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CONTROLLED FUNCTIONAL DIFFERENTIAL EQUATIONS: APPROXIMATE AND EXACT ASYMPTOTIC TRACKING WITH PRESCRIBED TRANSIENT PERFORMANCE

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Abstract. A tracking problem is considered in the context of a class $\mathcal{S}$ of multi-input, multi-output, nonlinear systems modelled by controlled functional differential equations. The class contains, as a prototype, all finite-dimensional, linear, m-input, m-output, minimum-phase systems with sign-definite "high-frequency gain". The first control objective is tracking of reference signals r by the output y of any system in $\mathcal{S}$: given $\lambda \geq 0$, construct a feedback strategy which ensures that, for every r (assumed bounded with essentially bounded derivative) and every system of class $\mathcal{S}$, the tracking error $e = y - r$ is such that, in the case $\lambda > 0$, $\limsup_{t\to\infty} \|e(t)\| < \lambda$ or, in the case $\lambda = 0$, $\lim_{t\to\infty} \|e(t)\| = 0$. The second objective is guaranteed output transient performance: the error is required to evolve within a prescribed performance funnel $\mathcal{F}_\varphi$ (determined by a function φ). For suitably chosen functions α, ν and θ, both objectives are achieved *via* a control structure of the form $u(t) = -\nu(k(t))\theta(e(t))$ with $k(t) = \alpha(\varphi(t)\|e(t)\|)$, whilst maintaining boundedness of the control and gain functions u and k. In the case $\lambda = 0$, the feedback strategy may be discontinuous: to accommodate this feature, a unifying framework of differential inclusions is adopted in the analysis of the general case $\lambda \geq 0$.

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1. Introduction

In precursors [6–8] to the present paper, an *approximate* tracking problem is addressed for various classes of systems. Let $\mathcal{S}$ be some given system class and let $\mathcal{R}$ be a class of reference signals. By approximate tracking, we mean attainment of the following: for any prescribed $\lambda > 0$, determine a *continuous* output feedback strategy which ensures that, for every system (with output y) in $\mathcal{S}$ and every reference signal $r \in \mathcal{R}$, (i) the tracking error $e = y - r$ is ultimately contained in the ball of radius λ centred at 0 (equivalently, $\limsup_{t\to\infty} \|e(t)\| < \lambda$), and (ii) the error e exhibits prescribed transient behaviour (that is, for some suitable prescribed function φ with $0 < \liminf_{t\to\infty} \varphi(t) < \infty$, we have $\|e(t)\| < 1/\varphi(t)$ for all $t > 0$). The present paper encompasses not only approximate tracking but also the problem of *asymptotic* tracking with prescribed transient behaviour: in the latter case, an output feedback strategy (possibly discontinuous) is sought which ensures that, for every system of class $\mathcal{S}$, every reference signal $r \in \mathcal{R}$ and some suitable prescribed function φ, with $\varphi(t) \to \infty$ as $t \to \infty$,

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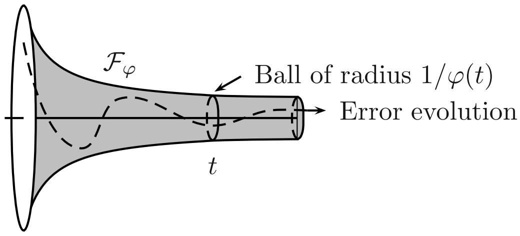

FIGURE 1. Performance funnel $\mathcal{F}_\varphi$.

we have $\|e(t)\| < 1/\varphi(t)$ for all $t > 0$ (and so $e(t) \to 0$ as $t \to \infty$). Both cases (approximate and asymptotic tracking) are analysed within a unified framework of functional differential inclusions.

The focus of our study will be nonlinear systems (akin to those considered in [6]), with control input $t \mapsto u(t) \in \mathbb{R}^m$, modelled by functional differential equations of the form

$$\dot{y}(t) = f(d(t), (Ty)(t), u(t)), \quad y|_{[-h,0]} = y^0 \in C([-h, 0], \mathbb{R}^m), \quad h \geq 0, \tag{1.1}$$

where f is continuous, T is a causal operator, d may be thought of as a continuous and bounded perturbation, and $h \geq 0$ quantifies the "memory" of the system. As in [6–8], the class $\mathcal{R}$ of reference signals is taken to be the space $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ of bounded locally absolutely continuous functions $r\colon \mathbb{R}_+ \to \mathbb{R}^m$ with essentially bounded derivative $\dot{r} \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$.

The paper is structured as follows. Section 2 formulates the control objectives and, in Section 3, a full description of the system class $\mathcal{S}$ is provided. Section 4 details the feedback structure, the potentially discontinuous nature of which leads to an interpretation of the closed-loop system in the form of a functional differential inclusion. An existence theory (which may be of independent interest) for functional differential inclusions of sufficient generality to encompass the closed-loop system is developed in Section 5. The main results on transient behaviour and asymptotic tracking for the closed-loop system are given in Section 6.

2. Control objectives and the performance funnel

The two control objectives are:

(i) tracking of any reference signal $r \in \mathcal{R} := W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ by the output y, that is, for arbitrary $\lambda \geq 0$, we seek an output feedback strategy which ensures that, for every $r \in \mathcal{R}$, every solution of the closed-loop system is bounded and the tracking error $e = y - r$ is such that either $\limsup_{t\to\infty} \|e(t)\| < \lambda$ if $\lambda > 0$ or $\lim_{t\to\infty} \|e(t)\| = 0$ if $\lambda = 0$;

(ii) prescribed transient behaviour of the tracking error.

Both objectives are captured in the concept of a performance funnel

$$\mathcal{F}_\varphi := \big\{ (t, e) \in \mathbb{R}_+ \times \mathbb{R}^m \big|\ \varphi(t)\, \|e\| < 1 \big\}$$

associated with a function φ (the reciprocal of which determines the funnel boundary) in

$$\Phi_\lambda := \Big\{ \varphi \in AC_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}) \ \Big|\ \ \varphi(0) = 0,\ \varphi(s) > 0\ \forall\ s > 0,\ \liminf_{s\to\infty} \varphi(s) = 1/\lambda,$$
$$\exists\ c > 0 : \quad \dot{\varphi}(s) \leq c[1 + \varphi(s)] \text{ for a.a. } s > 0 \Big\},$$

with the convention that, if $\lambda = 0$, then $1/\lambda := \infty$ (and so $\varphi(t) \to \infty$ as $t \to \infty$). Here, $AC_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ denotes the space of locally absolutely continuous functions $\mathbb{R}_+ \to \mathbb{R}$.

If a feedback structure can be devised which ensures that, for every system of the underlying class and every $r \in \mathcal{R}$, the graph of the tracking error $e = y - r$ is properly contained in $\mathcal{F}_\varphi$ in the sense that $\sup_{t\in\mathbb{R}_+} \varphi(t)\|e(t)\| < 1$ then the tracking objective (i) is attained, and (ii) transient behaviour is governed by the choice of φ:

for example, if $\lambda > 0$ and φ is chosen as the function $t \mapsto \min\{t/\tau, 1\}/\lambda$, then the prescribed tracking accuracy $\lambda > 0$ is achieved within the prescribed time $\tau > 0$.

The intuition underpinning the feedback structure proposed below is an intrinsic high-gain property of the system class which ensures that, if $(t, e(t))$ approaches the funnel boundary, then the control input attains values sufficiently large to preclude boundary contact.

3. Class of systems

For $m \in \mathbb{N}$ and an interval $I \subset \mathbb{R}$, $C(I, \mathbb{R}^m)$ denotes the space of continuous functions $I \to \mathbb{R}^m$. If I is an interval of the form $[-h, a)$ or $[-h, a]$, $0 < a < \infty$, and $x \in C(I, \mathbb{R}^m)$, then, for each $\sigma \in J := I \backslash [-h, 0)$, we define the function $x_\sigma \in C([-h, \infty), \mathbb{R}^m)$ by

$$x_\sigma(t) := \begin{cases} x(t), & t \in [-h, \sigma], \\ x(\sigma), & t > \sigma. \end{cases}$$

For $h, t \in \mathbb{R}_+$, $w \in C([-h, t], \mathbb{R}^m)$, $\tau > t$ and $\delta > 0$, define

$$\mathcal{C}(w; h, t, \tau, \delta) := \left\{ v \in C([-h, \tau], \mathbb{R}^m) \;\middle|\; v|_{[-h,t]} = w,\ \|v(s) - w(t)\| \le \delta \quad \forall\, s \in [t, \tau] \right\},$$

that is, the space of all continuous extensions v of $w \in C([-h, t], \mathbb{R}^m)$ to the interval $[-h, \tau]$ with the property that $\|v(s) - w(t)\| \le \delta$ for all $s \in [t, \tau]$.

We first define a class of operators $\mathcal{T}_h$, parameterized by $h \ge 0$.

Definition 3.1 (operator class $\mathcal{T}_h$)**.** An operator T is said to be of class $\mathcal{T}_h$ if, and only if, the following hold:

(i) For some $q \in \mathbb{N}$, $T\colon C([-h, \infty), \mathbb{R}^m) \to L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^q)$.

(ii) T is a causal operator: for all $x, y \in C([-h, \infty), \mathbb{R}^m)$ and all $\tau > 0$

$$x(t) = y(t)\ \forall\, t \in [-h, \tau] \implies (Tx)(t) = (Ty)(t)\ \forall\, t \in [0, \tau].$$

(iii) For each $t \ge 0$ and each $w \in C([-h, t], \mathbb{R}^m)$, there exist $\tau > t$, $\delta > 0$ and $c_0 > 0$ such that

$$\text{ess-sup}_{s\in[t,\tau]} \|(Tx_\tau)(s) - (Ty_\tau)(s)\| \ \le\ c_0 \ \sup_{s\in[t,\tau]} \|x(s) - y(s)\| \quad \forall\, x, y \in \mathcal{C}(w; h, t, \tau, \delta).$$

(iv) For every $c_1 > 0$, there exists $c_2 > 0$ such that, for all $y \in C([-h, \infty), \mathbb{R}^m)$,

$$\sup_{t\in[-h,\infty)} \|y(t)\| \le c_1 \quad \Longrightarrow \quad \|(Ty)(t)\| \le c_2 \quad \text{for a.a. } t \ge 0.$$

Remark 3.2. Property (iii) is a technical assumption of local Lipschitz type which is used in establishing well-posedness of the closed-loop system (defined later in Sect. 4.1). We will have occasion to give meaning to Tx, for a function $x \in C(I, \mathbb{R}^m)$ on a bounded interval I of the form $[-h, a)$ or $[-h, a]$, where $0 < a < \infty$. This we do by showing that T "localizes", in a natural way, to an operator $\tilde{T}\colon C(I, \mathbb{R}^m) \to L^\infty_{\text{loc}}(J, \mathbb{R}^q)$, where $J := I \setminus [-h, 0)$. In particular, and invoking causality, we may define $\tilde{T}x \in L^\infty_{\text{loc}}(J, \mathbb{R}^q)$ by the property

$$\tilde{T}x|_{[0,\sigma]} = Tx_\sigma|_{[0,\sigma]} \quad \forall\, \sigma \in J.$$

Henceforth, we will not distinguish notationally an operator T and its "localisation" $\tilde{T}$: the correct interpretation being clear from context. For example, with this convention in place, we may reinterpret the lefthand side of the displayed inequality in property (iii) above as $\text{ess-sup}_{s\in[t,\tau]} \|(Tx)(s) - (Ty)(s)\|$, where $T = \tilde{T}\colon C([-h, \tau], R^m) \to L^\infty_{\text{loc}}([0, \tau], \mathbb{R}^q)$ now represents a "localization" of the original causal operator $T\colon C([-h, \infty), \mathbb{R}^m) \to L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^q)$.

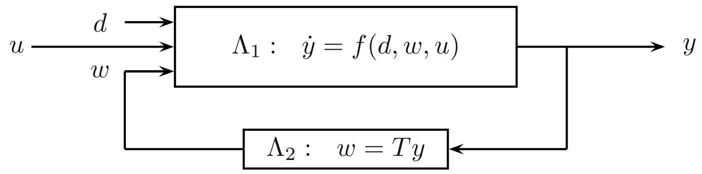

FIGURE 2. System of class $\mathcal{S}$.

We are now in a position to define the system class.

Definition 3.3 (system class $\mathcal{S}$)**.** The class $\mathcal{S}$ is comprised of m-input $(u(t) \in \mathbb{R}^m)$, m-output $(y(t) \in \mathbb{R}^m)$, nonlinear systems (f, d, T) of the form (1.1), satisfying the following assumptions.

(A1) The function $f\colon \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous.
(A2) For each compact set $\mathcal{K} \subset \mathbb{R}^p \times \mathbb{R}^q$, the continuous function $\gamma_{\mathcal{K}}\colon \mathbb{R} \to \mathbb{R}$, given by

$$\gamma_{\mathcal{K}}(s) := \min \big\{ \langle v, f(l, w, sv) \rangle \,\big|\; (l, w) \in \mathcal{K}, \;\; \|v\| = 1 \big\}, \tag{3.1}$$

is such that either (i) $\limsup_{s\to\infty} \gamma_{\mathcal{K}}(s) = \infty$, or (ii) $\limsup_{s\to -\infty} \gamma_{\mathcal{K}}(s) = \infty$.
(A3) $d \in C(\mathbb{R}_+, \mathbb{R}^p)$ is bounded.
(A4) $T\colon C([-h, \infty), \mathbb{R}^m) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^q)$ is of class $\mathcal{T}_h$.

3.1. Prototypical subclasses of $\mathcal{S}$

3.1.1. *Linear prototype*

With reference to Figure 2, a system (1.1) of class $\mathcal{S}$ can be thought of as an interconnection of two subsystems. The dynamical subsystem Λ_1, which can be influenced directly by the control input u, is also driven by a disturbance d and by the output w from the subsystem Λ_2, formulated as a causal operator mapping the the signal y to w (an internal quantity, unavailable for feedback purposes).

To illustrate this, consider the prototype class $\mathcal{L}$ of finite-dimensional, minimum-phase, m-input $(u(t) \in \mathbb{R}^m)$, m-output $(y(t) \in \mathbb{R}^m)$ linear systems (A, B, C) with sign-definite high-frequency gain, in the sense that either CB or $-CB$ is positive definite (symmetry of CB is not assumed). The minimum-phase property is characterized by

$$\det \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \neq 0 \quad \text{for all } s \in \mathbb{C}_+ := \{ s \in \mathbb{C} |\; \mathrm{Re}(s) \geq 0 \}. \tag{3.2}$$

Specifically,

$$\mathcal{L} = \big\{ (A, B, C) |\; A \in \mathbb{R}^{n\times n},\; B \in \mathbb{R}^{n\times m},\; C \in \mathbb{R}^{m \times n},\; m, n \in \mathbb{N},\; m \leq n,\; CB \text{ sign definite, } (3.2) \text{ holds} \big\}.$$

It is well known (see for example [4], Lem. 2.1.3) that, for each $(A, B, C) \in \mathcal{L}$ (and assuming $m < n$), there exists a similarity transformation which takes the system into the form

$$\left.\begin{aligned} \dot{y}(t) &= A_1 y(t) + A_2 z(t) + CBu(t), & y(0) &= y^0, \\ \dot{z}(t) &= A_3 y(t) + A_4 z(t), & z(0) &= z^0, \end{aligned}\right\} \tag{3.3}$$

where, by the minimum-phase property, A_4 is a Hurwitz matrix. Defining the function d (continuous and bounded) and operator T (linear) by

$$d(t) := A_2 \big(\exp(A_4 t)\big) z^0, \qquad (Ty)(t) := A_1 y(t) + A_2 \int_0^t \big(\exp A_4(t - s)\big) A_3 y(s) \mathrm{d}s\,, \tag{3.4}$$

we see that the original system $(A, B, C) \in \mathcal{L}$ can be recast in the form of the (linear) functional differential equation

$$\dot{y}(t) = d(t) + (Ty)(t) + CB\,u(t), \quad y(0) = y^0 \in \mathbb{R}^m,$$

which is of the form (1.1) with $h = 0$ and $f\colon \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $(l, w, v) \mapsto l + w + CB\,v$. Clearly, Assumption (A1) holds. Since A_4 is Hurwitz, we see that (A3) and (A4) (with $h = 0$) are valid. It remains to show that (A2) also holds. Recall that CB is sign definite and so either (i) CB is positive definite, which we write symbolically as $CB > 0$, or (ii) $-CB > 0$. Let $\mathcal{K} \subset \mathbb{R}^m \times \mathbb{R}^m$ be compact and define

$$c_{\mathcal{K}} := \min\{\langle v, l + w\rangle \mid (l, w) \in \mathcal{K},\ \|v\| = 1\}.$$

Now, observe that

$$\begin{aligned} CB > 0 \quad &\Longrightarrow \quad \min\{\langle v, CBv\rangle \mid \ \|v\| = 1\} = \tfrac{1}{2}\|(CB + B^TC^T)^{-1}\|^{-1} \\ -CB > 0 \quad &\Longrightarrow \quad \min\{\langle v, CBv\rangle \mid \ \|v\| = 1\} = -\tfrac{1}{2}\|CB + B^TC^T\| \end{aligned}$$

Therefore,

$$\begin{aligned} &\text{(i) } CB > 0,\ s \geq 0 \implies \gamma_{\mathcal{K}}(s) \geq c_{\mathcal{K}} + \tfrac{1}{2}s\|(CB + B^TC^T)^{-1}\|^{-1} && \text{and so (A2)(i) holds,} \\ &\text{(ii) } -CB > 0,\ s \leq 0 \implies \gamma_{\mathcal{K}}(s) \geq c_{\mathcal{K}} - \tfrac{1}{2}s\|CB + B^TC^T\| && \text{and so (A2)(ii) holds.} \end{aligned}$$

3.1.2. *Systems with input nonlinearity*

To illustrate the generality afforded by Assumption (A2), consider a single-input, single-output ($m = 1$) system (3.3) of class $\mathcal{L}$ with a nonlinearity g in the input channel

$$\left.\begin{aligned} \dot{y}(t) &= A_1 y(t) + A_2 z(t) + \beta\, g(u(t)), && y(0) = y^0, \\ \dot{z}(t) &= A_3 y(t) + A_4 z(t), && z(0) = z^0, \end{aligned}\right\} \tag{3.5}$$

where $\beta := CB$ is now a non-zero real number. We assume only that $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous unbounded function with bounded even part, for example, $g\colon v \mapsto (1 + v)\cos v$. Such a function can influence/reverse the polarity of an input signal $u(\cdot)$ in a manner unpredictable by a controller. Defining d and T as in (3.4), system (3.5) can be expressed as

$$\dot{y}(t) = d(t) + (Ty)(t) + \beta g(u(t)), \quad y(0) = y^0 \in \mathbb{R},$$

which again is of form (1.1). Assumptions (A1), (A3) and (A4) clearly hold. Define g_o and g_e to be the odd and even parts, respectively, of the function βg. To see that (A2) holds, let $\mathcal{K} \subset \mathbb{R} \times \mathbb{R}$ be compact, define $c_{\mathcal{K}}$ as above, and observe that, since $v g_o(sv) = g_o(s)$ for all $|v| = 1$ and all $s \in \mathbb{R}$,

$$\gamma_{\mathcal{K}}(s) = \min\{v(l + w + g_e(sv)) \mid (l, w) \in \mathcal{K},\ |v| = 1\} + g_o(s) \geq c_{\mathcal{K}} - |g_e(s)| + g_o(s) \quad \forall\ s. \tag{3.6}$$

Since the function g_o is odd and unbounded, there must exist an unbounded monotone sequence (s_n) (either strictly increasing or strictly decreasing) such that $g_o(s_n) \to \infty$ as $n \to \infty$ which, together with boundedness of g_e and (3.6), ensures $\gamma_{\mathcal{K}}(s_n) \to \infty$ as $n \to \infty$.

3.1.3. *Nonlinear systems*

Now consider a further generalization of systems of form (3.5) to nonlinear systems of the form

$$\left.\begin{aligned} \dot{y}(t) &= f_1(y(t), z(t)) + g(u(t)), && y(0) = y^0 \in \mathbb{R}, \\ \dot{z}(t) &= f_2(y(t), z(t)), && z(0) = z^0 \in \mathbb{R}^p, \end{aligned}\right\} \tag{3.7}$$

with f_1 continuous, f_2 locally Lipschitz, and (as above) g continuous and unbounded with bounded even part (we have absorbed the parameter $\beta \neq 0$ in g). Temporarily regarding y as an independent input to the second subsystem in (3.7), denote the unique solution of the initial-value problem $\dot{z} = f_2(y, z)$, $z(0) = z^0$, by $z(\cdot\,; z^0, y)$. If we now assume that the second subsystem in (3.7) is input-to-state stable (ISS) (see [13]), then, for each $z^0 \in \mathbb{R}^p$, we may define an operator $C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R} \times \mathbb{R}^p)$ by

$$(Ty)(t) := (y(t), z(t; z^0, y)) \quad \forall\, t \in \mathbb{R}_+.$$

This operator T is of class $\mathcal{T}_0$ (Assumption (A4) holds with $h = 0$, $m = 1$ and $q = p + 1$). System (3.7) may be expressed as the functional differential equation

$$\dot{y}(t) = f_1((Ty)(t)) + g(u(t)), \quad y(0) = y^0,$$

which is of the form (1.1) with $h = 0$ and $f\colon (l, w, v) \mapsto f_1(w) + g(v)$. Evidently, Assumption (A1) holds, Assumption (A3) is vacuous, and Assumption (A2) holds by the argument (*mutatis mutandis*) used in Section 3.1.2.

3.1.4. *Systems with delays and hysteresis*

Finally, we remark that nonlinear delay elements are incorporated in the operator class $\mathcal{T}_h$, see for example [12], whilst the class $\mathcal{T}_0$ encompasses a wide range of hysteresis operators, including many physically motivated effects: as observed in [5], examples such as relay hysteresis, elastic-plastic hysteresis, backlash hysteresis, Prandtl and Preisach operators (for background, see [2,10]) are of class $\mathcal{T}_0$.

4. Feedback control

We proceed to make precise the proposed output feedback structure. Let $\lambda \geq 0$ and $\varphi \in \Phi_\lambda$. Let $\nu\colon \mathbb{R} \to \mathbb{R}$ be any continuous function with the properties

$$\limsup_{k\to\infty} \nu(k) = +\infty \quad \text{and} \quad \liminf_{k\to\infty} \nu(k) = -\infty\,, \tag{4.1}$$

for example, $\nu\colon k \mapsto k \cos k$. Let $\alpha\colon [0, 1) \to \mathbb{R}_+$ be a continuous unbounded injection, for example, $\alpha\colon s \mapsto s/(1 - s)$. Define

$$\mu := \begin{cases} \dfrac{1}{2\sup_{t\in\mathbb{R}_+} \varphi(t)}\,, & \text{if } \varphi \text{ is bounded,} \\ 0, & \text{otherwise.} \end{cases}$$

If $\mu > 0$, let $\mathrm{sat}_\mu\colon \mathbb{R}^m \to \mathcal{B} := \{v \in \mathbb{R}^m |\ \|v\| \leq 1\}$ be any continuous function with the property that $\mathrm{sat}_\mu(e) = \|e\|^{-1}e$ for all $\|e\| > \mu$, in which case the control strategy takes the form

$$u(t) = -\nu(k(t))\mathrm{sat}_\mu(y(t) - r(t)), \qquad k(t) = \alpha(\varphi(t)\|y(t) - r(t)\|).$$

In the case $\mu = 0$, the control strategy is given formally by

$$u(t) = -\nu(k(t))\|y(t) - r(t)\|^{-1}(y(t) - r(t)), \qquad k(t) = \alpha(\varphi(t)\|y(t) - r(t)\|). \tag{4.2}$$

We accommodate each case and the (potential) discontinuity in (4.2) by embedding the control in a set-valued map θ_μ, defined as follows:

$$\theta_\mu(e) = \begin{cases} \{e\|e\|^{-1}\}, & \text{if} \quad \|e\| > \mu, \\ \mathcal{B}, & \text{if} \quad \|e\| \leq \mu, \end{cases}$$

and interpret both control strategies in the following unified, set-valued sense:

$$u(t) \in -\nu(k(t))\theta_\mu(y(t)-r(t)), \qquad k(t) = \alpha(\varphi(t)\|y(t)-r(t)\|). \tag{4.3}$$

The role of the function ν is similar to that of a "Nussbaum" [11] function, commonly invoked in adaptive control, see, for example, [4]. If, for a given linear system (A, B, C) of prototype class $\mathcal{L}$, the polarity of the sign-definite high-frequency gain CB is known *a priori*, then the term $\nu(k(t))$ in (4.3) can be replaced by $k(t)$ if CB is positive definite or by $-k(t)$ if $-CB$ is positive definite.

Care must be exercised in making sense of the closed-loop initial-value problem given by (1.1) and (4.3). The central issue is to establish that $\varphi(t)\|y(t)-r(t)\| \in \mathrm{dom}(\alpha) = [0, 1)$ for all $t \in \mathbb{R}_+$. This we proceed to demonstrate.

4.1. **Closed-loop system**

Let $\lambda \geq 0$, $\varphi \in \Phi_\lambda$, $r \in \mathcal{R}$ and let $\mathcal{D} \subset \mathbb{R}_+ \times \mathbb{R}^m$ denote the set

$$\{(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^m \mid \ \varphi(t)\|\xi - r(t)\| < 1\}.$$

Let $(f, d, T) \in \mathcal{S}$. The conjunction of (1.1) with (4.3) yields the following closed-loop initial-value problem

$$\dot y(t) \in F(t, y(t), (Ty)(t)), \qquad y|_{[-h,0]} = y^0 \in C([-h, 0], \mathbb{R}^m), \tag{4.4}$$

where the set-valued map $(t, y, w) \mapsto F(t, y, w) \subset \mathbb{R}^m$, given by

$$F(t,y,w) := \big\{ f(d(t), w, u) \ \big| \ \ u \in -\nu\big(\alpha(\varphi(t)\|y-r(t)\|)\big)\theta_\mu(y-r(t))\big\}, \tag{4.5}$$

is upper semicontinuous on $\mathcal{D} \times \mathbb{R}^q$ with non-empty, convex, compact values. By a *solution* of (4.4) we mean a function $y \in C(I, \mathbb{R}^m)$ on some interval I of the form $[-h, \rho]$, $0 < \rho < \infty$ or $[-h, \omega)$, $0 < \omega \leq \infty$, such that $y|_{[-h,0]} = y^0$, $y|_J$ is locally absolutely continuous, with $(t, y(t)) \in \mathcal{D}$ for all $t \in J$ and $\dot y(t) \in F(t, y(t), (Ty)(t))$ for almost all $t \in J$, where $J := I \setminus [-h, 0)$. A solution is said to be *maximal* if it has no proper right extension that is also a solution. A solution defined on $[-h, \infty)$ is said to be *global*. We will demonstrate that the control objectives are achieved by establishing that: (i) the initial-value problem (4.4) has a solution; (ii) every solution can be extended to a maximal solution; (iii) every maximal solution is global. Facts (i) and (ii) are a consequence (Cor. 5.2) of the existence theory (Thm. 5.1) developed in Section 5 below; fact (iii) is the essence of the main result in Theorem 6.1. Before proceeding to establish these facts, some commentary on the case $\lambda = 0$ is warranted.

4.1.1. *Commentary on the asymptotic tracking problem*

Assume that $\lambda = 0$, in which case we have $\mu = 0$, and so the formal control structure (4.2) is potentially discontinuous. However, this need not always be the case. For example, with

$$\nu \colon k \mapsto k\cos(ak) \quad \text{and} \quad \alpha \colon s \mapsto \frac{s}{1-s},$$

where $a > 0$, the feedback (4.2) is, in fact, continuous on the domain $\mathcal{D}$: in particular, the control takes the form

$$u(t) = \psi(t, y(t) - r(t)), \tag{4.6}$$

with $\psi \in C(\mathcal{D}, \mathbb{R}^m)$ given by

$$\psi(t,\xi) := -\cos\left(\frac{a\varphi(t)\|\xi\|}{1-\varphi(t)\|\xi\|}\right)\left(\frac{\varphi(t)\xi}{1-\varphi(t)\|\xi\|}\right) \quad \forall\ (t,\xi) \in \mathcal{D}, \tag{4.7}$$

in which case the map F in (4.4) is singleton valued.

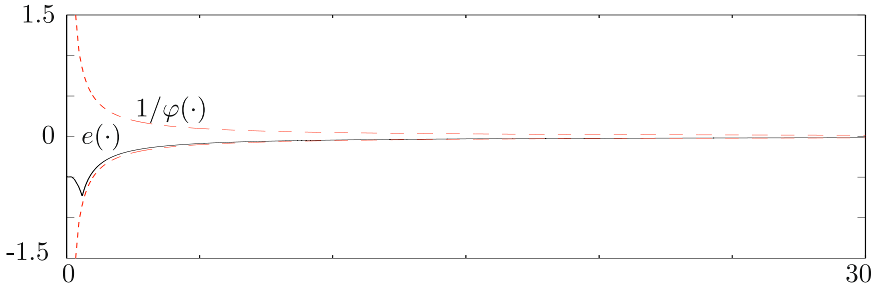

FIGURE 3. The funnel and tracking error e.

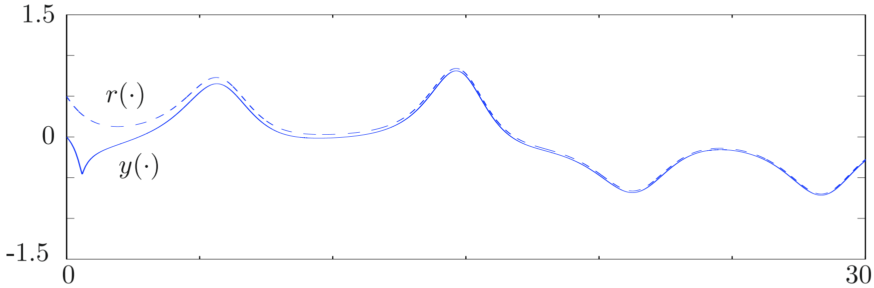

FIGURE 4. The reference signal r and output y.

Example. Consider a single-input, single-output system (3.7) of the nonlinear prototype class, with $f_1, f_2 \colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ given by

$$f_1(y,z) = z \sin y, \quad f_2(y,z) = -z|z| + y, \quad g(u) = u^{1/3}. \tag{4.8}$$

As reference signal $r \in \mathcal{R}$, we take $r = \zeta_1/2$, where ζ_1 is the first component of the (chaotic) solution of the following Lorenz system of equations:

$$\left.\begin{array}{ll} \dot\zeta_1(t) = \zeta_2(t) - \zeta_1(t), & \zeta_1(0) = 1, \\ \dot\zeta_2(t) = c_0\zeta_1(t) - c_1\zeta_2(t) - \zeta_1(t)\zeta_3(t), & \zeta_2(0) = 0, \\ \dot\zeta_3(t) = \zeta_1(t)\zeta_2(t) - c_2\zeta_3(t), & \zeta_3(0) = 3, \end{array}\right\} \tag{4.9}$$

with parameter values $c_0 = 28/10$, $c_1 = 1/10$ and $c_2 = 8/30$. It is well known that the unique global solution of (4.9) is bounded with bounded derivative, see for example [15].

Adopting control parameters $a = 1/4$ and $\varphi\colon t \mapsto 2t$, Figures 3–5 depict the behaviour of the closed-loop system with zero initial state.

There are, of course, practical issues relating to the synthesis of the control strategy (4.6)–(4.7). Whilst later analysis will establish the fact that $\sup_{t\in\mathbb{R}_+} \varphi(t)\|y(t) - r(t)\| < 1$, and so boundedness of the control function u is assured, practical computation of $u(t)$ for large t may encounter numerical ill-conditioning insofar as it involves the product of "large" and "small" quantities (since $\varphi(t) \to \infty$ and $\|y(t) - r(t)\| \to 0$ as $t \to \infty$). These practical issues are not addressed in this paper (the purpose of which is to highlight those performance characteristics that are attainable in principle): however, we remark that the ill-conditioning associated with the case $\mu = 0$ may be circumvented (at the expense of some degradation in performance) on setting $\lambda > 0$ and replacing unbounded φ by a bounded function $\varphi \in \Phi_\lambda$ with $\liminf_{t\in\mathbb{R}_+} \varphi(t) = 1/\lambda$, in which case, the guaranteed performance is weakened to that of approximate tracking, as quantified by $\limsup_{t\to\infty} \|y(t) - r(t)\| < \lambda$.

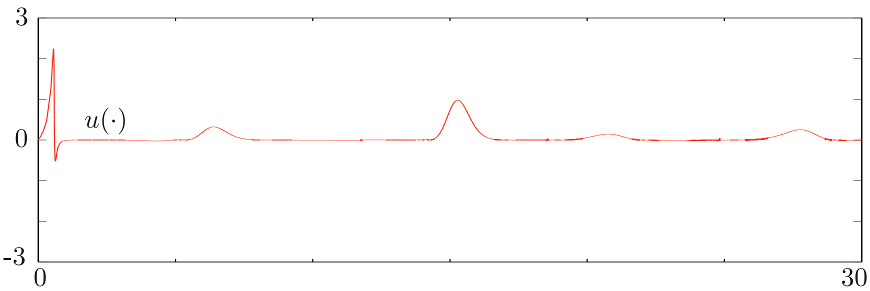

FIGURE 5. The control u.

5. EXISTENCE THEORY

Here, we present an existence theory of sufficient generality to encompass (4.4). Let $\mathcal{D}$ be a domain in $\mathbb{R}_+ \times \mathbb{R}^m$, that is, a non-empty, connected, relatively open subset of $\mathbb{R}_+ \times \mathbb{R}^m$. Let $(t,y,w) \mapsto G(t,y,w) \subset \mathbb{R}^m$ be upper semicontinuous on $\mathcal{G} := \mathcal{D} \times \mathbb{R}^q$, with non-empty, convex and compact values. Let $h \geq 0$ and $T\colon C([-h,\infty),\mathbb{R}^m) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^q)$ be a causal operator of class $\mathcal{T}_h$. For $t_0 \geq 0$, consider the initial-value problem

$$\dot{y}(t) \in G(t, y(t), (Ty)(t)), \quad y|_{[-h,t_0]} = y^0 \in C([-h,t_0],\mathbb{R}^m), \quad (t_0, y^0(t_0)) \in \mathcal{D}. \tag{5.1}$$

We emphasize that, for reasons which will become apparent in the proof of Theorem 5.1 below, the parameter $t_0 \geq 0$ has been incorporated in (5.1): this necessitates the obvious generalization of the earlier concept of a solution introduced in the context of (4.4) wherein $t_0 = 0$. Specifically, by a *solution* of (5.1) we mean a function $y \in C(I,\mathbb{R}^m)$ for some interval I of the form $[-h,\rho]$, $t_0 < \rho < \infty$ or $[-h,\omega)$, $t_0 < \omega \leq \infty$, such that $y|_{[-h,t_0]} = y^0$, $y|_J$ is locally absolutely continuous, $\dot{y}(t) \in G(t,y(t),(Ty)(t))$ for almost all $t \in J$, and $(t,y(t)) \in \mathcal{D}$ for all $t \in J$, where $J := I \setminus [-h,t_0)$. Again, a solution is said to be *maximal* if it has no proper right extension that is also a solution.

Theorem 5.1. *For each $t_0 \geq 0$ and $y^0 \in C([-h,t_0],\mathbb{R}^m)$ with $(t_0,y^0(t_0)) \in \mathcal{D}$,*

(i) *the initial-value problem (5.1) has a solution;*
(ii) *every solution can be extended to a maximal solution $y\colon [-h,\omega) \to \mathbb{R}^m$;*
(iii) *if $y\colon [-h,\omega) \to \mathbb{R}^m$ is a maximal solution of (5.1) and $\omega < \infty$, then, for every $\sigma \in [t_0,\omega)$ and every compact set $\mathcal{K} \subset \mathcal{D}$, there exists $t \in [\sigma,\omega)$ such that $(t,y(t)) \notin \mathcal{K}$.*

A proof of this result can be found in the Appendix.

Corollary 5.2. *Let $(f,d,T) \in \mathcal{S}$, $\lambda \geq 0$ and $\varphi \in \Phi_\lambda$. Then, for every reference signal $r \in \mathcal{R}$ and all initial data $y^0 \in C([-h,0],\mathbb{R}^m)$, application of the feedback (4.3) to the system (1.1) yields the initial-value problem (4.4)–(4.5) which has a solution and every solution can be extended to a maximal solution $y\colon [-h,\omega) \to \mathbb{R}^m$, $0 < \omega \leq \infty$. Furthermore, if $y\colon [-h,\omega) \to \mathbb{R}^m$ is a maximal solution and there exists a compact set $\mathcal{K} \subset \mathcal{D}$ such that $(t,y(t)) \in \mathcal{K}$ for all $t \in [\sigma,\omega)$, then $\omega = \infty$.*

Proof. Defining the domain $\mathcal{D} := \{(t,y) \in \mathbb{R}_+ \times \mathbb{R}^m \mid \varphi(t)\,\|y - r(t)\| < 1\}$, we identify the initial-value problem (4.4)–(4.5) as a particular case of (5.1) (with $G = F$ and $t_0 = 0$):

$$\dot{y}(t) \in F(t, y(t), (Ty)(t)), \quad y|_{[-h,0]} = y^0 \in C([-h,0],\mathbb{R}^m), \quad (0, y^0(0)) \in \mathcal{D}, \tag{5.2}$$

where $F(t,y,w) = \{f(d(t),w,u) \mid u \in -\nu(\alpha(\varphi(t)\|y - r(t)\|))\theta_\mu(y - r(t))\}$.

An application of Theorem 5.1 completes the proof. $\square$

6. Main result

We now arrive at the main result, statement (ii) of which asserts that the output of the closed-loop system evolves within the performance funnel and is bounded away from the funnel boundary.

Theorem 6.1. *Let $(f, d, T, h) \in \mathcal{S}$, $\lambda \geq 0$ and $\varphi \in \Phi_\lambda$. Then for every reference signal $r \in \mathcal{R}$ and all initial data $y^0 \in C([-h, 0], \mathbb{R}^m)$, application of the feedback (4.3) to the system (1.1) yields the closed-loop initial-value problem (4.4)–(4.5) which has a solution and each solution can be extended to a maximal solution $y\colon [-h, \omega) \to \mathbb{R}^m$. Every maximal solution $y\colon [-h, \omega) \to \mathbb{R}^m$ has the properties:*

(i) $\omega = \infty$;
(ii) $\sup_{t\in\mathbb{R}_+} \varphi(t)\|y(t) - r(t)\| < 1$;
(iii) *the function $k\colon t \mapsto \alpha(\varphi(t)\|y(t) - r(t)\|)$ is bounded.*

Remark 6.2. The conjunction of assertions (i) and (ii) ensures that both control objectives are attained. Assertion (iii) implies boundedness of the control. In the case where $\varphi(t) \to \infty$ as $t \to \infty$, assertion (ii) implies asymptotic tracking: $\|y(t) - r(t)\| \to 0$ as $t \to \infty$.

Proof. Let $r \in \mathcal{R}$ and $y^0 \in C([-h, 0], \mathbb{R}^m)$. By Corollary 5.2, the closed-loop initial-value problem (4.4)–(4.5) has a solution and every solution can be maximally extended. Let $y\colon [-h, \omega) \to \mathbb{R}^m$ be a maximal solution of (4.4). Defining $e(t) = y(t) - r(t)$ for all $t \in [0, \omega)$, we have

$$\dot e(t) + \dot r(t) \in F(t, e(t) + r(t), (Ty)(t)) \quad \text{for a.a. } t \in [0, \omega). \tag{6.1}$$

Since $(t, y(t)) \in \mathcal{D}$ for all $t \in [0, \omega)$, it follows that $\varphi(t)\|e(t)\| < 1$ for all $t \in [0, \omega)$. By properties of $\varphi \in \Phi_\lambda$, we may infer boundedness of the function e. Furthermore, since $r \in \mathcal{R}$ is bounded, we may conclude that y is bounded. Invoking Assumptions (A3) and (A4) (in particular, property (iv) of the operator class $\mathcal{T}_h$), we deduce the existence of a non-empty, compact set $\mathcal{K} \subset \mathbb{R}^p \times \mathbb{R}^q$ such that $(d(t), (Ty)(t)) \in \mathcal{K}$ for almost all $t \in [0, \omega)$. With this set, we associate the function $\gamma_\mathcal{K}$, defined as in (3.1). Writing

$$\Sigma := \{t \in [0, \omega) \mid \|e(t)\| > \mu\}, \quad \text{and} \quad k(t) := \alpha(\varphi(t)\|e(t)\|) \quad \forall\, t \in [0, \omega),$$

we have

$$\begin{aligned} t \in \Sigma \implies &\langle e(t), f(d(t), (Ty)(t), -\nu(k(t))\|e(t)\|^{-1} e(t))\rangle \\ &\leq -\|e(t)\| \;\min\{\langle u, f(v, w, \nu(k(t))u)\rangle \mid \; (v, w) \in \mathcal{K},\; \|u\| = 1\} \\ &= -\|e(t)\|\gamma_\mathcal{K}(\nu(k(t))). \end{aligned} \tag{6.2}$$

Noting that

$$t \in \Sigma \quad \implies \quad F(t, e(t) + r(t), (Ty)(t)) = \big\{f(d(t), (Ty)(t), -\nu(k(t))\|e(t)\|^{-1} e(t))\big\},$$

we may infer from (6.2) that

$$\langle e(t), v\rangle \leq -\gamma_\mathcal{K}(\nu(k(t)))\|e(t)\| \quad \forall\, v \in F(t, e(t) + r(t), (Ty)(t)), \quad \forall\, t \in \Sigma.$$

Therefore, by (6.1) and essential boundedness of $\dot r$, there exists $c_0 > 0$ such that

$$\langle e(t), \dot e(t)\rangle \leq \big[c_0 - \gamma_\mathcal{K}(\nu(k(t)))\big]\,\|e(t)\| \quad \text{for a.a. } t \in \Sigma. \tag{6.3}$$

By Assumption A2, either (i) $\limsup_{s\to+\infty} \gamma_\mathcal{K}(s) = \infty$, or (ii) $\limsup_{s\to-\infty} \gamma_\mathcal{K}(s) = \infty$. Therefore, there exists an unbounded sequence $(s_n) \subset \mathbb{R}$, which is either strictly increasing (in case (i)) or strictly decreasing (in case (ii)), such that the sequence $(\gamma_\mathcal{K}(s_n))$ is unbounded and strictly increasing, with $\gamma_\mathcal{K}(s_n) > 0$ for all $n \in \mathbb{N}$. By properties (4.1) and continuity of ν, for every $a, b \in \mathbb{R}$ the set $\{\kappa > a | \ \nu(\kappa) = b\}$ is non-empty. Let $k_1 \in \{\kappa > \alpha(\frac{1}{2}) | \ \nu(\kappa) = s_1\}$ be arbitrary and define the strictly-increasing unbounded sequence (k_n) in $(\alpha(\frac{1}{2}), \infty)$ by the recursion $k_{n+1} := \inf\{\kappa > k_n | \ \nu(\kappa) = s_{n+1}\}$, and so $\gamma_{\mathcal{K}}(\nu(k_n)) = \gamma_{\mathcal{K}}(s_n) \to \infty$ as $n \to \infty$.

We proceed to prove boundedness of k. Seeking a contradiction, suppose k is unbounded (in which case, $\mathrm{im}(k) = \mathrm{im}(\alpha) = [\alpha(0), \infty)$). For each $n \in \mathbb{N}$, define

$$\tau_n := \inf\{t \in [0, \omega) | \ k(t) = k_{n+1}\} \quad \text{and} \quad \sigma_n := \sup\{t \in [0, \tau_n] | \ \gamma_{\mathcal{K}}(\nu(k(t))) = \gamma_{\mathcal{K}}(\nu(k_n))\}.$$

We briefly digress to assemble some useful facts.

Proposition 6.3. (a) $\sigma_n < \tau_n \ \ \forall \, n \in \mathbb{N}$. (b) $k(\sigma_n) < k(\tau_n) \ \ \forall \, n \in \mathbb{N}$. (c) $k(t) \geq k_n \ \ \forall \, t \in [\sigma_n, \tau_n] \ \ \forall \, n \in \mathbb{N}$. (d) $\gamma_{\mathcal{K}}(\nu(k(t))) \geq \gamma_{\mathcal{K}}(\nu(k_n)) > 0 \ \ \ \forall \, t \in [\sigma_n, \tau_n] \ \ \forall \, n \in \mathbb{N}$. (e) $[\sigma_n, \tau_n] \subset \Sigma \ \ \forall \, n \in \mathbb{N}$.

Proof. (a) Suppose, for contradiction, that $\sigma_n = \tau_n$ for some $n \in \mathbb{N}$. Then,

$$\gamma_{\mathcal{K}}(s_{n+1}) = \gamma_{\mathcal{K}}(\nu(k_{n+1})) = \gamma_{\mathcal{K}}(\nu(k(\tau_n))) = \gamma_{\mathcal{K}}(\nu(k(\sigma_n))) = \gamma_{\mathcal{K}}(\nu(k_n)) = \gamma_{\mathcal{K}}(s_n),$$

which contradicts strict monotonicity of the sequence $(\gamma_{\mathcal{K}}(s_n))$.

(b) Suppose, for contradiction, that $k(\sigma_n) \geq k(\tau_n) = k_{n+1}$ for some $n \in \mathbb{N}$. Then, since $k(0) = \alpha(0) < \alpha(1/2) < k_{n+1}$, there exists $s \leq \sigma_n < \tau_n$ such that $k(s) = k_{n+1}$, whence the contradiction: $\tau_n = \inf\{t \in [0, \omega) | \ k(t) = k_{n+1}\} \leq s < \tau_n$.

(c) Suppose, for contradiction, that, for some $n \in \mathbb{N}$ and $t \in [\sigma_n, \tau_n]$, $k(t) < k_n$. Then, since $k(\tau_n) = k_{n+1}$, there exists $s \in (\sigma_n, \tau_n]$ such that $k(s) = k_n$. Invoking the definition of σ_n, we arrive at a contradiction: $\sigma_n < s \leq \sigma_n$.

(d) Suppose, for contradiction, that, for some $n \in \mathbb{N}$ and $t \in [\sigma_n, \tau_n]$, $\gamma_{\mathcal{K}}(\nu(k(t))) < \gamma_{\mathcal{K}}(\nu(k_n))$. Since

$$\gamma_{\mathcal{K}}(\nu(k_n)) = \gamma_{\mathcal{K}}(s_n) < \gamma_{\mathcal{K}}(s_{n+1}) = \gamma_{\mathcal{K}}(\nu(k_{n+1})) = \gamma_{\mathcal{K}}(\nu(k(\tau_n))),$$

it follows that, for some $s \in (\sigma_n, \tau_n]$, $\gamma_{\mathcal{K}}(\nu(k(s))) = \gamma_{\mathcal{K}}(\nu(k_n))$, which contradicts the definition of σ_n.

(e) Suppose, for contradiction, that, for some $n \in \mathbb{N}$, there exists $t \in [\sigma_n, \tau_n]$ such that $t \notin \Sigma$, then $\|e(t)\| \leq \mu$. Note that $\alpha(0) < \alpha(1/2)$ and, if $\mu > 0$, then $\alpha(\mu\varphi(t)) \leq \alpha(1/2)$. Therefore, we arrive at a contradiction.

$$\alpha(1/2) < k_n \leq k(t) = \alpha(\varphi(t)\|e(t)\|) \leq \alpha(1/2). \qquad \square$$

We now return to the proof of Theorem 6.1. From assertions (c) and (d) of Proposition 6.3, we may infer that

$$\frac{1}{2} < \alpha^{-1}(k_n) \leq \alpha^{-1}(k(t)) = \varphi(t)\|e(t)\| < 1 \quad \forall \, t \in [\sigma_n, \tau_n] \ \ \forall \, n \in \mathbb{N}, \tag{6.4}$$

where $\alpha^{-1}\colon [\alpha(0), \infty) \to [0, 1)$ is the inverse of the bijection $\alpha\colon [0, 1) \to \mathrm{im}(\alpha)$, and

$$-\,2\varphi^2(t)\|e(t)\|\gamma_{\mathcal{K}}(\nu(k(t))) \leq -\varphi(t)\gamma_{\mathcal{K}}(\nu(k(t))) \quad \forall \, t \in [\sigma_n, \tau_n] \ \ \forall \, n \in \mathbb{N}. \tag{6.5}$$

By properties of $\varphi \in \Phi_\lambda$, there exists $c_1 > 0$ such that $\dot{\varphi}(t) \leq c_1[1 + \varphi(t)]$ for almost all t which, together with (6.3), yields, for almost all $t \in \Sigma$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\big[\varphi(t)\|e(t)\|\big]^2 &= 2\,\varphi(t)\dot{\varphi}(t)\,\|e(t)\|^2 + 2\,\varphi^2(t)\,\langle e(t), \dot{e}(t)\rangle \\ &\leq \ 2c_1\,\varphi(t)[1 + \varphi(t)]\|e(t)\|^2 + 2\,\varphi^2(t)\|e(t)\|\big(c_0 - \gamma_{\mathcal{K}}(\nu(k(t)))\big). \end{aligned}$$

Invoking (6.4), (6.5) and boundedness of e, we may conclude the existence of $c_2 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\varphi(t)\|e(t)\|\big]^2 \leq \varphi(t)\big[c_2 - \gamma_{\mathcal{K}}(\nu(k(t)))\big] \quad \text{for a.a. } t \in [\sigma_n, \tau_n], \quad \forall \, n \in \mathbb{N}. \tag{6.6}$$

Fix $n \in \mathbb{N}$ sufficiently large so that $c_2 - \gamma_{\mathcal{K}}(\nu(k_n)) < 0$. Recalling that $\gamma_{\mathcal{K}}(\nu(k(t))) \geq \gamma_{\mathcal{K}}(\nu(k_n))$ for all $t \in [\sigma_n, \tau_n]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\varphi(t)\|e(t)\|\right]^2 < 0 \quad \text{for a.a. } t \in [\sigma_n, \tau_n]$$

and so $\varphi(\tau_n)\|e(\tau_n)\| < \varphi(\sigma_n)\|e(\sigma_n)\|$. Therefore,

$$k(\tau_n) = \alpha\big(\varphi(\tau_n)\|e(\tau_n)\|\big) < \alpha\big(\varphi(\sigma_n)\|e(\sigma_n)\|\big) = k(\sigma_n),$$

which contradicts assertion (b) of Proposition 6.3. This proves boundedness of k (and so $\nu \circ k \colon t \mapsto \nu(\alpha(\varphi(t)\|y(t) - r(t)\|))$ is also bounded). By boundedness of $t \mapsto k(t) = \alpha(\varphi(t)\|e(t)\|)$, it follows that $\sup_{t\in[0,\omega)} \varphi(t)\|y(t) - r(t)\| < 1$, equivalently, there exists $\varepsilon \in (0,1)$ such that $\varphi(t)\|y(t) - r(t)\| \leq 1 - \varepsilon$ for all $t \in [0,\omega)$.

Finally, we show that $\omega = \infty$. By boundedness of y, there exists $c_3 > 0$ such that $\|y(t)\| \leq c_3$ for all $t \in [0,\omega)$. Suppose $\omega < \infty$. Then

$$\tilde{\mathcal{K}} := \{(t,v) \in \mathbb{R}_+ \times \mathbb{R}^m \,|\; \varphi(t)\|v - r(t)\| \leq 1 - \varepsilon,\ \|v\| \leq c_3,\ t \in [0,\omega]\}$$

is a compact subset of $\mathcal{D}$ with the property $(t, y(t)) \in \tilde{\mathcal{K}}$ for all $t \in [0,\omega)$, which contradicts assertion (iii) of Theorem 5.1. Therefore, $\omega = \infty$. This completes the proof. □

Remark 6.4. To paraphrase Wonham [17], p. 210, the *internal model principle* states that every "good" regulator must incorporate a model of the outside world (in the sense that the feedback loop incorporates a suitably reduplicated model of the dynamic structure of the exogenous signals which the closed-loop system is required to track). In the context of linear systems with linear regulators (see [16,17]), "good" means "structurally stable"; in a more general context of smooth nonlinear systems (see [14]), "good" amounts to a "signal detection" property. In effect, "good" implies some robustness property of the closed loop. The feedback structure proposed in the present paper ensures tracking of any signal of class $W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^m)$, yet it does not contain a model capable of replicating this class of signals. For consistency with the internal model principle, one must therefore conclude that the closed-loop system of the present paper lacks certain robustness properties. This perceived lack of robustness may stem from the potential singularity introduced *via* the injection α in the closed loop or from the unbounded nature of the funnel function φ. It is not unreasonable to expect that the adoption of a bounded function φ (with attendant reduction in performance from asymptotic to approximate tracking) might induce some robustness in the closed loop. However, in the absence of a rigorous robustness analysis, the results of the paper are mainly of a theoretical nature, serving to illustrate those performance characteristics that are attainable, in principle, under weak assumptions on the plant data.

A. Appendix: proof of Theorem 5.1

Let X be a normed vector space. The open ball of radius $\varepsilon > 0$ centred at $x \in X$ is denoted by $\mathbb{B}_\varepsilon(x)$ (the ambient space X being clear from context), $\overline{\mathbb{B}}_\varepsilon(x)$ denotes the closure of $\mathbb{B}_\varepsilon(x)$: if $x = 0$, then, for simplicity, we write $\mathbb{B}_\varepsilon$ in place of $\mathbb{B}_\varepsilon(0)$.

We record the following properties of G:

$$\left.\begin{array}{ll}
\text{(a)} & \operatorname{graph}(G) := \{(z,\zeta) \,|\;\ \zeta \in G(z),\ z \in \mathcal{G}\} \text{ is closed;} \\[4pt]
\text{(b)} & \text{if } \mathcal{K} \subset \mathcal{G} \text{ is compact, then } G(\mathcal{K}) := \cup_{z\in\mathcal{K}} G(z) \text{ is compact;} \\[4pt]
\text{(c)} & \text{for each } \varepsilon > 0, \text{ there exists a locally Lipschitz function } g\colon \mathcal{G} \to \mathbb{R}^m \\
 & \text{such that } \operatorname{graph}(g) \subset \operatorname{graph}(G) + \mathbb{B}_\varepsilon\,.
\end{array}\right\} \tag{A.1}$$

For (a) see [1], Proposition 2, p. 41, for (b) see [1], Proposition 3, p. 42, for (c) see [1], Theorem 1, p. 84.

To facilitate the proof of the general result in Theorem 5.1, we first establish a variant in the restricted context wherein G is a singleton-valued map $G\colon (t,y,w) \mapsto \{g(t,y,w)\}$ and $g\colon \mathcal{G} \to \mathbb{R}^m$ is locally Lipschitz.

Lemma A.1. *Let $g\colon \mathcal{G} \to \mathbb{R}^m$ be a locally Lipschitz function. For $t_0 \geq 0$ and $y^0 \in C([-h,t_0],\mathbb{R}^m)$, the initial-value problem*

$$\dot{y}(t) = g(t, y(t), (Ty)(t)), \quad y|_{[-h,t_0]} = y^0 \in C([-h,t_0],\mathbb{R}^m), \quad (t_0, y^0(t_0)) \in \mathcal{D}, \tag{A.2}$$

has a unique maximal solution, $y\colon [-h,\omega) \to \mathbb{R}^m$. Furthermore, if $\omega < \infty$, then, for every $\sigma \in [t_0,\omega)$ and every compact set $\mathcal{K} \subset \mathcal{D}$, there exists $t \in [\sigma,\omega)$ such that $(t,y(t)) \notin \mathcal{K}$.

Proof. **Step 1:** Existence of a unique solution on a small interval.

By property (iii) of $T \in \mathcal{T}_h$, there exist $\delta > 0$, $c_0 > 0$ and $\tau > t_0$ such that

$$\|(Ty)(t) - (Tz)(t)\| \leq c_0 \max_{s\in[t_0,\tau]} \|y(s) - z(s)\| \text{ for a.a. } t \in [t_0,\tau] \text{ and all } y,z \in \mathcal{C}(y^0; h, t_0, \tau, \delta). \tag{A.3}$$

Without loss of generality, we may assume that $\delta \in (0,1)$ and $\tau - t_0 > 0$ are sufficiently small so that $[t_0,\tau] \times \overline{\mathbb{B}}_\delta(y^0(t_0)) \subset \mathcal{D}$. For each $\rho \in (t_0,\tau]$, define $C_\rho := \mathcal{C}(y^0, h, t_0, \rho, \delta)$ which, equipped with the metric

$$(y,z) \mapsto \beta_\rho(y,z) := \sup_{t\in[-h,\rho]} \|y(t) - z(t)\|,$$

is a complete metric space. Observe that, if $y \in C_\rho$, then $(t,y(t)) \in \mathcal{D}$ for all $t \in [t_0,\rho]$. For each $\rho \in (t_0,\tau]$, define the operator Z_ρ on C_ρ by

$$(Z_\rho y)(t) := \begin{cases} y^0(t), & t \in [-h,t_0], \\ y^0(t_0) + \int_{t_0}^t g(s,y(s),(Ty)(s))\mathrm{d}s, & t \in (t_0,\rho). \end{cases}$$

We proceed to show that Z_ρ is a contraction. Define $c_1 := \max_{s\in[-h,t_0]} \|y^0(s)\| + \delta$. By property (iv) of T, there exists $c_2 > 0$ such that

$$\sup_{t\in[-h,\tau]} \|y(t)\| < c_1 \quad \Longrightarrow \quad \|(Ty)(t)\| < c_2 \quad \text{for a.a. } t \in [t_0,\tau].$$

By the local Lipschitz property of g, there exists a constant $c_3 > 0$ such that, for all $t \in [t_0,\tau]$,

$$\|g(t,y,w) - g(t,z,x)\| \leq c_3 \Big[\|y - z\| + \|w - x\|\Big] \quad \forall\, y,z \in \mathbb{B}_{c_1}, \quad \forall\, w,x \in \mathbb{B}_{c_2}.$$

Write

$$g^* := \max\{\|g(t,y,w)\| \mid (t,y,w) \in [t_0,\tau] \times \overline{\mathbb{B}}_\delta(y^0(t_0)) \times \overline{\mathbb{B}}_{c_2}\}.$$

Fix $\rho^* \in (t_0,\tau]$ sufficiently close to t_0 so that

$$(\rho^* - t_0)(g^* + (c_0 + 1)c_3) < \delta.$$

Let $\rho \in (t_0,\rho^*]$ and $y \in C_\rho$. By definition, $(Z_\rho y)|_{[-h,t_0]} = y^0$ and

$$\begin{aligned} \|(Z_\rho y)(t) - y^0(t_0)\| &= \left\| \int_{t_0}^t g(s,y(s),(Ty)(s))\mathrm{d}s \right\| \\ &\leq \int_{t_0}^\rho \|g(s,y(s),(Ty)(s))\|\mathrm{d}s \leq (\rho - t_0)g^* < \delta \quad \forall\, t \in [t_0,\rho]. \end{aligned}$$

Therefore $(Z_\rho y)(\cdot) \in C_\rho$. Furthermore,

$$\begin{aligned}\beta_\rho(Z_\rho y, Z_\rho z) &= \sup_{t\in[t_0,\rho]} \left\| \int_{t_0}^{t} \left[g(s,y(s),(Ty)(s)) - g(s,z(s),(Tz)(s))\right]\mathrm{d}s\right\| \\ &\le \int_{t_0}^{\rho} \|g(s,y(s),(Ty)(s)) - g(s,z(s),(Tz)(s))\|\mathrm{d}s \\ &\le (\rho - t_0)c_3 \left[\operatorname*{ess-sup}_{s\in[t_0,\rho]} \|(Ty)(s) - (Tz)(s)\| + \beta_\rho(y,z)\right] \\ &\le (c_0+1)(\rho-t_0)c_3\, \beta_\rho(y,z) \quad \forall\, y,z \in C_\rho,\end{aligned}$$

wherein the last inequality follows by (A.3). Since $(c_0+1)(\rho-t_0)c_3 < \delta < 1$, we may infer that $Z_\rho \colon C_\rho \to C_\rho$ is a contraction. By the contraction mapping theorem, Z_ρ has a unique fixed point. Thus we have shown that, for each $\rho \in (t_0, \rho^*]$, the initial-value problem (A.2) has a unique solution $y \in C_\rho$. We stress that the uniqueness property of y holds only in relation to solutions in the restricted class C_ρ: there may exist another solution on the interval $[-h,\rho]$ which is not contained in the space C_ρ. However, the following argument establishes uniqueness of the solution on a sufficiently small interval. Let y^* (not necessarily in C_{ρ^*}) be a solution on $[-h,\rho^*]$. Define

$$\Delta := \{t \in [t_0,\rho^*] \mid \|y^*(t) - y^0(t_0)\| = \delta\}, \qquad \rho := \begin{cases} \inf\Delta, & \Delta \ne \emptyset, \\ \rho^*, & \Delta = \emptyset. \end{cases}$$

Clearly $\rho > t_0$ and $y := y^*|_{[-h,\rho]}$ is in C_ρ. Therefore, y is the unique solution of (A.2) on the interval $[-h,\rho]$.

Step 2: Extended uniqueness: any two solutions must coincide on the intersection of their domains.

Let $y_1 \colon I_1 \to \mathbb{R}^m$ and $y_2 \colon I_2 \to \mathbb{R}^m$ be solutions of (A.2) and, without loss of generality, assume $I_2 \subset I_1$. For contradiction, suppose that $y_1|_{I_2} \ne y_2$. Let $t^* := \inf\{t \in I_2 \mid y_1(t) \ne y_2(t)\}$. By the result in Step 1, the solutions y_1 and y_2 must coincide on some interval $[-h,\rho]$, with $\rho > t_0$. Therefore, $t^* > t_0$. An application of the result of Step 1 in the context of an initial-value problem of the form (A.2), with t^* replacing t_0 and initial function $y_1|_{[-h,t^*]} \in C([-h,t^*],\mathbb{R}^m)$ replacing y^0, yields the existence of a unique solution $y \in C([-h,\rho],\mathbb{R}^m)$ for some $\rho > t^*$. It follows that $y_1(t) = y_2(t) = y(t)$ for all $t \in [-h,\rho]$, contradicting the definition of t^*.

Step 3: Existence of a unique maximal solution.

Let $\mathcal{P}$ be the set of all $\rho > t_0$ such that there exists a solution y_ρ of (A.2) on the interval $[-h,\rho]$. By Step 1, we know that $\mathcal{P} \ne \emptyset$. Let $\omega := \sup\mathcal{P}$ and define $y \colon [-h,\omega) \to \mathbb{R}^m$ by the property

$$y|_{[-h,\rho]} = y_\rho \quad \forall\, \rho \in \mathcal{P}.$$

The function y is well-defined since, by Step 2, for all $\rho_1,\rho_2 \in \mathcal{P}$, we have $y_{\rho_2} = y_{\rho_1}|_{[-h,\rho_2]}$ whenever $\rho_2 \le \rho_1$. Clearly y is a maximal solution and uniqueness follows by Step 2.

Step 4: Assume that $y \colon [-h,\omega) \to \mathbb{R}^m$ is a maximal solution with $\omega < \infty$. Seeking a contradiction, suppose there exist $\sigma \in [t_0,\omega)$ and a compact set $\mathcal{K} \subset \mathcal{D}$ such that $(t,y(t)) \in \mathcal{K}$ for all $t \in [\sigma,\omega)$. Then y is bounded and, by property (iv) of $T \in \mathcal{T}_h$, Ty is essentially bounded. Therefore, the function $t \mapsto (t,y(t),(Ty)(t))$ is essentially bounded and so, by continuity of g, it follows that $\dot{y}$ is essentially bounded on the interval $[t_0,\omega)$. Therefore y is uniformly continuous on $[-h,\omega)$ and so extends to $y^* \in C([-h,\omega],\mathbb{R}^m)$. By compactness of $\mathcal{K}$, we have $(\omega, y^*(\omega)) \in \mathcal{K} \subset \mathcal{D}$. An application of the result of Step 1 in the context of an initial-value problem of the form (A.2), with ω replacing t_0 and y^* replacing y^0, yields the existence of a unique solution $y^e \in C([-h,\rho],\mathbb{R}^m)$ for some $\rho > \omega$, with $y^e|_{[-h,\omega)} = y$. This contradicts maximality of y. □

We are now in a position to prove the existence of a solution to the problem (5.1).

Proof of Theorem 5.1

Proof. (i) Let $(\varepsilon_n) \subset (0,1)$ be a monotonically decreasing sequence with $\varepsilon_n \to 0$ as $n \to \infty$. By property (A.1c), for each $n \in \mathbb{N}$, there exists a locally Lipschitz function $g_n \colon \mathcal{G} \to \mathbb{R}^m$ with

$$\mathrm{graph}(g_n) \subset \mathrm{graph}(G) + \mathbb{B}_{\varepsilon_n}. \tag{A.4}$$

By Lemma A.1, for each $n \in \mathbb{N}$, the initial-value problem

$$\dot y(t) = g_n(t, y(t), (Ty)(t)), \quad y|_{[-h,t_0]} = y^0 \in C([-h,t_0],\mathbb{R}^m), \quad (t_0, y^0(t_0)) \in \mathcal{D},$$

has a unique maximal solution which we denote by $y_n \colon [-h,\omega_n) \to \mathbb{R}^m$.

Recalling that $\mathcal{D}$ is a relatively open subset of $\mathbb{R}_+ \times \mathbb{R}^m$ and invoking property (iii) of $T \in \mathcal{T}_h$, we may choose $\delta > 0$ sufficiently small and $\omega^* > t_0$ sufficiently close to t_0 so that

$$[t_0, \omega^*] \times \overline{\mathbb{B}}_\delta(y^0(t_0)) =: \mathcal{K}_0 \subset \mathcal{D},$$

and there exists $c_0 > 0$ such that

$$\operatorname*{ess\text{-}sup}_{t\in[t_0,\omega^*]} \|(Ty)(t) - (Tz)(t)\| \le c_0 \max_{t\in[t_0,\omega^*]} \|y(t) - z(t)\| \quad \forall\, y, z \in \mathcal{C}(y^0; h, t_0, \omega^*, \delta). \tag{A.5}$$

For each $n \in \mathbb{N}$, define

$$\omega_n^* := \min\{\omega^*, \omega_n\},\ \ \Delta_n := \{t \in [t_0, \omega_n^*) \mid \|y_n(t) - y^0(t_0)\| = \delta\},\ \ \rho_n := \begin{cases} \inf \Delta_n, & \text{if } \Delta_n \neq \emptyset, \\ \omega_n^*, & \text{if } \Delta_n = \emptyset. \end{cases}$$

We claim that $\rho_n < \omega_n$ for all $n \in \mathbb{N}$. Suppose otherwise, then there exists $n \in \mathbb{N}$ such that $\rho_n = \omega_n$. It follows that $\Delta_n = \emptyset$ and so $\omega_n = \omega_n^* \le \omega^*$. Therefore, $(t, y_n(t)) \in \mathcal{K}_0 \subset \mathcal{D}$ for all $t \in [t_0, \omega_n)$, contradicting the final assertion of Lemma A.1. Therefore, $\rho_n < \omega_n$ for all $n \in \mathbb{N}$. Furthermore, for each $n \in \mathbb{N}$, $y_n(t) \in \overline{\mathbb{B}}_\delta(y^0(t_0))$ for all $t \in [t_0, \rho_n]$ and so

$$\|y_n(t)\| \le c_1 := \max_{s\in[-h,t_0]} \|y^0(s)\| + \delta \quad \text{for all } t \in [-h, \rho_n] \text{ and all } n \in \mathbb{N}.$$

By property (iv) of $T \in \mathcal{T}_h$, there exists $c_2 > 0$ such that

$$\|(Ty_n)(t)\| \le c_2 \quad \text{for a.a. } t \in [t_0, \rho_n] \ \text{ and all } n \in \mathbb{N}.$$

Write $\mathcal{K}_1 := \mathcal{K}_0 \times \overline{\mathbb{B}}_{c_2}$ and observe

$$(t, y_n(t), (Ty_n)(t)) \in \mathcal{K}_1 \quad \text{for a.a. } t \in [t_0, \rho_n] \text{ and all } n \in \mathbb{N}.$$

By property (A.1b) of G, the set $\mathcal{K}_2 := G(\mathcal{K}_1)$ is compact. Let $c_3 := 1 + \max_{v\in\mathcal{K}_2} \|v\|$. Then, in view of (A.4),

$$\|g_n(t, y, w)\| < c_3 \quad \text{for all } (t, y, w) \in \mathcal{K}_1 \text{ and all } n \in \mathbb{N}. \tag{A.6}$$

Therefore,

$$\begin{aligned} \|y_n(\rho_n) - y^0(t_0)\| \le \int_{t_0}^{\rho_n} \|\dot y_n(t)\| \mathrm{d}t &= \int_{t_0}^{\rho_n} \|g_n(t, y_n(t), (Ty_n)(t))\| \mathrm{d}t \\ &< c_3 |\rho_n - t_0| \quad \forall\, n \in \mathbb{N}. \end{aligned} \tag{A.7}$$

Next, define $\rho := \inf_{n\in\mathbb{N}} \rho_n \geq t_0$. Seeking a contradiction, suppose $\rho = t_0$. Fix $n \in \mathbb{N}$ sufficiently large so that $c_3|\rho_n - t_0| < \delta$ and $\rho_n < \omega^*$. Recalling that $\rho_n < \omega_n$, we have $\rho_n < \min\{\omega^*, \omega_n\} = \omega_n^*$ and so $\Delta_n \neq \emptyset$ and we arrive at a contradiction:

$$\delta = \|y_n(\rho_n) - y^0(t_0)\| < c_3|\rho_n - t_0| < \delta.$$

Therefore $\rho \in (t_0, \omega^*]$. For each $n \in \mathbb{N}$, define

$$z_n := y_n|_{[t_0,\rho]} \quad \text{and} \quad w_n := (Ty_n)|_{[t_0,\rho]}.$$

For all $t \in [t_0, \rho]$, $(z_n(t)) \subset \overline{\mathbb{B}}_\delta(y^0(t_0))$ and by (A.6),

$$\|\dot{z}_n(t)\| < c_3 \quad \text{for a.a. } t \in [t_0, \rho] \text{ and all } n \in \mathbb{N}. \tag{A.8}$$

Therefore, the sequence $(z_n) \subset C([t_0, \rho], \mathbb{R}^m)$ is uniformly bounded and equicontinuous. By the Arzelà-Ascoli theorem, and extracting a subsequence if necessary, we may assume that (z_n) converges uniformly to $z \in C([t_0, \rho], \mathbb{R}^m)$.

To complete the proof of assertion (i), we adopt an argument akin to that used in the proof of [3], Theorem 3.1.7 and [9], Theorem 2D.5.

By weak*-compactness of the unit ball in $L^\infty([t_0, \rho], \mathbb{R}^m)$ (Alaoglu's theorem), together with (A.8), the sequence $(\dot{z}_n) \subset L^\infty([t_0, \rho], \mathbb{R}^m) \subset L^1([t_0, \rho], \mathbb{R}^m)$ has a subsequence (which we do not relabel) with weak*-limit $v \in L^\infty([t_0, \rho], \mathbb{R}^m)$, that is,

$$\lim_{n\to\infty} \int_{t_0}^{\rho} \langle p(t), \dot{z}_n(t)\rangle \,\mathrm{d}t = \int_{t_0}^{\rho} \langle p(t), v(t)\rangle \,\mathrm{d}t \quad \forall\, p \in L^1([t_0, \rho], \mathbb{R}^m) \tag{A.9}$$

and so, *a fortiori*, the sequence $(\dot{z}_n)$ converges weakly in $L^1([t_0, \rho], \mathbb{R}^m)$ to v. Let $\{e_1, ..., e_m\}$ be a basis for $\mathbb{R}^m$. For $k = 1, ..., m$ and $t \in [t_0, \rho]$, define $p_{k,t} \in L^1([t_0, \rho], \mathbb{R}^m)$ by

$$p_{k,t}(s) := \begin{cases} e_k, & s \in [t_0, t] \\ 0, & \text{otherwise.} \end{cases}$$

Setting $p = p_{k,t}$ ($k = 1, ..., m$ and $t \in [t_0, \rho]$) in (A.9) and integrating, we may now conclude that

$$z(t) = \lim_{n\to\infty} z_n(t) = y^0(t_0) + \int_{t_0}^{t} v(s)\mathrm{d}s \quad \forall\, t \in [t_0, \rho].$$

Therefore, $z \in AC([t_0, \rho], \mathbb{R}^m)$ (the space of absolutely continuous functions $[t_0, \rho] \to \mathbb{R}^m$) and $\dot{z}(t) = v(t)$ for almost all $t \in [t_0, \rho]$.

Let $y \in C([-h, \rho], \mathbb{R}^m)$ denote the concatenation of y^0 and z, and write $w := (Ty)|_{[t_0,\rho]}$. Therefore, $y|_{[-h,t_0]} = y^0$, $y|_{[t_0,\rho]} = z \in AC([t_0, \rho], \mathbb{R}^m)$ and, to conclude that y is a solution of the initial-value problem (5.1), it suffices to show that $\dot{z}(t) \in G(t, z(t), w(t))$ for almost all $t \in [t_0, \rho]$.

By (A.5), we have

$$\|w_n(t) - w(t)\| \leq c_0 \max_{s\in[t_0,\rho]} \|z_n(s) - z(s)\| \quad \text{for a.a. } t \in [t_0, \rho] \text{ and all } n \in \mathbb{N}. \tag{A.10}$$

Therefore, for almost all $t \in [t_0, \rho]$, $w_n(t) \to w(t)$ as $n \to \infty$. Moreover,

$$\int_{t_0}^{\rho} \|w_n(t) - w(t)\|\mathrm{d}t \leq c_0|\rho - t_0| \max_{s\in[t_0,\rho]} \|z_n(s) - z(s)\| \to 0 \quad \text{as } n \to \infty.$$

Therefore, (w_n) converges (strongly) in $L^1([t_0, \rho], \mathbb{R}^m)$ to w.

Define the function $\sigma\colon \mathcal{K}_1 \times \mathbb{R}^m \to \mathbb{R}$ by

$$\sigma(t,\eta,\xi,q) := \max\{\langle q,\zeta\rangle |\ \zeta \in G(t,\eta,\xi)\}.$$

Observe that, for each $(t,\eta,\xi) \in \mathcal{K}_1$, $q \mapsto \sigma(t,\eta,\xi,q)$ is the support function for the compact and convex set $G(t,\eta,\xi)$ (and so is globally Lipschitz). Therefore, to establish that $\dot z(t) \in G(t,z(t),w(t))$ for almost all $t \in [t_0,\rho]$, it suffices to show that

$$\langle q,\dot z(t)\rangle \le \sigma(t,z(t),w(t),q) \quad \text{for a.a. } t \in [t_0,\rho] \text{ and all } q \in \mathbb{R}^m. \tag{A.11}$$

By continuity of the maps $q \mapsto \langle q,\zeta\rangle$ and $q \mapsto \sigma(t,\eta,\xi,q)$ for all $\zeta \in \mathbb{R}^m$ and all $(t,\eta,\xi) \in \mathcal{K}_1$, (A.11) holds if, any only if,

$$\langle q,\dot z(t)\rangle \le \sigma(t,z(t),w(t),q) \quad \text{for a.a. } t \in [t_0,\rho] \text{ and all } q \in \mathbb{Q}^m, \tag{A.12}$$

where $\mathbb{Q}^m \subset \mathbb{R}^m$ is the set of vectors in $\mathbb{R}^m$ with rational coordinates. We proceed to establish (A.12). First, we show that, for each $q \in \mathbb{R}^m$, the map $(t,\eta,\xi) \mapsto \sigma(t,\eta,\xi,q)$ is upper semicontinuous on $\mathcal{G}$. Let $q \in \mathbb{R}^m$ and $(t,\eta,\xi) \in \mathcal{K}_1$ be arbitrary and define

$$\sigma^* := \limsup_{(t',\eta',\xi')\to(t,\eta,\xi)} \sigma(t',\eta',\xi',q).$$

Let $\big((t_k,\eta_k,\xi_k)\big) \subset \mathcal{K}_1$ be a sequence converging to (t,η,ξ) such that $\sigma(t_k,\eta_k,\xi_k,q) \to \sigma^*$ as $k \to \infty$. For each $k \in \mathbb{N}$, by compactness of $G(t_k,\eta_k,\xi_k)$ there exists $\zeta_k \in G(t_k,\eta_k,\xi_k)$ such that $\langle q,\zeta_k\rangle = \sigma(t_k,\eta_k,\xi_k,q)$. The resulting sequence (ζ_k) is contained in the compact set $\mathcal{K}_2 = G(\mathcal{K}_1)$ and so has a subsequence converging to $\zeta \in \mathcal{K}_2$. By property (A.1a), the graph of G is closed and so we may infer that $\zeta \in G(t,\eta,\xi)$. Therefore,

$$\limsup_{(t',\eta',\xi')\to(t,\eta,\xi)} \sigma(t',\eta',\xi',q) = \lim_{k\to\infty} \sigma(t_k,\eta_k,\xi_k,q) = \lim_{k\to\infty} \langle q,\zeta_k\rangle = \langle q,\zeta\rangle \le \sigma(t,\eta,\xi,q),$$

whence upper semicontinuity of $\sigma(\cdot,\cdot,\cdot,q)$.

For $p \in L^\infty([t_0,\rho],\mathbb{R}^m)$,

$$|\sigma(t,z_n(t),w_n(t),p(t))| \le \max_{v\in\mathcal{K}_2} \|v\|\|p(t)\| \le c_3\|p(t)\| \quad \text{for a.a. } t \in [t_0,\rho] \text{ and all } n \in \mathbb{N}.$$

Furthermore, in view of (A.4),

$$\begin{aligned}\langle p(t),\dot z_n(t)\rangle &= \langle p(t), g_n(t,z_n(t),w_n(t))\rangle \\ &\le \sigma(t,z_n(t),w_n(t),p(t)) + \varepsilon_n\|p(t)\| \quad \text{for a.a. } t \in [t_0,\rho] \text{ and all } n \in \mathbb{N},\end{aligned}$$

and so

$$\int_{t_0}^{\rho} \big[\langle p(t),\dot z_n(t)\rangle - \varepsilon_n\|p(t)\|\big]\,\mathrm{d}t \le \int_{t_0}^{\rho} \sigma(t,z_n(t),w_n(t),p(t))\,\mathrm{d}t \quad \forall\ n \in \mathbb{N}.$$

Taking the limit superior as $n \to \infty$, invoking Fatou's lemma and upper semicontinuity of $\sigma(\cdot,\cdot,\cdot,q)$, we have

$$\int_{t_0}^{\rho} \langle p(t),\dot z(t)\rangle\,\mathrm{d}t \le \int_{t_0}^{\rho} \sigma(t,z(t),w(t),p(t))\,\mathrm{d}t. \tag{A.13}$$

Let $q \in \mathbb{Q}^m$ and let $t \in [t_0,\rho)$ be a Lebesgue point for the integrable functions $\dot z$ and $t \mapsto \sigma(t,z(t),w(t),q)$. For $\tau > 0$, define $p \in L^\infty([t_0,\rho],\mathbb{R}^m)$ by

$$p(s) := \begin{cases} q/\tau, & s \in [t,t+\tau] \cap [t_0,\rho], \\ 0, & \text{otherwise.}\end{cases}$$

By (A.13), we have

$$\frac{1}{\tau}\int_t^{t+\tau}\big[\sigma(s,z(s),w(s),q)-\langle q,\dot z(s)\rangle\big]\,\mathrm{d}s\ge 0\quad\forall\,\tau>0.$$

Passage to the limit as $\tau\to 0$ yields $\langle q,\dot z(t)\rangle\le\sigma(t,z(t),w(t),q)$, which is valid for all $t\in[t_0,\rho]\setminus\mathcal{N}(q)$, where $\mathcal{N}(q)$ is a set of measure zero which may depend on $q\in\mathbb{Q}^m$. Since $\mathbb{Q}^m$ is countable, $\cup_{q\in\mathbb{Q}^m}\mathcal{N}(q)$ has measure zero and so we may conclude that (A.12) (and hence (A.11)) holds. We have now shown that $y\colon[-h,\rho]\to\mathbb{R}^m$ is a solution of (5.1), whence assertion (i).

(ii) Let $y\in C(I_y,\mathbb{R}^m)$ be a solution of (5.1). Define

$$\mathcal{A}:=\left\{(I,z)\mid\ I_y\subset I,\ \ z\in C(I,\mathbb{R}^m)\text{ is a solution of (5.1) with } z|_{I_y}=y\right\}.$$

On this non-empty set define a partial order $\preceq$ by

$$(I_1,z_1)\preceq(I_2,z_2)\quad\Longleftrightarrow\quad\sup I_1\le\sup I_2\quad\text{and}\quad z_2|_{I_1}=z_1.$$

We proceed to show that $\mathcal{A}$ has a maximal element, that is, an element $(I^*,z^*)\in\mathcal{A}$ such that, for all $(I,z)\in\mathcal{A}$, $(I^*,z^*)\preceq(I,z)$ implies $(I,z)=(I^*,z^*)$, in which case $z^*\in C(I^*,\mathbb{R}^m)$ is a solution of (5.1) and is a maximal extension of the solution $y\in C(I_y,\mathbb{R}^m)$. Let $\mathcal{O}$ be a totally ordered subset of $\mathcal{A}$. Let $\omega:=\sup\{\sup I|(I,z)\in\mathcal{O}\}$ and let $z^*\colon[-h,\omega)\to\mathbb{R}^m$ be defined by the property that, for every $(I,z)\in\mathcal{O}$, $z^*|_I=z$. Then (ω,z^*) is in $\mathcal{A}$ and is an upper bound for $\mathcal{O}$. By Zorn's Lemma, it follows that $\mathcal{A}$ contains at least one maximal element. This establishes assertion (ii).

(iii) Assume $y\in C([-h,\omega),\mathbb{R}^m)$ is a maximal solution of (5.1) and that $\omega<\infty$. Seeking a contradiction, suppose there exist $\sigma\in[t_0,\omega)$ and compact $\mathcal{K}\subset\mathcal{D}$ such that $(t,y(t))\in\mathcal{K}$ for all $t\in[\sigma,\omega)$. By boundedness of y and property (iv) of $\mathcal{T}_h$, we conclude that Ty is bounded. Therefore, the function $t\mapsto(t,y(t),(Ty)(t))$ is essentially bounded and so by property (A.1b) of G, it follows that $\dot y$ is essentially bounded on $[\sigma,\omega)$. Therefore, y is uniformly continuous on $[-h,\omega)$ and so extends to a function $y^*\in C([-h,\omega],\mathbb{R}^m)$. By compactness of $\mathcal{K}$, we have $(\omega,y^*(\omega))\in\mathcal{K}\subset\mathcal{D}$. An application of Assertion (i) of the theorem (with ω and y^* replacing t_0 and y^0, respectively) yields the existence of a solution $y^e\in C([-h,\rho],\mathbb{R}^m)$ for some $\rho>\omega$, with $y^e|_{[-h,\omega)}=y$. This contradicts maximality of y. □

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