \Longrightarrow a \leq \mu_B(\Gamma \cup \Gamma_1) \leq b$.

We separate the proof into 2 steps now. In the first step we prove that exactly one of $\boxed{\text{I}}$-$\boxed{\text{IV}}$ holds. In the second step we construct the appropriate embeddings and market models which achieve the upper bounds on the prices.

Step 1:
This step is divided into 4 cases. We start with technical lemmas which are proved after the cases are considered.

Lemma 4.0.2. *If $\rho_+(\gamma_+(w_0)) < \rho_-(w_0)$ for some w_0, then $\rho_+(\gamma_+(w)) < \rho_-(w)$ for all $w \leq w_0$. Similarly, if $\rho_-(\gamma_-(z_0)) > \rho_+(z_0)$ for some z_0, then $\rho_-(\gamma_-(z)) > \rho_+(z)$ for all $z \geq z_0$*

Lemma 4.0.3. *If $\overline{b} \geq \rho_-(0)$ and $\underline{b} \leq \rho_+(\infty)$ then at least of one of the functions $\gamma_\pm$ is bounded on its domain. In particular at most one of $\boxed{I}$ and $\boxed{II}$ may be true.*

Lemma 4.0.4. $\boxed{IV}$ *implies not* $\boxed{III}$. $\boxed{III}$ *implies not (*$\boxed{I}$ *or* $\boxed{II}$*).*

Case (a): $\overline{b} \geq \rho_-(0), \underline{b} \leq \rho_+(\infty)$

We note first of all that the case $\boxed{\text{IV}}$ is not possible, and that the second half of the conditions for $\boxed{\text{I}}$ and $\boxed{\text{II}}$ are trivially true. Suppose neither $\boxed{\text{I}}$ or $\boxed{\text{II}}$ hold. If we have $\gamma_-(\overline{b}) = 0$ then $\boxed{\text{III}}$ holds with $w_0 = 0$ and if $\gamma_+(\underline{b}) = \infty$ then $\boxed{\text{III}}$ holds with $w_0 = \underline{b}$. We may thus assume that $\gamma_+(\cdot)$ is bounded above, and $\gamma_-(\cdot)$ is bounded away from zero, which in turn implies:

$$\gamma_-(\gamma_+(\underline{b})) < \underline{b} \text{ and } \gamma_-(\gamma_+(0)) > 0.$$

The function $\gamma_-(\gamma_+(\cdot))$ is continuous and increasing on $(0, \underline{b}]$ and thus we must have w_0 such that $\gamma_-(\gamma_+(w_0)) = w_0$. Finally, suppose that for such a w_0 we in fact have $\rho_-(w_0) > \rho_+(\gamma_+(w_0))$, then Lemma 4.0.2 implies $\rho_-(0) > \rho_+(\gamma_+(0)) = \overline{b}$, contradicting our assumptions. That only one of the cases $\boxed{\text{I}}$–$\boxed{\text{III}}$ holds now follows from Lemmas 4.0.3 and 4.0.4.

Case (b): $\overline{b} \geq \rho_-(0), \underline{b} > \rho_+(\infty)$

It follows that neither $\boxed{\text{II}}$ or $\boxed{\text{IV}}$ are possible. Observe that Lemma 4.0.2 implies that $\rho_-(w) \leq \rho_+(\gamma_+(w))$ for all $w \leq \underline{b}$ — if this were not true, then $\overline{b} = \rho_+(\gamma_+(0)) < \rho_-(0)$. Suppose further that $\boxed{\text{I}}$ does not hold. If $\gamma_-(\overline{b}) = 0$ then $\boxed{\text{III}}$ holds with $w_0 = 0$.
So assume instead that $\gamma_-(\cdot)$ is bounded away from zero, and therefore that $w < \gamma_-(\gamma_+(w))$ for w close to zero. If we show also that $\gamma_-(\gamma_+(\rho_+(\infty))) \leq \rho_+(\infty)$ then by continuity of $\gamma_-(\gamma_+(\cdot))$ there exists a suitable w_0 for which $\boxed{\text{III}}$ holds. Let $\Gamma_1 = (\rho_+(\infty), \infty)$ and $\Gamma_2 = (\rho_+(\gamma_+(\rho_+(\infty))), \gamma_+(\rho_+(\infty)))$. We have by definition $\mu_B(\Gamma_1) = \overline{b} = \mu_B(\Gamma_2)$ so that $\mu_B(\Gamma_1 \setminus \Gamma_2) = \overline{b}$, since $\Gamma_2 \subset \Gamma_1$. Let $\Gamma = \big(\rho_+(\infty), \rho_-(\rho_+(\infty))\big) \cup \big(\gamma_+(\rho_+(\infty)), \infty\big)$ and note that $\mu_B(\Gamma) = \overline{b}$ is equivalent to $\gamma_-(\gamma_+(\rho_+(\infty))) = \rho_+(\infty)$. Noting that $\rho_-(w) \leq \rho_+(\gamma_+(w))$ for all $w \leq \underline{b}$ implies $\rho_-(\rho_+(\infty)) \leq \rho_+(\gamma_+(\rho_+(\infty)))$, and using Lemma 4.0.1 have

$$\mu_B(\Gamma) = \mu_B\Big(\big(\Gamma_1 \setminus \Gamma_2\big) \setminus \big(\rho_-(\rho_+(\infty)), \rho_+(\gamma_+(\rho_+(\infty)))\big)\Big) \geq \overline{b},$$

which implies that $\gamma_-(\gamma_+(\rho_+(\infty))) \leq \rho_+(\infty)$. As previously, it remains to note that Lemma 4.0.4 implies exclusivity of $\boxed{\text{III}}$ and $\boxed{\text{I}}$.

Case (c): $\overline{b} < \rho_-(0), \underline{b} \leq \rho_+(\infty)$

This case is essentially identical to Case (b) above.

Case (d): $\overline{b} < \rho_-(0), \underline{b} > \rho_+(\infty)$

Note that we now cannot have either of $\boxed{\text{I}}$ or $\boxed{\text{II}}$. Suppose further that $\boxed{\text{IV}}$ does not hold — or rather, the weaker:

$$\begin{aligned} \rho_+(\rho_-(0)) &> \underline{b}, \\ \rho_-(\rho_+(\infty)) &< \overline{b}. \end{aligned}$$

Let $\Gamma_w = (w, \rho_-(w)) \cup (\overline{b}, \infty)$ and observe that $\mu_B(\Gamma_w)$ decreases as w decreases, provided $\rho_-(w) < \overline{b}$. We have

$$\mu_B(\Gamma_{\rho_+(\infty)}) \geq \mu_B((\rho_+(\infty), \infty)) = \overline{b}.$$

Our assumption $\rho_-(\rho_+(\infty)) < \overline{b} < \rho_-(0)$ implies that $\rho_-^{-1}(\overline{b}) \in (0, \rho_+(\infty))$ so that $\Gamma_{\rho_-^{-1}(\overline{b})} \supsetneq (\rho_+(\infty), \infty)$ and in consequence $\mu_B(\Gamma_{\rho_-^{-1}(\overline{b})}) < \overline{b}$. Using continuity of $w \to \mu_B(\Gamma_w)$ we conclude that there exists a $w_1 \in (\rho_-^{-1}(\overline{b}), \rho_+(\infty)]$ with $\mu_B(\Gamma_{w_1}) = \overline{b}$, or equivalently $\gamma_-(\overline{b}) = w_1$. A symmetric argument implies that $\gamma_+(\underline{b}) > 0$. We conclude, as in Case (a), that there exists a w_0 such that $\gamma_-(\gamma_+(w_0)) = w_0$.

It remains to show that if $\boxed{\text{IV}}$ does not hold, and the first half of the condition for $\boxed{\text{III}}$ holds, then so too does the second condition. Suppose w_0 is a point satisfying $\gamma_-(\gamma_+(w_0)) = w_0$ and suppose for a contradiction that $\rho_-(w_0) > \rho_+(\gamma_+(w_0))$. Since the sets $(\rho_+(\gamma_+(w_0)), \gamma_+(w_0))$ and $(w_0, \rho_-(w_0)) \cup (\gamma_+(w_0), \infty)$ are both centred at $\overline{b}$, and overlap, it follows that $\mu_B([w_0, \infty)) > \overline{b}$ and in consequence $\rho_+(\infty) < w_0$. Symmetric arguments imply that $\rho_-(0) > \gamma_+(w_0)$. Applying $\rho_-(\cdot), \rho_+(\cdot)$ to these inequalities, we further deduce that

$$\begin{aligned}\rho_-(w_0) &< \rho_-(\rho_+(\infty))\\ \rho_+(\rho_-(0)) &< \rho_+(\gamma_+(w_0))\end{aligned}$$

which, together with the assumption that $\rho_-(w_0) > \rho_+(\gamma_+(w_0))$, implies:

$$\rho_+(\rho_-(0)) < \rho_+(\gamma_+(w_0)) < \rho_-(w_0) < \rho_-(\rho_+(\infty)),$$

contradicting $\boxed{\text{IV}}$ not holding. □

Proof of Lemma 4.0.2. Consider $w < w_0$ with $\rho_+(\gamma_+(w_0)) < \rho_-(w_0)$. The latter implies $\mu_B\big((w_0, \rho_+(\gamma_+(w_0)))\big) < \underline{b}$, so that $\mu_B\big((0, \gamma_+(w_0))\big) < \underline{b}$. Suppose now that $\rho_+(\gamma_+(w)) \geq \rho_-(w)$. As ρ_- is decreasing and γ_+ is increasing we have $\rho_-(w_0) < \rho_-(w) \leq \rho_+(\gamma_+(w))$ and $\overline{b} < \gamma_+(w) < \gamma_+(w_0)$. We then have

$$\begin{aligned}\underline{b} &= \mu_B\big((0, \rho_-(w)) \cup (\rho_+(\gamma_+(w)), \gamma_+(w))\big)\\ &= \mu_B\Big((0, \gamma_+(w_0)) \setminus \big[\big(\rho_-(w_0), \rho_+(\gamma_+(\omega))\big) \cup \big((\gamma_+(w), \gamma_+(w_0)\big)\big]\Big)\\ &\leq \mu_B\big((0, \gamma_+(w_0))\big) < \underline{b},\end{aligned}$$

which gives the desired contradiction. □

Proof of Lemma 4.0.3. Define $\Gamma = (0, \underline{b}) \cup (\overline{b}, \infty)$ and consider $\mu_B(\Gamma)$. If both $\boxed{\text{I}}$ and $\boxed{\text{II}}$ hold, or more generally if $\gamma_-(z_0) = 0$ and $\gamma_+(w_0) = \infty$ for some $z_0 \geq \overline{b}$, $w_0 \leq \underline{b}$, we have:

$$\mu_B((0, \rho_-(0)) \cup (\overline{b}, \infty)) \geq \mu_B((0, \rho_-(0)) \cup (z_0, \infty)) = \overline{b} \tag{49}$$

and

$$\mu_B((0, \underline{b}) \cup (\rho_+(\infty), \infty)) \leq \mu_B((0, w_0) \cup (\rho_+(\infty), \infty)) = \underline{b}. \tag{50}$$

Now suppose $\mu_B(\Gamma) < \overline{b}$. Then:

$$\mu_B(\Gamma \cup (\underline{b}, \rho_-(0))) < \overline{b}$$

contradicting (49) and similarly, if $\mu_B(\Gamma) > \underline{b}$

$$\mu_B(\Gamma \cup (\rho_+(\infty), \overline{b})) > \underline{b},$$

contradicts (50). □

Proof of Lemma 4.0.4. $\boxed{\text{IV}} \Longrightarrow$ not $\boxed{\text{III}}$

Assume both $\boxed{\text{IV}}$ and $\boxed{\text{III}}$ hold. From the definition of $\gamma_+(w_0)$ and $\rho_-(w_0)$ we have that $\mu_B(\Gamma_+) = \underline{b}$ for $\Gamma_+ = \big(0, \rho_-(w_0)\big) \cup \big(\rho_+(\gamma_+(w_0)), \gamma_+(w_0)\big)$. This implies that $\gamma_+(w_0) \geq \rho_-(0)$ since otherwise

$$\mu_B(\Gamma_+) = \mu_B\Big(\big(0, \rho_-(0)\big) \setminus \big\{\big(\rho_-(w_0), \rho_+(\gamma_+(w_0))\big) \cup \big(\gamma_+(w_0), \rho_-(0)\big)\big\}\Big) > \underline{b},$$

where we also used the assumption $\rho_-(w_0) \leq \rho_+(\gamma_+(w_0))$. Likewise, using $\gamma_-(\gamma_+(w_0)) = w_0$ we see that $w_0 \leq \rho_+(\infty)$. Applying ρ_- to the last inequality and using our assumptions we obtain

$$\rho_+(\rho_-(0)) < \rho_-(\rho_+(\infty)) \leq \rho_-(w_0) \leq \rho_+(\gamma_+(w_0)).$$

In consequence, $\gamma_+(w_0) < \rho_-(0)$ which gives the desired contradiction.

$\boxed{\text{III}} \Longrightarrow$ not ($\boxed{\text{I}}$ or $\boxed{\text{II}}$)

Suppose $\boxed{\text{III}}$ and $\boxed{\text{II}}$ hold together. Let $w_1 < \underline{b}$ be the point given by $\boxed{\text{II}}$ such that $\gamma_+(w_1) = \infty$ and w_0 the point in $\boxed{\text{III}}$ such that $\gamma_-(\gamma_+(w_0)) = w_0$. Naturally, as $\gamma_-(\gamma_+(w_1)) = \gamma_-(\infty) = \underline{b} > w_1$ we have that $w_0 < w_1$. Observe also that $\mu_B((0, \rho_-(w_1)) \cup (\rho_+(\infty), \infty)) = \underline{b}$ and $\mu_B(\mathbb{R}) = S_0 \in (\underline{b}, \overline{b})$ which readily imply $\underline{b} < \rho_-(w_1) < \rho_+(\infty) < \overline{b}$. Let us further denote $r = \min\{\rho_-(w_0), \rho_+(\infty)\}$ and $R = \max\{\rho_-(w_0), \rho_+(\infty)\}$ so that finally, using our assumptions,

$$w_0 < w_1 < \underline{b} < \rho_-(w_1) < r \leq R \leq \rho_+(\gamma_+(w_0)) \leq \overline{b} \leq \gamma_+(w_0). \tag{51}$$

By definition we have

$$\int_{\rho_+(\infty)}^{\infty} (\overline{b}-u)\mu(\mathrm{d}u) = 0 = \int_{w_0}^{\rho_-(w_0)} (\overline{b}-u)\mu(\mathrm{d}u) + \int_{\rho_+(\gamma_+(w_0))}^{\infty} (\overline{b}-u)\mu(\mathrm{d}u).$$

Subtracting these two quantities we arrive at

$$\int_{R}^{\rho_+(\gamma_+(w_0))} (\overline{b}-u)\mu(\mathrm{d}u) = \int_{w_0}^{r} (\overline{b}-u)\mu(\mathrm{d}u), \qquad \text{and using (51) we deduce}$$

$$\mu\Big(\big(R,\rho_+(\gamma_+(w_0))\big)\Big) > \mu\Big((w_0,r)\Big). \tag{52}$$

Using the properties of our functions again we have

$$\int_{-\infty}^{w_1} (u-\underline{b})\mu(\mathrm{d}u) + \int_{\rho_+(\infty)}^{\infty} (u-\underline{b})\mu(\mathrm{d}u) = 0 = \int_{-\infty}^{w_0} (u-\underline{b})\mu(\mathrm{d}u) + \int_{\rho_+(\gamma_+(w_0))}^{\gamma_+(w_0)} (u-\underline{b})\mu(\mathrm{d}u),$$

which after subtracting, using $\int_{w_0}^{w_1}(u-\underline{b})\mu(\mathrm{d}u) = -\int_{\rho_-(w_1)}^{\rho_-(w_0)}(u-\underline{b})\mu(\mathrm{d}u)$, yields

$$\int_{\rho_+(\infty)}^{\rho_+(\gamma_+(w_0))} (u-\underline{b})\mu(\mathrm{d}u) - \int_{\rho_-(w_1)}^{\rho_-(w_0)} (u-\underline{b})\mu(\mathrm{d}u) + \int_{\gamma_+(w_0)}^{\infty} (u-\underline{b})\mu(\mathrm{d}u) = 0. \tag{53}$$

The last term in (53) is positive and for the first two terms, using (52), we have

$$\begin{aligned}\int_{R}^{\rho_+(\gamma_+(w_0))} (u-\underline{b})\mu(\mathrm{d}u) \quad &\geq \quad (R-\underline{b})\mu\Big(\big(R,\rho_+(\gamma_+(w_0))\big)\Big)\\ &> \quad (R-\underline{b})\mu\Big((\rho_-(w_1),r)\Big) \geq \int_{\rho_-(w_1)}^{r} (u-\underline{b})\mu(\mathrm{d}u).\end{aligned} \tag{54}$$

This readily implies that the left hand side of (53) is strictly positive leading to the desired contradiction. The case when $\boxed{\text{III}}$ and $\boxed{\text{I}}$ hold together is similar.

Step 2: Construction of relevant embeddings.
Our strategy is now as follows. For each of the four exclusive cases $\boxed{\text{I}}$-$\boxed{\text{IV}}$ we construct a stopping time τ which solves the Skorokhod embedding problem for μ and such that for the price process $S_t := B_{\frac{t}{T-t}\wedge\tau}$ the appropriate superhedge $\overline{H}^{I} - \overline{H}^{IV}$ is in fact a perfect hedge. The stopping time τ will be a composition of stopping times, each of which is a solution to an embedding problem for a (rescaled) restriction of μ to appropriate intervals.

Suppose that $\boxed{\text{II}}$ holds.
This embedding is closely related to the classical Azéma-Yor embedding [AY79] used in the work of Brown, Hobson and Rogers [BHR01] on one-sided barrier options. Let τ_1 be a UI embedding, in $(B_t)_{t\geq 0}$ with $B_0 = S_0$, of

$$\nu^1 = \mu|_{(w_0,\rho_+(\infty))} + p\delta_{\underline{b}}, \quad \text{where } p = \big(1-\mu((w_0,\rho_+(\infty)))\big),$$

which is centred in S_0. Let $\nu^2 = \frac{1}{p}\mu|_{\mathbb{R}_+\setminus(w_0,\rho_+(\infty))}$, which is a probability measure with $\nu^2_B(\mathbb{R}_+) = \underline{b}$, and let τ_2 be the Azéma-Yor embedding (cf. Obłój [Obł04, Sec. 5]) of ν^2, i.e.

$$\tau_2 = \inf\big\{t>0 : \overline{B}_t \geq \nu^2_B([B_t,\infty))\big\},$$

which is a UI embedding of ν^2 when $B_0 = \underline{b}$. Note that $\nu^2_B([x,\infty)) = \mu_B([x,\infty))$ for $x \geq \rho_+(\infty)$ and that

$$\{\overline{B}_{\tau_2} \geq \overline{b}\} = \{B_{\tau_2} \geq \rho_+(\infty)\}, \quad \text{since } \nu^2_B((\rho_+(\infty),\infty)) = \overline{b}.$$

We define our final embedding as follows: we first embed ν^1 and then the atom in $\underline{b}$ is diffused into ν^2 using the Azéma-Yor procedure, i.e.

$$\tau := \tau_1 \mathbf{1}_{B_{\tau_1}\neq\underline{b}} + \tau_2\circ\tau_1 \mathbf{1}_{B_{\tau_1}=\underline{b}}, \tag{55}$$

where $B_0 = S_0$. Clearly, τ is a UI embedding of μ and $S_t := B_{\frac{t}{T-t}\wedge\tau}$ defines a model for the stock price which matches the given prices of calls and puts, i.e. $S_T \sim \mu$. Furthermore, $\{\overline{S}_T \geq \overline{b}\} = \{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}\} = \{S_T \geq \rho_+(\infty)\}$ and it follows that

$$\mathbf{1}_{\overline{S}_T\geq\overline{b},\,\underline{S}_T\leq\underline{b}} = \overline{H}^{II}(\rho_+(\infty)).$$

Suppose that $\boxed{\text{I}}$ holds.

This is a mirror image of $\boxed{\text{II}}$. We first embed $\nu^1 = \mu|_{(\rho_-(0),z_0)} + p\delta_{\overline{b}}$, with $p = 1 - \mu((\rho_-(0), z_0))$. Then the atom in $\overline{b}$ is diffused into $\mu|_{\mathbb{R}_+\setminus(\rho_-(0),z_0)}$ using the reversed Azéma-Yor stopping time (cf. Obłój [Obł04, Sec. 5.3]). The resulting stopping time τ and the stock price model $S_t := B_{\frac{t}{T-t}\wedge\tau}$ satisfy $\mathbf{1}_{\overline{S}_T\geq\overline{b},\, \underline{S}_T\leq\underline{b}} = \overline{H}^I(\rho_-(0))$.

Suppose that $\boxed{\text{III}}$ holds.

We describe the embedding in words before writing it formally. We first embed μ on $(\rho_-(w_0), \rho_+(\gamma_+(w_0)))$ or we stop when we hit $\overline{b}$ or $\underline{b}$. If we hit $\overline{b}$ then we embed μ on $(\gamma_+(w_0), \infty)$ or we run until we hit $\underline{b}$. Likewise, if we first hit $\underline{b}$ then we embed μ on $(0, w_0)$ or we run till we hit $\overline{b}$. Finally, from $\underline{b}$ and $\overline{b}$ we embed the remaining bits of μ.

We now formalise these ideas. Let

$$\nu^1 = p\delta_{\underline{b}} + \mu|_{(\rho_-(w_0),\rho_+(\gamma_+(w_0)))} + \big(1 - p - \mu(\rho_-(w_0), \rho_+(\gamma_+(w_0)))\big) \tag{56}$$

where p is chosen so that $\nu^1_B(\mathbb{R}_+) = S_0$. Define two more measures

$$\begin{aligned}\nu^2 &= \mu\big([w_0, \rho_-(w_0)]\big)\delta_{\underline{b}} + \mu|_{(\gamma_+(w_0),\infty)}\\ \nu^3 &= \mu|_{(0,w_0)} + \mu\big([\rho_+(\gamma_+(w_0)), \gamma_+(w_0)]\big)\delta_{\overline{b}}\end{aligned} \tag{57}$$

and note that by definition $\nu^2_B(\mathbb{R}_+) = \overline{b}$ and $\nu^3_B(\mathbb{R}_+) = \underline{b}$. Furthermore, as the barycentre of

$$\nu^3 + \mu|_{((\rho_-(w_0),\rho_+(\gamma_+(w_0)))} + \nu^2$$

is equal to the barycentre of μ, and from the uniqueness of p in (56), we deduce that

$$\nu^3(\mathbb{R}_+) = p, \text{ and } \nu^2(\mathbb{R}_+) = q = \big(1 - p - \mu(\rho_-(w_0), \rho_+(\gamma_+(w_0)))\big).$$

Let τ_1 be a UI embedding of ν^1 (for $B_0 = S_0$), τ_2 be a UI embedding of $\frac{1}{q}\nu^2$ (for $B_0 = \overline{b}$) and τ_3 be a UI embedding of $\frac{1}{p}\nu^3$ (for $B_0 = \underline{b}$). Further, let τ_4 and τ_5 be UI embeddings of respectively

$$\frac{1}{\mu\big((w_0, \rho_-(w_0))\big)}\mu|_{(w_0,\rho_-(w_0))} \text{ and } \frac{1}{\mu\big((\rho_+(\gamma_+(w_0)), \gamma_+(w_0))\big)}\mu|_{(\rho_+(\gamma_+(w_0)),\gamma_+(w_0))},$$

where the starting points are respectively $B_0 = \underline{b}$ and $B_0 = \overline{b}$. We are ready to define our stopping time. Let $B_0 = S_0$ and write $H_z = \inf\{t : B_t = z\}$. We put

$$\begin{aligned}\tau :=& \tau_1 \mathbf{1}_{\tau_1 < H_{\underline{b}}\wedge H_{\overline{b}}}\\ &+ \tau_2\circ\tau_1 \mathbf{1}_{H_{\overline{b}}=\tau_1}\mathbf{1}_{\tau_2\circ\tau_1<H_{\underline{b}}}\\ &+ \tau_4\circ\tau_2\circ\tau_1 \mathbf{1}_{H_{\overline{b}}=\tau_1}\mathbf{1}_{H_{\underline{b}}=\tau_2\circ\tau_1}\\ &+ \tau_3\circ\tau_1 \mathbf{1}_{H_{\underline{b}}=\tau_1}\mathbf{1}_{\tau_3\circ\tau_1<H_{\overline{b}}}\\ &+ \tau_5\circ\tau_3\circ\tau_1 \mathbf{1}_{H_{\underline{b}}=\tau_1}\mathbf{1}_{H_{\overline{b}}=\tau_3\circ\tau_1},\end{aligned} \tag{58}$$

and it is immediate from the properties of our measures that $B_\tau \sim \mu$ and $(B_{t\wedge\tau})$ is a UI martingale. Furthermore, with $S_t := B_{\frac{t}{T-t}\wedge\tau}$, we see that

$$\mathbf{1}_{\overline{S}_T\geq\overline{b},\, \underline{S}_T\leq\underline{b}} = \overline{H}^{III}\big(\gamma_+(w_0), \rho_+(\gamma_+(w_0)), \rho_-(w_0), w_0\big), \quad a.s.$$

Finally, suppose that $\boxed{\text{IV}}$ holds.

In this case, we initially run to $\{\underline{b}, \overline{b}\}$ without stopping any mass. Then, from $\overline{b}$, we either run to $\underline{b}$ or embed μ on $(\rho_-(0), \infty)$. The mass which is at $\underline{b}$ after the first step is run to either $\overline{b}$ or used to embed μ on $(0, \rho_+(\infty))$. The mass which remains at $\underline{b}$ and $\overline{b}$ is then used to embed the remaining part of μ on $(\rho_+(\infty), \rho_-(0))$.

To begin with, we define the measures

$$\begin{aligned}\nu^1 &= \left[\frac{S_0 - \underline{b}}{\overline{b} - \underline{b}} - \mu((\rho_-(0), \infty))\right]\delta_{\underline{b}} + \mu|_{(\rho_-(0),\infty)},\\ \nu^2 &= \left[\frac{\overline{b} - S_0}{\overline{b} - \underline{b}} - \mu((0, \rho_+(\infty))\right]\delta_{\overline{b}} + \mu|_{(0,\rho_+(\infty))}.\end{aligned}$$

Then ν^1 is a measure, since $\mu((\rho_-(0),\infty)) < \frac{S_0-\underline{b}}{\overline{b}-\underline{b}}$: noting that $\overline{b} < \rho_-(0)$, we get

$$\begin{aligned} 0 &= \int_0^{\rho_-(0)} (u-S_0)\mu(\mathrm{d}u) + \int_{\rho_-(0)}^{\infty} (u-S_0)\mu(\mathrm{d}u) \\ &\geq (\underline{b}-S_0)\mu((0,\rho_-(0))) + (\overline{b}-S_0)\mu((\rho_-(0),\infty)) \end{aligned}$$

and the statement follows. Moreover, we can see that $\nu^1_B(\mathbb{R}_+) = \overline{b}$:

$$\begin{aligned} &\int_{\rho_-(0)}^{\infty} (u-\overline{b})\mu(\mathrm{d}u) + \frac{S_0-\underline{b}}{\overline{b}-\underline{b}}(\underline{b}-\overline{b}) - \mu((\rho_-(0),\infty))(\underline{b}-\overline{b}) \\ &\quad = \int_{\rho_-(0)}^{\infty} (u-\underline{b})\mu(\mathrm{d}u) + (\underline{b}-S_0) \\ &\quad = (S_0-\underline{b}) - \int_0^{\rho_-(0)} (u-\underline{b})\mu(\mathrm{d}u) + (\underline{b}-S_0) = 0. \end{aligned}$$

Similar results hold for ν^2.

Consequently, we can construct the first stages of the embedding. The final stage is to run from $\underline{b}$ and $\overline{b}$ to embed the remaining mass. Of course, it does not matter exactly how we do this from the optimality point of view, since these paths have already struck both barriers, but we do need to check that the embedding is possible. It is clear that the means and probabilities match, but unlike the previously considered cases, we now have initial mass in two places, and the existence of a suitable embedding is not trivial. To resolve this, we note the following: suppose we can find a point $z^* \in (\rho_+(\rho_-(0)), \rho_-(\rho_+(\infty)))$ such that $\nu^1(\{\underline{b}\}) = \mu((\rho_+(\infty), z^*))$. Then because $\mu_B((\rho_+(\infty), \rho_-(\rho_+(\infty)))) = \underline{b}$, we can find $z_1 \in (\rho_+(\infty), z^*)$ such that the measure

$$\nu^3 = \mu|_{(\rho_+(\infty),z_1)} + (\nu^1(\{\underline{b}\}) - \mu((\rho_+(\infty), z_1)))\delta_{z^*}$$

has barycentre $\underline{b}$. There is a similar construction for ν^4 and a point z_2 which will embed mass from ν^2 at $\overline{b}$ to μ on $(z_2, \rho_-(0))$, and an atom at z^*. In the final stage, we can then embed the mass from z^* to (z_1, z_2).

It remains to show that we can find such a point z^*. To do this, we check that there is sufficient mass being stopped at $\underline{b}$ at the end of the second step (i.e. which has already hit $\overline{b}$.) Specifically, we need to show that

$$\frac{S_0-\underline{b}}{\overline{b}-\underline{b}} - \mu((\rho_-(0),\infty)) \geq \mu((\rho_+(\infty), \rho_+(\rho_-(0)))).$$

Rearranging, and using the definitions of the functions ρ_+ and ρ_-, this is equivalent to

$$\begin{aligned} (S_0-\underline{b}) &\geq \int_{\rho_+(\infty)}^{\infty} (u-\underline{b})\,\mu(\mathrm{d}u) - \int_{\rho_+(\rho_-(0))}^{\rho_-(0)} (u-\underline{b})\mu(\mathrm{d}u) \\ &\geq \int_{(\rho_+(\infty),\rho_+(\rho_-(0)))\cup(\rho_-(0),\infty)} (u-\underline{b})\,\mu(\mathrm{d}u) \\ &\geq (S_0-\underline{b}) - \int_{(0,\rho_+(\infty))\cup(\rho_+(\rho_-(0)),\rho_-(0))} (u-\underline{b})\,\mu(\mathrm{d}u). \end{aligned}$$

Using the definitions of the appropriate functions, this can be seen to be equivalent to

$$0 \leq \int_{\rho_+(\rho_-(0))}^{\rho_-(\rho_+(\infty))} (u-\underline{b})\,\mu(dy),$$

which follows since $\rho_+(\rho_-(0)) > \underline{b}$. The construction of the appropriate stopping time, and its optimality follow as previously. This ends the proof of Theorem 2.3.2. □

In order to prove Theorem 2.3.4 we start with an auxiliary lemma.

Lemma 4.0.5. *Either we may construct an embedding of μ under which the process never hits both $\overline{b}$ and $\underline{b}$, or*

$$\begin{aligned} \inf\{v \in [\underline{b}, \overline{b}] : \psi(v) < \infty\} &\geq \inf\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\} &(59)\\ \sup\{v \in [\underline{b}, \overline{b}] : \psi(v) < \infty\} &\geq \sup\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\} &(60)\end{aligned}$$

and we may then write

$$\underline{v} = \inf\{v \in [\underline{b}, \overline{b}] : \psi(v) < \infty\} \leq \sup\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\} = \overline{v}, \tag{61}$$

where $\underline{v}, \overline{v}$ are given in (41).

Proof. We begin by showing that if $\theta(v) = -\infty$ for all $v \in [\underline{b}, \overline{b}]$ then there exists an embedding of μ which does not hit both $\underline{b}$ and $\overline{b}$.

For $w \geq \overline{b}$, define $\alpha_*(w)$ to be the mass that must be placed at $\underline{b}$ in order for the barycentre of this mass plus μ on $(\overline{b}, w)$ to be $\overline{b}$, so $\alpha_*(w)$ satisfies

$$\alpha_*(w)\underline{b} + \int_{\overline{b}}^{w} u\,\mu(\mathrm{d}u) = \overline{b}\left(\alpha_*(w) + \mu((\overline{b}, w))\right).$$

If follows that $\alpha_*(w)$ exists, although there is no guarantee that it is less than $1 - \mu((\overline{b}, w))$. In addition, define $\beta^*(w)$ to be

$$\beta^*(w) = \inf\left\{\beta \in [\underline{b}, \overline{b}] : \int_{(\underline{b},\beta)\cup(\overline{b},w)} u\,\mu(\mathrm{d}u) = \overline{b}\int_{(\underline{b},\beta)\cup(\overline{b},w)} \mu(\mathrm{d}u)\right\}.$$

If this is finite, then it is the point at which $\mu_B((\underline{b}, \beta^*(w)) \cup (\overline{b}, w)) = \overline{b}$. Note also that $\beta^*(w)$ is increasing as a function of w, and is continuous when $\beta^*(w) < \infty$. Define

$$\begin{aligned} p_*(w) &= \mu((\overline{b}, w)) + \alpha_*(w)\\ p^*(w) &= \mu((\underline{b}, \beta^*(w)) \cup (\overline{b}, w)).\end{aligned}$$

Suppose initially that $\beta^*(w) \leq \overline{b}$ for all $w \geq \overline{b}$. Then we may assign the following interpretations to these quantities: $p_*(w)$ is the smallest amount of mass that we can start at $\overline{b}$ and run to embed μ on $(\overline{b}, w)$, $(\underline{b}, \cdot)$ and an atom at $\underline{b}$, and $p^*(w)$ is the largest amount of mass that we may do this with: the smallest amount is attained by running all the mass below $\overline{b}$ to $\underline{b}$, while the largest probability is attained by running all this mass to $(\underline{b}, \beta^*(w))$. The assumption that $\beta^*(w) \leq \overline{b}$ implies that this upper bound does not run out of mass to embed. Moreover, by adjusting the size of the atom at $\underline{b}$, we can embed an atom of any size between $p_*(w)$ and $p^*(w)$ from $\overline{b}$ in this way. Recalling the definition of $\theta(v)$, we conclude that there exists v such that $\theta(v) = w$ if and only if $p_*(w) \leq \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}} \leq p^*(w)$. Finally, note that the functions $p_*(w)$ and $p^*(w)$ are both increasing in w, and further that $p_*(\overline{b}) = 0$. Consequently, if there is no v such that $\theta(v) > -\infty$, and $\beta^*(w) \leq \overline{b}$ for all $w \geq \overline{b}$, we must have $p^*(\infty) := \lim_{w\to\infty} p^*(w) < \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$.

So suppose $p^*(\infty) < \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$. We now construct an embedding as follows: from S_0, we initially run to either $\overline{b}$ or

$$b_* = \frac{S_0 - p^*(\infty)\overline{b}}{1 - p^*(\infty)}.$$

Since $p^*(\infty) < \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$, then $b_* \in (\underline{b}, S_0)$, and the probability that we hit $\overline{b}$ before b_* is $p^*(\infty)$. In addition, by the definition of $\beta^*(w)$, we deduce that the set $(\underline{b}, \beta^*(\infty)] \cup [\overline{b}, \infty)$ is given mass $p^*(\infty)$ by μ, and that the barycentre of μ on this set is $\overline{b}$. We may therefore embed the paths from $\overline{b}$ to this set, and the paths from b_* to the remaining intervals, $[0, \underline{b}] \cup (\beta^*(\infty), \overline{b})$ and we note no paths will hit both $\overline{b}$ and $\underline{b}$.

So suppose instead that $\beta^*(w_0) = \overline{b}$ for some w_0, with $p^*(w_0) \leq \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$. (If the latter condition does not hold, then using the fact that $\beta^*(w)$ is left-continuous and increasing, we can find a w such that $\beta^*(w) < \overline{b}$

and $p^*(w) = \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$, and therefore, by the arguments above, there exists v with $\theta(v) > -\infty$.) We may then continue to construct measures with barycentre $\overline{b}$, which are equal to μ on $(\underline{b}, w)$ for $w > w_0$, and have a compensating atom at $\underline{b}$. As we increase w, eventually either w reaches ∞, or the mass of the measure reaches $\frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$. In the latter case, we know $\theta(\overline{b}) = w$, contradicting $\theta(v) = -\infty$ for all $v \in [\underline{b}, \overline{b}]$. So consider the former case: we obtained that the measure which is μ on $(\underline{b}, \infty)$ with a further atom at $\underline{b}$ to give barycentre $\overline{b}$ has total mass (p say) less than $\frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$. We show that this is impossible: divide μ into its restriction to $(0, z)$ and $[z, \infty)$, where z is chosen so that $\mu([z, \infty)) = p$. Then $z < \underline{b}$ and the barycentre of the restriction to $[z, \infty)$ is strictly smaller than the barycentre of the measure with the mass on $[z, \underline{b})$ placed at $\underline{b}$, which is the measure described above, and which has barycentre $\overline{b}$. Additionally, the barycentre of the lower restriction of μ must be strictly smaller than $\underline{b}$. Moreover, we may calculate the barycentre of μ by considering the barycentre of the two restrictions; since μ has mean (and therefore barycentre) S_0, we must have:

$$S_0 = (1-p)\mu_B((0,z)) + p\mu_B([z,\infty)) < (1-p)\underline{b} + p\overline{b} < \underline{b}\frac{\overline{b} - S_0}{\overline{b} - \underline{b}} + \overline{b}\frac{S_0 - \underline{b}}{\overline{b} - \underline{b}} = S_0,$$

which is a contradiction.

We conclude that, if $\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\}$ is empty, there is an embedding of μ which does not hit both $\overline{b}$ and $\underline{b}$. A similar result follows for $\psi(v)$. In particular, if we assume that there is no such embedding, then there exists v such that $\psi(v) < \infty$, and (not necessarily the same) v such that $\theta(v) > -\infty$. We now wish to show that (59) holds. Suppose not. Then:

$$v_* := \inf\{v \in [\underline{b}, \overline{b}] : \psi(v) < \infty\} < \inf\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\} =: v^*.$$

Moreover, we can deduce from the definition of $\theta(v)$ that since $v^* > \underline{b}$, we must have $\theta(v^*) = \infty$. Now consider the barycentre of the measure which is taken by running from S_0 to $\underline{b}$ and $\overline{b}$, and then from $\underline{b}$ to $(\psi(v_*), \underline{b}) \cup (v_*, \overline{b})$, with a compensating mass at $\overline{b}$, so that the measure has barycentre $\underline{b}$, and from $\overline{b}$ to $(\underline{b}, v^*) \cup (\overline{b}, \theta(v^*)) = (\underline{b}, v^*) \cup (\overline{b}, \infty)$, with a compensating mass at $\underline{b}$, so that the measure has barycentre $\overline{b}$. Then the whole law of the resulting process must have mean S_0, since this can be done in a uniformly integrable way, but the resulting distribution is at least μ on $(\psi(v_*), \infty)$ (it is twice μ on (v_*, v^*), has atoms at $\underline{b}$ and $\overline{b}$ and is μ elsewhere), and zero on $(0, \psi(v_*))$, so must have mean greater than S_0, which is a contradiction. A similar argument shows (60). Hence, we may conclude (still under the assumption that there is no embedding which never hits both $\underline{b}$ and $\overline{b}$) that the equalities in (61) hold. It remains to show the inequality when $\underline{v}, \overline{v} \in (\underline{b}, \overline{b})$. However this is now almost immediate: the forms of $\overline{v}, \underline{v}$ imply that μ gives mass $\frac{\overline{b} - S_0}{\overline{b} - \underline{b}}$ to the set $(\psi(\underline{v}), \underline{b}) \cup (\underline{v}, \overline{b})$ and it gives mass $\frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$ to the set $(\underline{b}, \overline{v}) \cup (\overline{b}, \theta(\overline{v}))$. If $\overline{v} < \underline{v}$, this implies that μ gives mass 1 to the set $(\psi(\underline{v}), \overline{v}) \cup (\underline{v}, \theta(\overline{v})) \subsetneq [0, \infty)$, contradicting the positivity of μ. $\square$

Proof of Theorem 2.3.4. From Lemma 4.0.5 case $\boxed{\text{IV}}$ and the last statement of the theorem follow. Assume from now on that $\underline{v} \le \overline{v}$. We note firstly that $\psi(v)$ and $\theta(v)$ are both continuous and decreasing on $[\underline{v}, \overline{v}]$, and consequently $\kappa(v)$ is also continuous and decreasing as a function of v on $[\underline{v}, \overline{v}]$. It follows that the three cases $\kappa(\underline{v}) < \underline{v}$, $\kappa(\overline{v}) > \overline{v}$ and the existence of $v_0 \in [\underline{v}, \overline{v}]$ such that $\kappa(v_0) = v_0$ are exclusive and exhaustive. We consider each case separately:

$\boxed{\text{I}}$ Suppose that there exists $v_0 \in [\underline{v}, \overline{v}]$ such that $\kappa(v_0) = v_0$. By the definition of $\psi(v)$, we can run all the mass initially from S_0 to $\{\underline{b}, \overline{b}\}$ and then embed (in a uniformly integrable way) from $\underline{b}$ to $(\psi(v_0), \underline{b}) \cup (v_0, \overline{b})$ and a compensating atom at $\overline{b}$ with the remaining mass, and similarly from $\overline{b}$ to $(\underline{b}, v_0) \cup (\overline{b}, \theta(v_0))$ with an atom at $\underline{b}$. The mass now at $\underline{b}$ and $\overline{b}$ can now be embedded in the remaining tails in a suitable way — the means and masses must agree, since the initial stages were embedded in a uniformly integrable manner, and the remaining mass all lies outside $[\underline{b}, \overline{b}]$. We denote τ the stopping time which achieves the embedding.

Now we compare both sides of the inequality in (14), where we choose $K_1 = \theta(v_0), K_2 = \psi(v_0)$ and therefore, as a consequence of the definition of $\kappa(v)$, we also have $K_3 = v_0$. The key observation is now that the mass is stopped only at points where the inequality is an equality: mass which hits $\overline{b}$ initially either stops in the interval $(\underline{b}, K_3) \cup (\overline{b}, K_1)$, when there is equality in (14), or it goes on to hit $\overline{b}$, and from this point also continues to the tails $(0, K_2) \cup (K_1, \infty)$, where there is again equality between both sides of (14). Taking expectations on the right of (14), we get the terms on the right of (42), and we conclude that (42) holds in the market model $S_t := B_{\tau \wedge \frac{t}{T-t}}$.

$\boxed{\text{II}}$ Suppose now that $\kappa(\overline{v}) > \overline{v}$. Then we must have $\overline{v} = \sup\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\}$ by Lemma 4.0.5, and then $\overline{v} < \kappa(\overline{v}) \leq \overline{b}$. So, by the definition of $\theta(v)$, since $\overline{v}$ exists and is less than $\overline{b}$, we must have

$$\int_{(\underline{b},\overline{v})\cup(\overline{b},\theta(\overline{v}))} u\,\mu(\mathrm{d}u) = \overline{b}\frac{S_0 - \underline{b}}{\overline{b} - \underline{b}} \text{ and } \mu((\underline{b}, \overline{v}) \cup (\overline{b}, \theta(\overline{v}))) = \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}},$$

and we can embed from $\overline{b}$ (having initially run to $\{\underline{b}, \overline{b}\}$) to $(\underline{b}, \overline{v}) \cup (\overline{b}, \theta(\overline{v}))$ *without* leaving an atom at $\underline{b}$. Similarly, we can also run from $\underline{b}$ to $(\psi(\overline{v}), \underline{b}) \cup (\overline{v}, \overline{b})$ with an atom at $\overline{b}$. The atom can then be embedded in the tails $(0, \psi(\overline{v})) \cup (\theta(\overline{v}), \infty)$ in a uniformly integrable manner. We now need to show that when we take $K_3 = \overline{v}, K_1 = \theta(\overline{v})$ and $K_2 = \psi(\overline{v})$ we get the required equality in (43). The main difference from the above case occurs in the case where we hit $\overline{b}$ initially and then hit $\underline{b}$: we no longer need equality in (14), since this no longer occurs in our optimal construction, however what remains to be checked is that the inequality does hold on this set. Specifically, we need to show that:

$$1 \geq \alpha_0 + \alpha_1(K_2 - S_0) - (\alpha_3 - \alpha_3 + \alpha_1)(K_2 - \overline{b}) + (\alpha_3 - \alpha_2)(K_2 - \underline{b})$$

Using (24) and (29), we see that this occurs when

$$\frac{(K_3 - K_2)(K_1 - \underline{b})}{(\overline{b} - K_2)(K_1 - K_3)} \leq 1$$

which rearranges to give:

$$K_3 \leq \overline{b}\frac{K_1 - \underline{b}}{(K_1 - \underline{b}) + (\overline{b} - K_2)} + \underline{b}\frac{\overline{b} - K_2}{(K_1 - \underline{b}) + (\overline{b} - K_2)}.$$

This is satisfied by our choice of $\overline{v}$ as K_3, and $K_1 = \theta(\overline{v}), K_2 = \psi(\overline{v})$.

$\boxed{\text{III}}$ This is symmetric to case $\boxed{\text{II}}$.

$\square$

5 Additional Figures

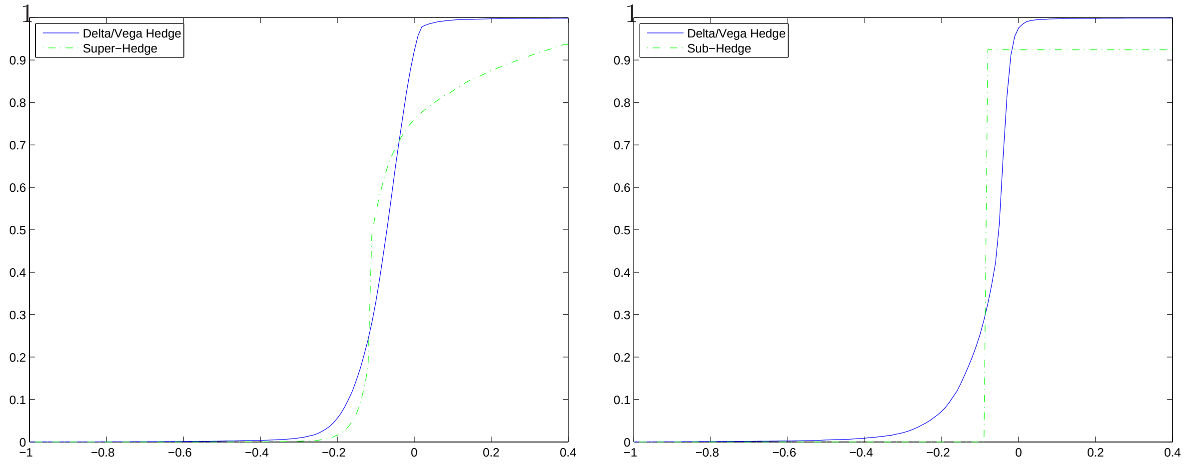

Figure 11: Cumulative distributions of hedging errors under different scenarios of a short position (*left*) and a long position (*right*) in a double touch option with barriers at 130 and 70 under the Heston model (47)–(48).

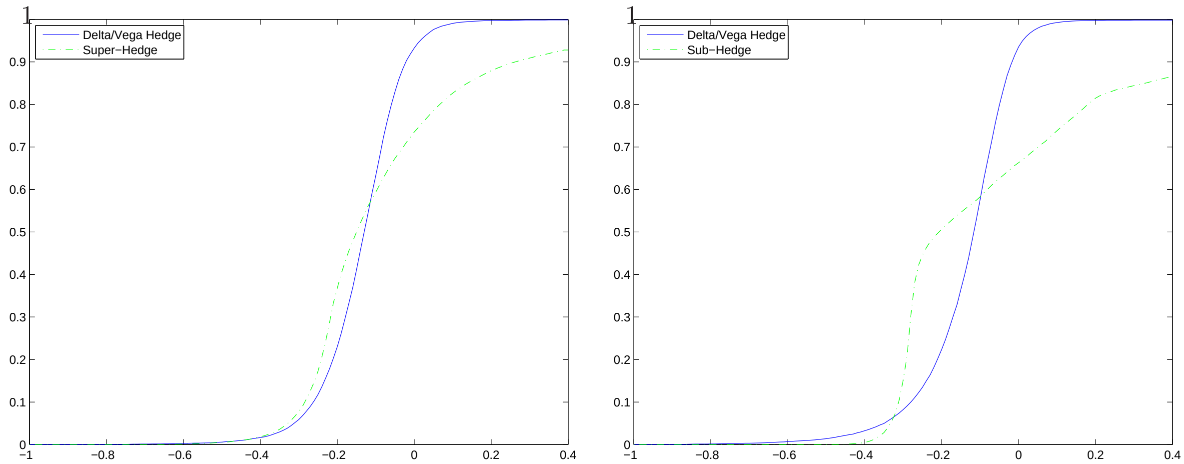

Figure 12: Cumulative distributions of hedging errors under different scenarios of a short position (*left*) and a long position (*right*) in a double touch option with barriers at 115 and 85 under the Heston model (47)–(48).

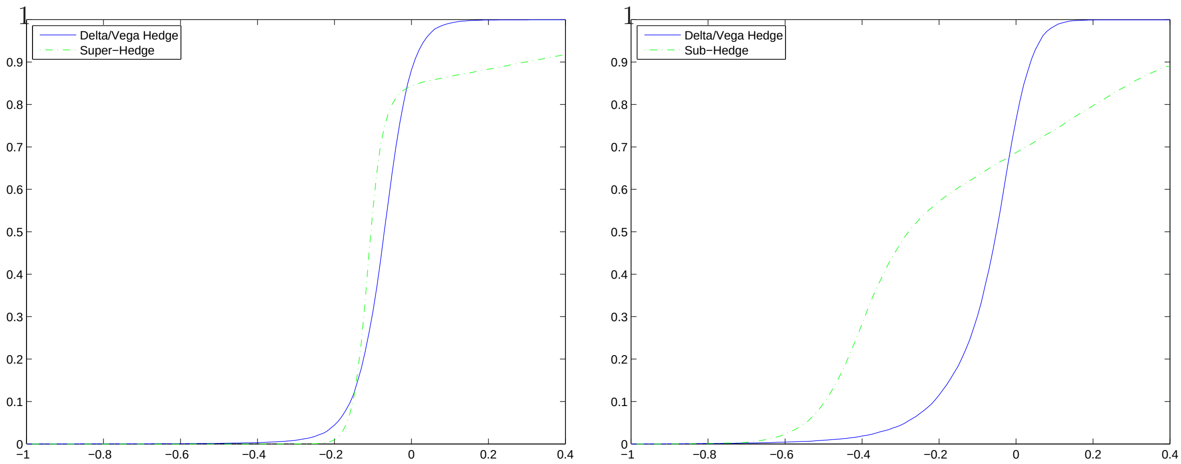

Figure 13: Cumulative distributions of hedging errors under different scenarios of a short position (*left*) and a long position (*right*) in a double touch option with barriers at 103 and 97 under the Heston model (47)–(48).

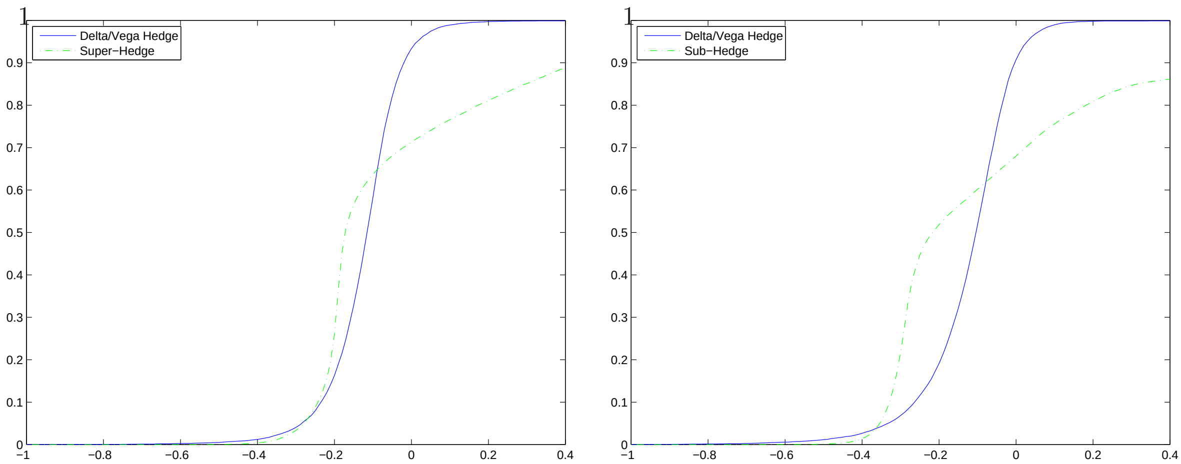

Figure 14: Cumulative distributions of hedging errors under different scenarios of a short position (*left*) and a long position (*right*) in a double touch option with barriers at 120 and 95 under the Heston model (47)–(48).

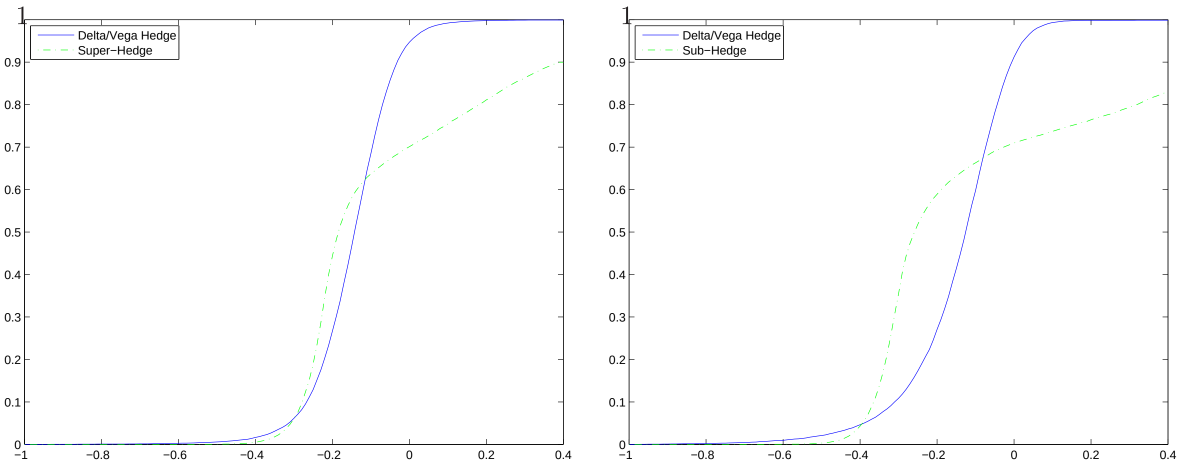

Figure 15: Cumulative distributions of hedging errors under different scenarios of a short position (*left*) and a long position (*right*) in a double touch option with barriers at 105 and 80 under the Heston model (47)–(48).

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