A Critical Case of the Circle Criterion

T. Fliegner†, H. Logemann‡ & E.P. Ryan§

Department of Mathematical Sciences, University of Bath,
Claverton Down, Bath BA2 7AY, United Kingdom

1 Introduction

Absolute stability, and its relation to the concept of positive-real transfer functions, permeates much of the classical and modern control literature (see, for example, [1]-[6], [8]-[18]). Of particular importance are absolute stability results of circle-criterion type, applicable in the context of a canonical feedback structure with a linear plant Λ in the forward path and a nonlinearity f in the feedback path (see Figure 1). Absolute stability criteria are not only useful for stability analysis, but they have also been used in the context of control synthesis, see, for example, [3, 4].

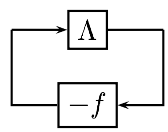

Figure 1: Feedback system with nonlinearity

In the present paper, we address absolute stability issues in the setting of finite-dimensional, single-input, single-output plants Λ which contain an integrator (and so are not asymptotically stable). We consider time-varying nonlinearities f satisfying a particular sector condition which posits the existence of constants $t_0 \geq 0$ and $\beta > 0$ such that $\xi f(t,\xi) \geq \beta f^2(t,\xi)$ for all $t \geq t_0$ and all $\xi \in \mathbb{R}$. Moreover, the lower gain of f, that is, $\inf\{|f(t,\xi)/\xi| : t \geq t_0,\ \xi \neq 0\}$, may be zero (as is the case for deadzone nonlinearities and bounded nonlinearities such as saturation). The conjunction of stable, but not asymptotically stable, linear plants Λ and nonlinearities f with possibly zero lower gain (a so-called *critical case* of the circle criterion) is the distinguishing feature of the paper.

2 A result of circle-criterion type

With reference to Figure 1, we consider real linear single-input single-output systems Λ of the form

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{b}u(t)\,, \quad z(0) = z^0\,; \quad y(t) = \mathbf{c}z(t) \tag{1}$$

with $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{b}, \mathbf{c}^T \in \mathbb{R}^n$, and with transfer function $\mathbf{G}$ given by $\mathbf{G}(s) := \mathbf{c}(sI - \mathbf{A})^{-1}\mathbf{b}$. The following hypothesis remains in force throughout this section.

(H) $\mathbf{A}$ has a simple eigenvalue at zero, every other eigenvalue of $\mathbf{A}$ has negative real part and $\lim_{s\to 0} s\mathbf{G}(s) > 0$.

†Now at: Dept. of Sciences and Liberal Arts, International University in Germany, D-76646 Bruchsal, Germany, email: thomas.fliegner@i-u.de

‡Email: hl@maths.bath.ac.uk

§Email: epr@maths.bath.ac.uk

Therefore, with zero input, Λ is stable but not asymptotically stable; moreover, it is readily verified that $\inf_{\omega\neq 0}\operatorname{Re}\mathbf{G}(i\omega) > -\infty$.

We now focus on stability properties of the linear system (1) under output feedback with time-varying nonlinearity in the feedback loop (recall Figure 1). First, we make precise the class $\mathcal{N}$ of allowable nonlinearities. A function

$$f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \quad (t,\xi) \mapsto f(t,\xi)$$

is deemed to be of class $\mathcal{N}$ if it is measurable in t and locally Lipschitz in ξ, uniformly with respect to t on bounded intervals, and there exists a non-negative function $c_f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that

$$|f(t,\xi)| \le c_f(t)\big[1+|\xi|\big], \quad \forall\, t\in\mathbb{R}_+\,,\ \forall\,\xi\in\mathbb{R}.$$

A function $f\in\mathcal{N}$ is said to be asymptotically autonomous with limit f_a if $f_a:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz and, for all $R>0$ and $\varepsilon>0$, there exists $\tau>0$ such that,

$$\text{ess-sup}_{t\ge\tau}|f(t,\xi)-f_a(\xi)| < \varepsilon\,, \quad \forall\,\xi\in[-R,R].$$

It follows from standard results in ordinary differential equations combined with Gronwall's lemma that, for $f\in\mathcal{N}$, the initial-value problem for the feedback system

$$\dot z(t) = \mathbf{A}z(t) - \mathbf{b}f(t,\mathbf{c}z(t)), \quad z(0)=z^0, \tag{2}$$

has a unique absolutely continuous solution defined on $\mathbb{R}_+$ (no finite escape time) which satisfies the differential equation in (2) for almost all $t\in\mathbb{R}_+$.

The next theorem is a stability result of circle-criterion type. For completeness, we have included therein (*viz.* Assertion (b) of Statement 1) a well-known classical result on exponential stability, see [17]. However, we emphasize the novelty of all other assertions of the theorem which pertain to feedback systems for which (i) the linear part contains an integrator (*i.e.*, we are considering a so-called particular or critical case in the terminology of [2, 13, 15], implying that the linear system is not asymptotically stable) and (ii) the "lower gain" $\inf\{|f(t,\xi)/\xi| :\ t\ge t_0,\ \xi\in\mathbb{R}^*\}$ of the nonlinearity f may be zero (which, for example, is the case for bounded nonlinearities such as saturation and deadzone). In fact, one of the motivations for studying this situation is its importance in the application to the low-gain integral control problem in the presence of input nonlinearities of saturation type (see Section 3).

Theorem 2.1. Assume that (H) holds. Let $0<\alpha<\infty$ be such that

$$1/\alpha + \inf_{\omega\neq 0}\operatorname{Re}\mathbf{G}(i\omega) > 0\,. \tag{3}$$

For $z^0\in\mathbb{R}^n$, denote the unique absolutely continuous solution of (2) (defined on $\mathbb{R}_+$) by $t\mapsto z(t,z^0)$.

1. If $f\in\mathcal{N}$ is such that, for some $t_0\ge 0$,

$$\xi f(t,\xi) \ge \frac{1}{\alpha}f^2(t,\xi)\,, \quad \forall\, t\ge t_0\,,\ \forall\,\xi\in\mathbb{R}, \tag{4}$$

then the following statements hold.

(a) There exists $M\ge 1$ such that

$$\|z(t+t_0,z^0)\| \le M\|z(t_0,z^0)\|\,, \quad \forall\,(t,z^0)\in\mathbb{R}_+\times\mathbb{R}^n\,.$$

(b) If there exists $\alpha_0 > 0$ such that $\alpha_0 \xi^2 \le \xi f(t,\xi)$ for all $t \ge t_0$ and all $\xi \in \mathbb{R}$, then there exist $M \ge 1$ and $\rho > 0$ such that

$$\|z(t+t_0, z^0)\| \le M e^{-\rho t} \|z(t_0, z^0)\|, \quad \forall\, (t, z^0) \in \mathbb{R}_+ \times \mathbb{R}^n .$$

(c) For each $z^0 \in \mathbb{R}^n$, the function $t \mapsto y(t) = \mathbf{c} z(t, z^0)$ is such that $y(\cdot) f(\,\cdot\,, y(\cdot)) \in L^1(\mathbb{R}_+)$ and $f(\,\cdot\,, y(\cdot)) \in L^2(\mathbb{R}_+)$.

(d) For each $z^0 \in \mathbb{R}^n$, $z(t, z^0)$ converges as $t \to \infty$ to a limit in $\ker \mathbf{A}$, that is,

$$z^\infty := \lim_{t \to \infty} z(t, z^0) \in \ker \mathbf{A},$$

and, moreover, $\dot{z}(\,\cdot\,, z^0) \in L^2(\mathbb{R}_+)$.

(e) If f is asymptotically autonomous with limit f_a, then, for each $z^0 \in \mathbb{R}^n$, $\lim_{t\to\infty} z(t, z^0) = z^\infty \in \ker \mathbf{A}$ is such that $\mathbf{c} z^\infty \in f_a^{-1}(0)$.

(f) If f is asymptotically autonomous with limit f_a and $f_a^{-1}(0) = \{0\}$, then 0 is a globally attractive equilibrium of the feedback system (2).

2. If $f \in \mathcal{N}$ is of the form $f(t,\xi) = k(t) g(\xi)$, where k is measurable, bounded and non-negative with $\limsup_{t\to\infty} k(t) \le 1$, g is locally Lipschitz with $\liminf_{\xi \to 0} g(\xi)/\xi > 0$ and

$$0 < \xi g(\xi) \le \alpha \xi^2, \quad \forall\, \xi \in \mathbb{R}^*,$$

then there exists $N \ge 1$ such that

$$\|z(t, z^0)\| \le N \|z^0\|, \quad \forall\, (t, z^0) \in \mathbb{R}_+ \times \mathbb{R}^n .$$

Furthermore, for each $r > 0$, there exist $L \ge 1$ and $\rho > 0$ such that, for all $z^0 \in \mathbb{R}^n$ with $\|z^0\| \le r$,

$$\|z(t, z^0)\| \le L e^{-\rho \int_0^t k(\tau) d\tau} \|z^0\|, \quad \forall\, t \in \mathbb{R}_+ .$$

In particular, if $k \notin L^1(\mathbb{R}_+)$, then $\lim_{t\to\infty} z(t, z^0) = 0$ for all $z^0 \in \mathbb{R}^n$; if $\liminf_{t\to\infty} k(t) > 0$, then 0 is a semi-globally exponentially stable equilibrium of system (2).

3 Discussion

Elaborating on earlier remarks, we reiterate that, with the exception of the well-known result in Statement 1(b) above, all other assertions of the theorem pertain to critical cases wherein the linear system contains an integrator and the nonlinearity may have zero lower gain. By contrast, in most of the circle-criterion type results available in the literature (such as [10, 14, 15, 16, 17]), the lower gain of the nonlinearity is either assumed to be positive, or, if the lower gain is allowed to be zero, the linear part is assumed to be asymptotically stable (and so does not contain an integrator): an exception is [18], where stability of the origin and Lagrange stability is proved for (2) without assuming that the lower gain of f is positive. However, none of the statements of Theorem 2.1 can be found in [18]. Also note that in contrast to most related results in the literature, Theorem 2.1 does not impose controllability or observability on the linear system. In [1], a "relaxed" circle criterion is given for the linear time-varying feedback system obtained from (2) by considering functions f of the form $f(t,\xi) = k(t)\xi$ (this is a special case of the setting in Statement 2 of Theorem 2.1). Assuming that the positive real condition (3) holds and

$0 \leq k(t) \leq \alpha$, Theorem 12 in [1] guarantees asymptotic stability of the origin, provided that k satisfies a certain additional condition: the point of interest in the present context is that the latter condition is satisfied by a large class of gain functions k with $\liminf_{t\to\infty} k(t) = 0$.*

One of the main motivations for developing, in Theorem 2.1, an absolute stability criterion for feedback structures of the form of Figure 1, wherein the linear plant Λ contains an integrator and the nonlinearity has zero lower gain, is their application in proving convergence and stability properties of low-gain integral feedback control for tracking of constant reference signals. Consider the closed-loop system in Figure 2, where Σ is an exponentially stable linear systems Σ with positive steady-state gain, φ and ψ are globally Lipschitz non-decreasing input/actuator and output/sensor nonlinearities, respectively, and $\kappa = \kappa(\cdot)$ is a positive gain function.

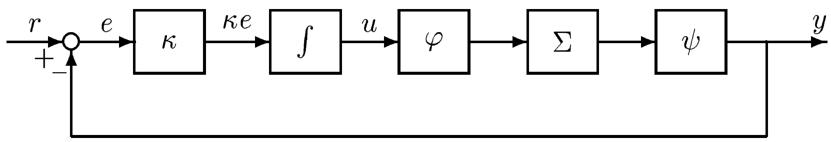

Figure 2: Integral control with input and output nonlinearities

Under an appropriate nonlinear transformation, the closed-loop system in Figure 2 has the structure of Figure 1, to which our absolute stability results apply. Under a natural feasability assumption on the reference value r, it is possible to deduce a threshold value $\kappa^* > 0$ such that performance of the closed-loop is assured for all gain functions κ with $\limsup_{t\to\infty} \kappa(t) < \kappa^*$ and $\lim_{t\to\infty} \int_0^t \kappa(s)ds = \infty$. Statement 2 of Theorem 2.1 is useful for estimating the speed of the convergence of $y(t)$ to the desired reference value r as $t \to \infty$. Whilst this result has significant overlap with those of [7], we stress that the absolute stability approach oulined above differs fundamentally from the arguments used in [7] and provides a natural and unified framework for investigations on low-gain integral control, see [8] for more details.

REFERENCES

[1] D. Aeyels, R. Sepulchre and J. Peuteman. Asymptotic stability conditions for time-variant systems and observability: uniform and non-uniform criteria, *Mathematics of Control, Signals, and Systems*, **11** (1998), 1–27.

[2] M.A. Aizerman and F.R. Gantmacher. *Absolute Stability of Regulator Systems*, Holden-Day, San Francisco, 1964.

[3] M. Arcak and P. Kokotović. Nonlinear observers: a circle criterion design and robustness analysis, *Automatica*, **37** (2001), 1923–1930.

[4] M. Arcak, M. Larsen and P. Kokotović. Circle and Popov criteria as tools for nonlinear feedback design, *Automatica*, **39** (2003), 643–650.

[5] M. Arcak and A. Teel. Input-to-state stability for a class of Lurie systems, *Automatica*, **38** (2002), 1945–1949.

[6] R.F. Curtain, H. Logemann and O. Staffans. Stability results of Popov-type for infinite-dimensional systems with applications to integral control, *Proc. London Math. Soc.*, **86** (2003), 779–816.

* We mention that the relaxed circle criterion [1, Theorem 12] is not correct in the generality stated in [1], wherein the only regularity assumption explicitly imposed on k is (Lebesgue) measurability: it is not difficult to construct counterexamples with continuous k. However, uniform continuity of k is sufficient for the assertions of Theorem 12 in [1] to hold.

[7] T. Fliegner, H. Logemann and E.P. Ryan. Low-gain integral control of continuous-time linear systems subject to input and output nonlinearities, *Automatica*, **39** (2003), 455–462.

[8] T. Fliegner, H. Logemann and E.P. Ryan. Absolute stability and integral control, submitted (2003).

[9] W. Hahn. *Stability of Motion*, Springer-Verlag, Berlin, 1967.

[10] H.K. Khalil. *Nonlinear Systems*, 2nd edition, Prentice-Hall, Upper Saddle River, NJ, 1996.

[11] U. Jönnson and A. Megretski. The Zames-Falb IQC for systems with integrators, *IEEE Trans. Autom. Control*, **45** (2000), 560–565.

[12] S. Lefschetz. *Stability of Nonlinear Control Systems*, Academic Press, New York, 1965.

[13] G.A. Leonov, D.V. Ponomarenko and V.B. Smirnova. *Frequency-Domain Methods for Nonlinear Analysis*, World Scientific, Singapore, 1996.

[14] R. Lozano, B. Brogliato, O. Egeland and B. Maschke. *Dissipative Systems Analysis and Control*, Springer-Verlag, London, 2000.

[15] K.S. Narendra and J.H. Taylor. *Frequency-Domain Criteria for Absolute Stability*, Academic Press, New York, 1973.

[16] S. Sastry. *Nonlinear Systems: Analysis, Stability and Control*, Springer-Verlag, New York, 1999.

[17] M. Vidyasagar. *Nonlinear Systems Analysis*, 2nd edition, Prentice Hall, Englewood Cliffs, NJ, 1993.

[18] J.L. Willems. *Stability Theory of Dynamical Systems*, Nelson, London, 1970.