[8] P. E. Caines and J. F. Zhang, "On the adaptive control of jump parameter systems via nonlinear filtering," *SIAM J. Control Optim.*, vol. 33, no. 6, pp. 1758–1777, 1995.
[9] F. Xue and L. Guo, "Necessary and sufficient conditions for adaptive stability of jump linear systems," *Commun. Inform. Syst.*, vol. 1, no. 2, pp. 205–224, 2001.
[10] T. W. Chen and B. Francis, *Optimal Sampled-Data Control Systems*. London, U.K.: Springer-Verlag, 1995.
[11] L. L. Yao and J. F. Zhang, "Sampled-data-based LQ control of stochastic linear continuous-time systems," *Sci. China (Series F)*, vol. 45, no. 5, pp. 383–396, 2002.
[12] D. Cheng and L. Zhang, "Adaptive control of switched systems," in *Proc. 42nd IEEE Conf. Decision Control*, Maui, HI, 2003, pp. 6260–6264.
[13] E. A. Coddington and R. Carlson, *Linear Ordinary Differential Equations*. Philadelphia, PA: SIAM, 1997.

Remarks on the L^p-Input Converging-State Property

E. P. Ryan

***Abstract*—Let $\mathbb{X} \subset \mathbb{R}^N$ and consider a system $\dot{x} = f(x,u)$, $f : \mathbb{X}\times\mathbb{R}^M \to \mathbb{R}^N$, with the property that the associated autonomous system $\dot{x} = f(x,0)$ has an asymptotically stable compactum C with region of attraction A. Assume that x is a solution of the former, defined on $[0,\infty)$, corresponding to an input function u. Assume further that, for each compact $K \subset \mathbb{X}$, there exists $k > 0$ such that $|f(z,v) - f(z,0)| \le k|v|$ for all $(z,v) \in K\times\mathbb{R}^M$. A simple proof is given of the following L^p-input converging-state property: if $u \in L^p$ for some $p \in [1,\infty)$ and x has an ω-limit point in A, then x approaches C.**

***Index Terms*—Asymptotic stability, converse Lyapunov theory, domain of attraction.**

I. Introduction

For a linear system $\dot{x} = Fx + Gu$, with F Hurwitz, the following properties are elementary: P1) if x is a solution on $\mathbb{R}_+ := [0,\infty)$ corresponding to an input $u \in L^\infty$ with $u(t) \to 0$ as $t \to \infty$, then $x(t) \to 0$ as $t \to \infty$; and P2) if x is a solution on $\mathbb{R}_+$ corresponding to an input $u \in L^p$ for some $p \in [1,\infty)$, then $x(t) \to 0$ as $t \to \infty$. Exploitation of these properties is widespread in the literature (on, for example, adaptive control, robustness to disturbances, and interconnected/cascaded systems). The question of nonlinear counterparts arises: to what extent do properties P1) and P2) persist in the context of a finite-dimensional nonlinear system $\dot{x} = f(x,u)$ under the hypothesis that 0 is an asymptotically stable equilibrium of the associated autonomous system $\dot{z} = f^*(z)$, where $f^*(\cdot) := f(\cdot,0)$? Even in the simplest of nonlinear systems satisfying the latter hypothesis, properties P1) and P2) may fail to hold. One such scalar system is given by $\dot{x} = -x + x^2u$ which, with initial data $x(0) = 1$ and input $u : t \mapsto 2e^{-t}$, has unbounded solution $x : t \mapsto e^t$.

In [1] (under the assumption that f is continuous and is locally Lipschitz in its first argument, uniformly with respect to its second argument in compact sets), a proof is provided of the following "well-known but hard-to-cite fact" [a nonlinear counterpart of the converging-input converging-state property P1)]. If a) x is a solution of the system $\dot{x} = f(x,u)$, defined on $\mathbb{R}_+$, corresponding to an input $u \in L^\infty$ with $u(t) \to 0$ as $t \to \infty$, b) 0 is an asymptotically stable equilibrium of the associated autonomous system $\dot{z} = f^*(z)$ with domain of attraction A, and c) x is K-recurrent for some compact $K \subset A$, then $x(t) \to 0$ as $t \to \infty$. Here, K-recurrence is the property that, for each $T > 0$, there exists $t > T$ such that $x(t) \in K$. The recurrence hypothesis c) is equivalent to positing that x has an ω-limit point in A. The nonlinear converging-input converging-state property in [1] has a closely-related antecedent in [2], Theorem 2 of which contains the essence of the result.

The purpose of the present note is to provide a nonlinear counterpart of the L^p-input converging-state property P2) (we use the term "input" in the general sense of either a control input or disturbance input): the essence is to identify conditions under which an input of bounded energy generates a converging state.[1] The main result subsumes the following: if a) x is a solution of the system $\dot{x} = f^*(x) + g(x)u$ (f^* and g locally Lipschitz), defined on $\mathbb{R}_+$, corresponding to an input $u \in L^p$ for some $p \in [1,\infty)$, b) the associated autonomous system $\dot{z} = f^*(z)$ has an asymptotically stable compactum C with domain of attraction A, and c) x has an ω-limit point in A, then x approaches C (in a sense made precise later).

II. Preliminaries

The Euclidean norm and inner product on $\mathbb{R}^N$ (or $\mathbb{R}^M$) are denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively. For $G \in \mathbb{R}^{N\times M}$, $|G| := \min_{|u|=1}|Gu|$. For a nonempty set $C \subset \mathbb{R}^N$, the function $d_C : \mathbb{R}^N \to \mathbb{R}_+$, given by $d_C(y) := \inf_{c\in C}|y-c|$, is the distance function for C and a function $x : \mathbb{R}_+ \to \mathbb{R}^N$ is said to approach C if $d_C(x(t)) \to 0$ as $t \to \infty$. Let $I \subset \mathbb{R}$ be such that $\mathbb{R}_+ \subset I$, let $\mathbb{X} \subset \mathbb{R}^N$ be nonempty and open, and let $x : I \to \mathbb{X}$. A point $z \in \mathrm{cl}(\mathbb{X})$ is an ω-limit point of x if there exists an unbounded sequence (t_n) in $\mathbb{R}_+$ with $x(t_n) \to z$ as $n \to \infty$; the ω-limit set of x is the set of all ω-limit points of x, this set is is denoted by $\Omega(x)$. For later convenience, we record some well-known properties of ω-limit sets (see, for example, [4]).

Proposition 2.1: For every function $x : \mathbb{R}_+ \to \mathbb{X}$ the following hold.

i) $\Omega(x)$ is closed.
ii) $\Omega(x) = \emptyset$ if, and only if, $|x(t)| \to \infty$ as $t \to \infty$.
iii) If x is continuous and $\Omega(x)$ is nonempty and compact, then x is bounded.
iv) If x is continuous and bounded, then $\Omega(x)$ is nonempty, compact, connected, and is approached by x.

III. The System

Denote, by $\mathcal{U} := L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^M)$, the space of locally integrable functions $u : \mathbb{R}_+ \to \mathbb{R}^M$. Let $\mathbb{X} \subset \mathbb{R}^N$ be nonempty and open. Let $f : \mathbb{X}\times\mathbb{R}^M \to \mathbb{R}^N$ be continuous and such that

$$\forall \text{ compact } K \subset \mathbb{X}\ \exists k > 0 : |f(x,u) - f(x,0)| \le k|u| \quad \forall (x,u) \in K\times\mathbb{R}^M \tag{1}$$

a canonical case being that wherein f is affine in the input, viz. $f(x,u) = f^*(x) + g(x)u$. We assume further that

$$f^*(\cdot) := f(\cdot,0) \text{ is locally Lipschitz.} \tag{2}$$

[1] In a context different from that of this note, asymptotic properties of solutions of systems with inputs of bounded energy (viz. the "bounded-energy weakly-converging-state," and the "bounded-energy frequently-bounded-state" properties) play a role in asymptotic characterizations of integral-input-to-state stability in [3].

Manuscript received November 17, 2004; revised February 22, 2005. Recommended by Associate Editor A. Astolfi.
The author is with the Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, U.K. (e-mail: epr@maths.bath.ac.uk).
Digital Object Identifier 10.1109/TAC.2005.851457

Consider the initial-value problem

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = \xi \in \mathbb{X}, \quad u \in \mathcal{U}. \tag{3}$$

For each $(\xi, u) \in \mathbb{X} \times \mathcal{U}$, (3) has a (forward) solution $x : [0, \omega) \to \mathbb{X}$ $(0 < \omega \leq \infty)$, that is, an absolutely continuous function, with $x(0) = \xi$, which satisfies the differential equation in (3) for almost all $t \in [0, \omega)$. Moreover, every solution can be extended into a maximal solution (a solution x is maximal if it does not have a proper right extension which is also a solution of (3)). A maximal solution $x : [0, \omega) \to \mathbb{X}$ is said to be *global* if $\omega = \infty$.

Remark 3.1: The hypotheses (1) and (2) on the continuous function f are not sufficient to ensure uniqueness of solutions. However, for each $(\xi, u) \in \mathbb{X} \times \mathcal{U}$, (3) has *unique* maximal solution if, in addition, one of the following holds [note that a) ⇒ b) ⇒ c)].

a) $f : (x, u) \mapsto f^*(x) + g(x)u$ where $g : \mathbb{X} \to \mathbb{R}^{N \times M}$ is locally Lipschitz.

b) For each compact $K \subset \mathbb{X}$, there exists $\kappa > 0$ such that

$$|f(x, u) - f(y, u)| \leq \kappa |u||x - y| \qquad \text{for all } x, y \in K \quad \text{and all } u \in \mathbb{R}^M.$$

c) For each compact $K \subset \mathbb{X}$ and $u \in \mathcal{U}$, there exists $v \in L^1_{\text{loc}}(\mathbb{R}_+)$ such that

$$|f(x, u(t)) - f(y, u(t))| \leq v(t)|x - y|q \qquad \text{for all } x, y \in K, \text{ and almost all } t \in \mathbb{R}_+.$$

▲

With (3), we associate the autonomous system

$$\dot{z}(t) = f^*(z(t)) \tag{4}$$

which, in view of (2), generates a flow φ. Let $C \subset \mathbb{X}$ be a nonempty and compact set. We assume that C is asymptotically stable with respect to (4) in the following sense: i) C is a φ-invariant set (i.e., $\mathbb{R} \times C \subset \text{dom}(\varphi)$ and $\varphi(\mathbb{R} \times C) = C$); ii) C *is stable* (i.e., for every open neighborhood $\mathcal{N}_0$ of C, there exists an open neighborhood $\mathcal{N}_1$ of C such that $\mathbb{R}_+ \times \mathcal{N}_1 \subset \text{dom}(\varphi)$ and $\varphi(t, \mathcal{N}_1) \subset \mathcal{N}_0$ for all $t \in \mathbb{R}_+$); and iii) C *is attractive* (i.e., there exists an open neighborhood $\mathcal{N}_2$ of C such that $\mathbb{R}_+ \times \mathcal{N}_2 \subset \text{dom}(\varphi)$ and, for all $\xi \in \mathcal{N}_2, \Omega(\varphi(\cdot, \xi))$ is a nonempty subset of C). The set $A := \{\xi \in X | \emptyset \neq \Omega(\varphi(\cdot, \xi)) \subset C\}$ is the domain of attraction of C and is an open set containing C (see [5, Ch. V, Th. 1.8]). By converse Lyapunov theory (see [6, Th. 3.2] and [7, Ths. 3.1 and 3.2]), there exists a smooth function $V : A \to \mathbb{R}_+$ with the properties shown in the equation at the bottom of the page.

IV. L^p-Input, Converging-State Property

For clarity, we reiterate the underlying hypotheses which remain in force throughout this section.

H1) $\mathbb{X} \subset \mathbb{R}^N$ is nonempty and open; $f : \mathbb{X} \times \mathbb{R}^M \to \mathbb{R}^N$ is continuous and such that (1) and (2) hold.

H2) $C \subset \mathbb{X}$ is a nonempty compact set and is asymptotically stable with respect to (4), with domain of attraction A.

First, we prove a technicality.

Proposition 4.1: Assume $(\xi, u) \in \mathbb{X} \times \mathcal{U}$ is such that (3) has a global solution $x : \mathbb{R}_+ \to \mathbb{X}$ with $\Omega(x) \cap A \neq \emptyset$. If $(\Omega(x) \cap A) \subset C$, then $\Omega(x) \subset C$.

We preface the proof with the remark that the assertion would be immediate if $\Omega(x) \neq \emptyset$ were known *a priori* to be a connected set: However, this is not the case. Connectedness may be inferred *a posteriori* from the proof (which establishes that the closed set $\Omega(x)$ is contained in C and so is compact which, together with Proposition 2.1iii)–iv), implies connectedness).

Proof: Since A is an open neighborhood of the compact set C, we may choose $\varepsilon > 0$ sufficiently small so that $K := \{y \in \mathbb{R}^N | d_C(y) = \varepsilon\}$ is a compact subset of A. Evidently, $K \cap C = \emptyset$. Assume $\Omega(x) \cap A \subset C$ and, seeking a contradiction, suppose $\Omega(x) \not\subset C$. Then, there exists $\zeta \in \Omega(x)$ with $\zeta \notin A$. Moreover, $d_C(\zeta) > \varepsilon$. Choose $z \in \Omega(x) \cap A \subset C$ arbitrarily (and so $d_C(z) = 0$). Let (s_n) and (t_n) be unbounded sequences in $\mathbb{R}_+$ with $x(s_n) \to z$ and $x(t_n) \to \zeta$ as $n \to \infty$. Passing to subsequences if necessary, we may assume $s_n < t_n < s_{n+1}, d_C(x(s_n)) < \varepsilon$ and $d_C(x(t_n)) > \varepsilon$ for all $n \in \mathbb{N}$. By continuity of x, there exists a sequence (τ_n) with $\tau_n \in (s_n, t_n)$ and $x(\tau_n) \in K$ for all $n \in \mathbb{N}$. By compactness of K, the sequence $(x(\tau_n))$ has a convergent subsequence with limit in K and so x has an ω-limit point in $K \subset A$. This contradicts the facts that $K \cap C = \emptyset$ and $(\Omega(x) \cap A) \subset C$. ■

We now arrive at the main result.

Theorem 4.2: Let $\xi \in \mathbb{X}, u \in L^p(\mathbb{R}_+, \mathbb{R}^M)$ for some $p \in [1, \infty)$, and assume that (3) has a global solution $x : \mathbb{R}_+ \to \mathbb{X}$. The following statements are equivalent:

i) x approaches C;
ii) x approaches a connected component of C;
iii) $\Omega(x) \cap A \neq \emptyset$.

Proof:
ii) $\Longrightarrow$ i). This is immediate.
i) $\Longrightarrow$ iii). Assume that i) holds. Then, by compactness of C, x is bounded and so $\Omega(x)$ is a nonempty subset of A.
iii) $\Longrightarrow$ ii). Assume that iii) holds. In view of Proposition 2.1, if $\Omega(x)$ is compact, then it is also connected and is approached by x. Therefore, to complete the proof, it suffices to establish compactness of $\Omega(x)$: this we do by showing that $(\Omega(x) \cap A) \subset C$ (in which case, by Proposition 4.1, $\Omega(x) \subset C$ and so, by compactness of C and closedness of the ω-limit set, $\Omega(x)$ is compact). Seeking a contradiction, suppose that $(\Omega(x) \cap A) \not\subset C$. Then there exists $z \in \Omega(x) \cap A$ with $d_C(z) > 0$. Let $V : A \to \mathbb{R}_+$ be a smooth function with properties (5). Since $z \in A \backslash C$, we have $V(z) > 0$. Define

$$A_1 := \{y \in A | V(z) \leq 2V(y) \leq 3V(z)\} \quad \text{and}$$
$$A_2 := \{y \in A | 3V(z) \leq 4V(y) \leq 5V(z)\}.$$

$$\left.\begin{array}{l} \text{i) } V(c) = 0 \quad \text{for all } c \in C \\ \text{ii) } V(y) > 0 \quad \text{and} \quad \langle \nabla V(y), f^*(y) \rangle < 0 \quad \text{for all } y \in A \backslash C \\ \text{iii) if } (y_n) \text{ is a sequence in } A \text{ which either converges to a boundary point of } A \\ \quad \text{or } |y_n| \to \infty \text{ as } n \to \infty, \text{ then } V(y_n) \to \infty \text{ as } n \to \infty. \end{array}\right\} \tag{5}$$

Clearly, $A_2 \subset A_1$ and, by properties (5) of V, each is a compact subset of $A\backslash C$. By property (1), there exists $\alpha > 0$ so that

$$|f(y,u) - f^*(y)| \le \alpha |u| \qquad \forall (y,u) \in A_1 \times \mathbb{R}^M.$$

Define

$$\beta := -\max_{y\in A_1} \langle \nabla V(y), f^*(y)\rangle > 0 \quad \text{and} \quad \gamma := \max_{y\in A_1} |\nabla V(y)| > 0.$$

If $p > 1$, let $q = p/(p-1)$ denote its conjugate exponent. Define $\delta > 0$ as follows:

$$\delta := \begin{cases} 1, & p = 1 \\ (q\beta/2)^{1/q}, & p \in (1,\infty). \end{cases}$$

Note that, if $p \in (1,\infty)$, then, by Young's inequality

$$\alpha\gamma|u| \le \frac{\delta^q}{q} + \frac{(\alpha\gamma|u|)^p}{p\delta^p} \qquad \forall u \in \mathbb{R}^M.$$

Therefore, in both possible cases of $p = 1$ and $p \in (1,\infty)$, we have

$$\alpha\gamma|u| \le \frac{\beta}{2} + \frac{(\alpha\gamma|u|)^p}{p\delta^p} \qquad \forall u \in \mathbb{R}^M.$$

Since $z \in \Omega(x) \cap \mathrm{int} A_2$, there exists an unbounded sequence (t_n) such that $x(t_n) \in A_2$ for all n. Since $u \in L^p$ and (t_n) is unbounded, there exists $k \in \mathbb{N}$ such that

$$\frac{(\alpha\gamma)^p}{p\delta^p} \int_{t_k}^{\infty} |u(t)|^p \, dt < \frac{V(z)}{4}. \tag{6}$$

For convenience, write $u_k(\cdot) := u(\cdot + t_k), x_k(\cdot) := x(\cdot + t_k)$ and observe that $\dot{x}_k(t) = f(x_k(t), u_k(t))$ for almost all $t \in \mathbb{R}_+$. Define $\mathcal{T} := \{t \in \mathbb{R}_+ | x_k(t) \in A_1\}$. We may now infer that

$$\begin{aligned}(V \circ x_k)'(t) &= \langle \nabla V(x_k(t)), f(x_k(t), u_k(t))\rangle \\ &\le \langle \nabla V(x_k(t)), f^*(x_k(t))\rangle \\ &\quad + |\nabla V(x_k(t))||f(x_k(t), u_k(t)) - f^*(x_k(t))| \\ &\le -\beta + \alpha\gamma|u_k(t)| \le -\frac{\beta}{2} \\ &\quad + \frac{|\alpha\gamma u_k(t)|^p}{p\delta^p} \quad \text{for a.a. } t \in \mathcal{T}. \end{aligned} \tag{7}$$

In view of (6) and (7), the following implication holds:

$$\begin{aligned}[a,b] \subset \mathcal{T} \Longrightarrow 0 &< V(x_k(b)) \\ &\le V(x_k(a)) - \frac{\beta|b-a|}{2} + \frac{V(z)}{4}. \end{aligned} \tag{8}$$

Since $\beta > 0$, it follows from (8) that $\mathcal{T} \neq \mathbb{R}_+$. Define $T := \inf\{t \in \mathbb{R}_+ | x_k(t) \notin A_1\} \in (0,\infty)$ (and so $[0,T] \subset \mathcal{T}$). Then, either $2V(x_k(T)) = 3V(z)$ or $2V(x_k(T)) = V(z)$. However, the former case cannot occur because, otherwise, (8) (with $[a,b] = [0,T]$ and noting that, since $x(t_k) = x_k(0) \in A_2$, we have $4V(x_k(0)) \le 5V(z)$) leads to the contradiction: $3V(z) = 2V(x_k(T)) \le 3V(z) - \beta T$. Therefore, $2V(x_k(T)) = V(z)$. Recalling that $x(t_n) \in A_2$ for all $n \in \mathbb{N}$, the following are well defined:

$$T_1 := \inf\{t \ge T | x_k(t) \in A_2\} \quad \text{and}$$
$$T_0 := \sup\{t \in [T, T_1] | 2V(x_k(t)) = V(z)\}.$$

Moreover, $[T_0, T_1] \subset \mathcal{T}, 2V(x_k(T_0)) = V(z)$ and $4V(x_k(T_1)) \ge 3V(z)$. Therefore, via (8), we arrive at the contradiction

$$\begin{aligned}3V(z) \le 4V(x_k(T_1)) &\le 4V(x_k(T_0)) \\ &\quad -2\beta|T_1 - T_0| + V(z) < 3V(z).\end{aligned}$$

We may now conclude that our original supposition is false and so $(\Omega(x) \cap A) \subset C$. ∎

Remarks 4.3:

1) Theorem 4.2 is akin to [8, Th. 4.4(ii)]; however, the latter applies in a more restricted context wherein 0 is assumed to be a *globally* asymptotically stable equilibrium of (4) and L^1 inputs u only are considered.
2) In the particular case of system (3) with $\mathbb{X} = \mathbb{R}^2$ and $f : (x,u) \mapsto f^*(x) + u$, it is shown in [9] that there exists a smooth vector field f^* such that $C = \{0\}$ is a globally asymptotically stable equilibrium of (4) and the following property holds: For every $\varepsilon > 0$ there is a function $u \in L^1$, with norm $\|u\|_1 = \int_0^\infty |u(t)|dt < \varepsilon$, such that $\dot{x} = f^*(x) + u$ has a global unbounded solution. In [10], this observation has been extended to examples of unbounded solutions in cases wherein $\{0\}$ is a globally exponentially stable equilibrium of (4) and u is exponentially decaying with $|u(0)| > 0$ arbitrarily small. Note that, for every such solution, the nature of the unboundedness must be such that $|x(t)| \to \infty$ as $t \to \infty$ (otherwise, the unbounded function x has nonempty ω-limit set whence, by Theorem 4.2, the contradiction: $x(t) \to 0$ as $t \to \infty$). Extrapolating this observation to the general case, we may deduce the following from Theorem 4.2.

Corollary 4.4: Assume that $A = \mathbb{X} = \mathbb{R}^N$. If x is a global solution of (3) with $u \in L^p$ for some $p \in [1,\infty)$, then the following dichotomy holds: Either $d_C(x(t)) \to 0$ as $t \to \infty$ or $|x(t)| \to \infty$ as $t \to \infty$.

3) It has been shown (by the scalar example in the Introduction) that, by itself, the hypothesis of asymptotic stability of a compactum C (with domain of attraction A) for the autonomous system (4) is not sufficient to guarantee convergence (as $t \to \infty$) to C of a global solution x of (3) with input $u \in L^p$. Theorem 4.2 establishes that the additional hypothesis that $\Omega(x) \cap A \neq \emptyset$ ensures convergence of x to C. With this latter hypothesis in place, it is reasonable to revisit the former to ask if asymptotic stability of C (that is, stability and attractivity) can be weakened to attractivity of C: in other words, can the assumption of stability of C be removed? This question is answered negatively by appealing to [11, Th. 1.6] which subsumes the following. Let $N = 2 = M$ and $f : (x,u) \mapsto f^*(x) + u$ with f^* is continuously differentiable. Assume further that z^* is a globally attractive equilibrium of (4) and that there exists a nontrivial homoclinic orbit O of (4) connecting z^* to itself (in which case z^* fails to be a stable equilibrium). Then, for each $x^0 \in O$, there exists a smooth exponentially-decreasing function $u : \mathbb{R}_+ \to \mathbb{R}^2$ such that, with initial data $x(0) = x^0$, the maximal solution x of (3) is global and $\Omega(x) = \mathrm{cl}(O)$.
4) Theorem 4.2 and Corollary 4.4 posit that x is a *global* solution of (3). There are, of course, *ad hoc* conditions that one could impose on system (3) to guarantee that maximal solutions are globally defined. For example, if $\mathbb{X} = \mathbb{R}^N$ and there exist continuous functions $\rho_0, \rho_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $|f(z,v)| \le \rho_0(|v|) + \rho_1(|v|)|z|$ for all $(z,v) \in \mathbb{X} \times \mathbb{R}^M$, then an application of Gronwall's Lemma confirms that essential boundedness of the input function u is sufficient to conclude that every maximal solution of (3) is global. A further example of a system of form (3), for which essential boundedness of the input function ensures that maximal solutions are globally defined, is described later. ▲

V. EXAMPLE

Let $\mathbb{X} = \mathbb{R}^N$ and $f : (x, u) \mapsto f^*(x) + g(x)u$ for some locally Lipschitz f^* and g. Assume that f^* is a gradient vector field in the sense that, for some continuously differentiable function (potential) $P : \mathbb{X} \to \mathbb{R}, f^*(x) = -\nabla P(x)$ for all $x \in \mathbb{X}$. Assume further that $f^*(0) = 0, f^*(x) \neq 0$ for all $x \neq 0$, and every sub-level set $\{x \in \mathbb{X} | P(x) \leq \lambda\}$ of P is compact. It is straightforward to verify that $C := \{0\}$ is an asymptotically stable equilibrium, with global domain of attraction $A = \mathbb{X}$, of the autonomous system (4). Now assume that the locally Lipschitz function $g : \mathbb{X} \to \mathbb{R}^{N \times M}$ is such that, for every sequence (z_n) in $\mathbb{X} \backslash C$

$$|z_n| \to \infty \text{ as } n \to \infty \implies \frac{|g(z_n)|}{|f^*(z_n)|} \to 0 \quad \text{as } n \to \infty. \tag{9}$$

We claim that, if $u \in L^\infty \cap L^p$ for some $p \in [1, \infty)$, then the unique (recall Remark 3.1) maximal solution $x : [0, \omega) \to \mathbb{X}$ of the initial-value problem $\dot{x} = f^*(x) + g(x)u, x(0) = \xi \in \mathbb{X}$, is global and approaches $C = \{0\}$. Observe that the claim is false if the hypothesis that $u \in L^\infty \cap L^p$ (for some $p \in [1, \infty)$) is weakened to $u \in L^p$, as the following scalar counterexample illustrates: $\dot{x} = -(1/2)x^3 + x^2 u, x(0) = 1$, which, for the (unbounded) input $u \in L^1$ (with norm $\|u\|_1 = 2$) given by $u(t) = 1/\sqrt{1-t}$ for all $t \in [0, 1)$ and $u(t) = 0$ for all $t \geq 1$, has unbounded solution $t \mapsto x(t) = 1/\sqrt{1-t}$ with maximal interval of existence $[0, 1)$.

We proceed to prove the claim. Let $\xi \in \mathbb{X}, u \in L^\infty \cap L^p$ for some $p \in [1, \infty)$ and write $\|u\|_\infty := \text{ess sup}_{t \in \mathbb{R}_+} |u(t)|$. By property (9), together with compactness of the sub-level sets of P, it follows that, for some $\lambda_0 \in \mathbb{R}$

$$-|f^*(z)| + |g(z)| \|u\|_\infty < 0 \quad \text{for all } z \text{ with } P(z) > \lambda_0. \tag{10}$$

For almost all $t \in I(\xi, u)$, we have

$$\begin{aligned} (P \circ x)'(t) &= \langle \nabla P(x(t)), \\ f^*(x(t)) + g(x(t))u(t) \rangle &\leq -|f^*(x(t))|^2 \\ &\quad + |f^*(x(t))| |g(x(t))| \|u\|_\infty \end{aligned}$$

which, in conjunction with (10), implies that $P(x(t)) \leq \lambda_1 := \max\{P(\xi), \lambda_0\}$ for all $t \in I(\xi, u)$ and so, by compactness of the λ_1-sub-level set, x is bounded. Therefore, $\omega = \infty$ and $\Omega(x) \neq \emptyset$. By Theorem 4.2, we infer that x approaches $C = \{0\}$.

ACKNOWLEDGMENT

The author would like to thank E. D. Sontag for drawing his attention to [2] and [11].

REFERENCES

[1] E. D. Sontag, "A remark on the converging-input converging-state property," *IEEE Trans. Autom. Control*, vol. 48, no. 2, pp. 313–314, Feb. 2003.

[2] L. Markus, "Asymptotically autonomous differential systems," in *Contributions to the Theory of Nonlinear Oscillations III*, S. Lefschetz, Ed. Princeton, NJ: Princeton Univ. Press, 1956, pp. 17–29.

[3] D. Angeli, B. Ingalls, E. D. Sontag, and Y. Wang, "Separation principles for input-output and integral-input-to-state stability," *SIAM J. Control Optim.*, vol. 43, pp. 256–276, 2004.

[4] H. W. Knobloch and F. Kappel, *Gewöhnliche Differentialgleichungen*, B. G. Teubner, Ed. Stuttgart, Germany, 1974.

[5] N. P. Bhatia and G. P. Szegö, *Stability Theory of Dynamical Systems*. New York: Springer-Verlag, 1970.

[6] F. W. Wilson Jr., "Smoothing derivatives of functions and applications," *Trans. Amer. Math. Soc.*, vol. 139, pp. 413–428, 1969.

[7] T. Nadzieja, "Construction of a smooth Lyapunov function for an asymptotically stable set," *Czech. Math. J.*, vol. 115, pp. 195–199, 1990.

[8] H. Logemann and E. P. Ryan, "Non-autonomous systems: Asymptotic behavior and weak invariance principles," *J. Diff. Equations*, vol. 189, pp. 440–460, 2003.

[9] E. D. Sontag and M. Krichman, "An example of a GAS system which can be destabilized by an integrable perturbation," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 1046–1049, Jun. 2003.

[10] A. R. Teel and J. Hespanha, "Examples of GES systems that can be driven to infinity by arbitrarily small additive decaying exponentials," *IEEE Trans. Autom. Control*, vol. 49, no. 8, pp. 1407–1410, Aug. 2004.

[11] H. R. Thieme, "Asymptotically autonomous differential equations in the plane," *Rocky Mountain J. Math.*, vol. 24, pp. 351–380, 1994.

Controllability, Observability, and Parameter Identification of Two Coupled Spin 1's

Domenico D'Alessandro

***Abstract*—In this note, we study the control theoretic properties of a couple of interacting spin 1's driven by an electromagnetic field. In particular, we assume that it is possible to observe the expectation value of the total magnetization and we study controllability, observability, and parameter identification of these systems. We give conditions for controllability and observability and characterize the classes of equivalent models which have the same input–output behavior. The analysis is motivated by the recent interest in three level systems in quantum information theory and quantum cryptography as well as by the problem of modeling molecular magnets as spin networks.**

***Index Terms*—Controllability, identification, observability, quantum control, spin dynamics.**

I. INTRODUCTION

In recent years, there have been several proposals to use three level systems, the so-called *qutrits*, in quantum information theory. The proposals concern the use of these systems as building blocks for protocols in quantum cryptography [13] and communication [7] as well as for the encoding of two logic qubits [16]. They also have been used to study fundamental questions in quantum mechanics such as entanglement measures [5], [9]. A study of control of three level systems was considered in [6]. From a quantum control perspective, a system of two coupled three level systems represents the next more difficult case after the well studied system of coupled spin $(1/2)$'s [11], [19], [21]. Motivation to study these systems also comes from the problem of modeling *molecular magnets*. These novel materials [3], [4], [14], [20] are of interest in many applications as nanosize magnets as well as for fundamental studies in quantum mechanics and biology. They are modeled as networks of interacting spins. Spin 1's are a very common example of three level systems. Examples are the nuclear spins of the naturally occurring isotopes ^{6}Li, ^{2}H, ^{14}N.

We shall study the control-related properties, namely *controllability, observability*, and *parameter identifiability*, for a pair of interacting spin 1's particles. To be more specific, we will consider an Heisenberg spin model with Hamiltonian given by

$$H(t) := i(A + B_x u_x(t) + B_y u_y(t) + B_z u_z(t)) \tag{1}$$

Manuscript received July 17, 2003; revised April 13, 2005. Recommended by Associate Editor J. M. A. Scherpen. This work was supported by the National Science Foundation Career Grant ECS-0 237 925.

The author is with the Systems and Control Theory, Department of Mathematics, Iowa State University, Ames, IA 50011 USA (e-mail: daless@iastate.edu).

Digital Object Identifier 10.1109/TAC.2005.851460